GEOMETRIC QUANTIZATION

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Abstract. Geometric Quantization (GQ) is a program for obtaining a quantum mechanical system given a classical mechanical system. The object of quantization need not be a physically feasible mechanical system, symplectic manifolds (with sufficiently nice properties) are also quantizable. For example, the quantization procedure for attaining spin originates from the 2-sphere, and not a cotangent bundle of some phase space. However, quantization is not a categorical functor between symplectic manifolds and quantum mechanical spaces, therefore this procedure may not seem completely natural. This expository paper assumes little background in differential geometry, and some familiarity with analytical mechanics, and quantum mechanics.

Contents

1. Introduction

Geometric Quantization aims to answer the question "What is the corresponding quantum mechanical system for a given classical system?" mathematically. We won't have a correspondence between the states of a classical system (points of a manifold) and the states of a quantum system (complex square integrable functions). The objects of interest to the observer are, as the name suggests, observables, and not states. States are simply the mathematical objects from which the observables, the relevant variables for physics, are derived from. For a physical theory, as long as the observables are consistent with experiment, the rest of the objects of the theory can be seen as appendices, which can later be related mathematically to other theories.

Therefore, the main objective of GQ is to find a mapping of observables. Simplest such mapping is taught in an introductory quantum mechanics course: *Canonical*

Quantization. Its simplicity is a compensation for its lack of generality, given it can only be used for flat phase spaces of finite dimensional systems. The procedure for q^i coordinates is as follows:

(1) Given a classical mechanical phase space $U \subset \mathbb{R}^{2n}$ with coordinates (q^i, p_i), take the space of square integrable complex functions on $\pi_Q(U)$ with the usual inner product, where π_Q is projection to q^i coordinates.

$$\mathcal{H} = L^2(U)$$

However the actual space of states is the projective space

$$\mathbb{P}\mathcal{H} = \mathcal{H}/\mathbb{C}$$

where the states are rays $\{\lambda\psi | \lambda \in \mathbb{C}\}$ in $\mathcal{H}$

(2) Given an observable $f(q^i, p_i) : \mathbb{R}^{2n} \to \mathbb{R}$, replace $q^i \mapsto q^i$ and $p_i \mapsto -i\partial_{q^i}$ to get $\hat{f} : \mathcal{H} \to \mathcal{H}$.

Some readers will be more familiar with the form $p_i \mapsto -i\hbar\partial_{q^i}$, but we will assume $\hbar = 1$ in this paper. The procedure for p_i coordinates is similar, only with π_P, $q^i \mapsto -i\partial_{p^i}$, and $p_i \mapsto p_i$.

The biggest shortcoming of Canonical Quantization is its failure to be coordinate free. The symmetries of a system are sometimes only apparent with coordinate changes or as we will define later, canonical transformations, which preserve the classical mechanics of the system, but is not compatible with this procedure. Therefore, we will try to find a quantization method using differential geometry objects to get a coordinate free procedure for quantization.

The structure of the paper is as follows:

(1) **Classical Mechanics**. We develop the geometric formalism of classical mechanics, however, we assume some familiarity with analytical mechanics (Lagrangian, Hamiltonian formalisms) for which the reader can use [1].
(2) **Quantum Mechanics**. We review the formalism of quantum mechanics, and go over the notation.
(3) **Quantization**. The properties a good quantization procedure should satisfy are laid out. The first naïve attempts in finding such a quantization are displayed to motivate the mathematical objects used in prequantization.
(4) **Prequantization**. A natural procedure of attaining a mapping of observables satisfying the required properties. However, the space which observables act on proves to be too big, which calls for the next step.
(5) **Polarization**. Since the space of states is $2n$ dimensional, but a quantum state depends on n variables, we need to choose which n variables on a $2n$ dimensional manifold are to be used.
(6) **Half-form Correction.** Introducing polarizations causes convergence problems. To aid this, we integrate over a smaller space, but this space doesn't have a natural choice for measure. Instead we pair wavefunctions with "half" n-forms, so when they are paired, they are integrable.

2. Classical Mechanics

There are many equivalent formulations of classical mechanics: Newtonian, Lagrangian, and the Hamiltonian, etc... In this paper, we will work exclusively with the Hamiltonian formalism, and present the connection between the Lagrangian and the Hamiltonian formalism.

Definition 2.1. *Configuration Space* Q of a system is the smooth n-dimensional manifold with the allowed physical configurations as its points.

If we work locally, a point q in Q will have a form $q = (q^1, ..., q^n)$, in which all q^i are generalized coordinates. By generalized coordinates we mean the variables which help us distinguish physical states from one another. The collection of all generalized coordinates should uniquely determine the state of the system. The angle θ of a pendulum swinging in a plane, the distance r of an orbiting body from a focus point and its angle ϕ from some axis in a plane are examples for generalized coordinates.

Definition 2.2. *Phase Space* T^*Q of a system is the cotangent bundle of its configuration space Q.

Elements of T^*Q are (q, p), with $q \in Q$ and $p \in T_q^*Q$, where q is the generalized position and p is the generalized momentum at that point. Again, if we work locally, points will take the form $(q, p) = (q^1, ..., q^n, p_1, ...p_n)$, where we are using the Einstein convention. Each p_i is the associated generalized momentum to q^i. For the angle a pendulum makes with the vertical, the corresponding generalized momentum would be the angular momentum.

The reader might raise the question why we use the cotangent bundle and not the tangent bundle which seems more natural for describing states with elements $(q, v) \in TQ$ describe generalized position and generalized velocity, the two having a more direct relation. Indeed, Lagrangian formulation uses the space TQ, however, the symmetries, i.e. the invariant observables are more apparent in the Hamiltonian formalism described with T^*Q and the relationship between the two will be established later.

Definition 2.3. An *observable* $f \in C^\infty(T^*Q)$ is a smooth function on the phase space.

Observables offer all the data a system has to offer. These might be the energy of the system, total momentum of the system, or a specific generalized momentum.

Hamilton's equations govern the time evolution of a given state in T^*Q.

Definition 2.4. The Hamiltonian H of a system is a prescribed observable of a given system.

Hamiltonian is prescribed: it is not derived (except from the lagrangian if it were given), but it is a natural observable which governs the physics of the system. If we were working locally (or in $\mathbb{R}^{2n}$), Hamilton's Equations take the form

$$\begin{aligned} \frac{dq^i}{dt} &= \frac{\partial H}{\partial p_i} \\ \frac{dp_i}{dt} &= -\frac{\partial H}{\partial q^i} \end{aligned} \tag{2.5}$$

where p and q are actually the local coordinates of a trajectory $x(t) = (q(t), p(t))$ in T^*Q.

We would like to state Hamilton's Equations (2.5) in a coordinate free manner. The solution trajectories are the integral curves of the vector field $X_H = (\frac{\partial H}{\partial p_i}, -\frac{\partial H}{\partial q^i})$. So given the Hamiltonian H of the system, we would like to obtain this vector field X_H without using coordinates.

We can obtain an object similar to this vector field, the covector field dH, which is locally $dH = \frac{\partial H}{\partial q^i} dq^i + \frac{\partial H}{\partial p_i} dp_i$. Therefore we need a map $\# : T^*M \to TM$[1] such that $\#(dH) = X_H$. We can take the dual approach and look for the inverse map $\flat : TM \to T^*M$ since the mapping is an isomorphism.

We try ω as a (2,0)-tensor field for our attempt at finding the map $\flat$. It turns out this tensor field is actually a 2-form: for conservation of energy to hold, the Hamiltonian should be constant along the solution trajectories, i.e. integral paths of X_H. Mathematically stated: $dH(X_H) = 0$ or $\omega(X_H, X_H) = 0$. Which implies that ω should be a form. Furthermore,

(1) ω **must be non-degenerate**: If $\omega(X, Y) = 0$ for all vector fields Y, then $X = 0$.

This is to ensure that the map $\flat : TM \to T^*M$ given by $\flat(X) = \omega(X, \cdot)$ is an isomorphism. If ω were to be degenerate, there would be some nonzero vector field $X \neq 0$ for which $\omega(X, \cdot)$ would be the zero map, thus $X \in \ker \flat$.

(2) ω **must be closed**: $d\omega = 0$.

This is to ensure that ω is invariant over time: $\rho_t^* \omega = \omega$ where ρ_t is the flow along X_H. We need $\mathcal{L}_{X_H} \omega = 0$. By Cartan's formula,

$$\mathcal{L}_{X_H} \omega = X_H \lrcorner d\omega + d(X_H \lrcorner \omega) = X_H \lrcorner d\omega + d(dH) = X_H \lrcorner d\omega$$

which vanishes if and only if $d\omega = 0$.

Therefore, to formulate Hamilton's Equations globally we need a non-degenerate, closed 2-form.

Definition 2.6. A *Symplectic Manifold* (M, ω) is a manifold equipped with a non-degenerate and closed symplectic form ω.

Usually $\dim M = 2n$ is included in the definition of a symplectic manifold, but we will show that this fact follows from the definition we gave. Before the proof, we will introduce a definition and a lemma.

Definition 2.7. Given a tangent space T_mM and a subspace $F \subseteq T_mM$, the *symplectic complement* $F^\perp$ is the set of vectors $X \in T_mM$ such that $\omega(X, Y) = 0$ for all vectors $Y \in F$.

$$F^\perp = \{X \in T_mM | \omega(X, Y) = 0 \quad \forall Y \in F\}$$

Lemma 2.8. *Let (M, ω) be a symplectic manifold and $F \subseteq T_mM$ for some $m \in M$. Then*

$$\dim F + \dim F^\perp = \dim T_mM$$

Proof. $\flat : T_mM \to T_m^*M$ is an isomorphism.
Restrict the image to F^* to get $\tilde{\flat} : T_mM \to F^*$.
By rank-nullity theorem, $\dim T_mM = \dim \ker \tilde{\flat} + \dim \operatorname{im} \tilde{\flat}$.
But $\ker \tilde{\flat} = F^\perp$ and $\dim \operatorname{im} \tilde{\flat} = \dim F^* = \dim F$. □

Theorem 2.9. *Let (M, ω) be a symplectic manifold. Then M is even dimensional.*

Proof. Choose a k-dimensional subspace $F \subset T_mM$ such that $\omega_m(X, Y)$ for all X, Y in F.
$F \subseteq F^\perp$ by construction, therefore we have $\dim F \leq \dim F^\perp = \dim T_mM - \dim F$.

[1] I sometimes abuse the notation by using TM instead of $\Gamma(TM)$. Whether TM is used correctly or not should be clear from the context.

$\dim F \leq \frac{1}{2} \dim T_m M$
If $F \neq F^\perp$, then choose a vector $X \in F^\perp - F$ and take the span of F and X.
By induction, there will be a $n \geq k$ subspace W such that $\dim W = n = \frac{1}{2} \dim T_m M$.
□

Theorem 2.10 (Darboux's Theorem[2])**.** *Let (M, ω) be a symplectic manifold. Then for each point $m \in M$, there is a neighborhood U with coordinate system (q^i, p_i) such that on U*

$$\omega = dq^i \wedge dp_i$$

We use Darboux's Theorem to check that we get Hamilton's Equations (2.5) when we work locally:

$$X_H = \frac{\partial H}{\partial p_i}\frac{\partial}{\partial q^i} - \frac{\partial H}{\partial q^i}\frac{\partial}{\partial p_i}$$

$$\omega(X_H,) = (dq_i \wedge dp^i)(X_H) = \frac{\partial H}{\partial p_i} dp_i - (-\frac{\partial H}{\partial q^i}) dq^i = \frac{\partial H}{\partial q^i} dq^i + \frac{\partial H}{\partial p_i} dp_i = dH$$

$$\omega(X_H,) = dH$$

Symplectic structure not only provides an isomorphism between TM and T^*M, but is actually a generalized Poisson Bracket.

Definition 2.11. The *Poisson Bracket* $\{\cdot, \cdot\} : C^\infty(M) \times C^\infty(M) \to C^\infty(M)$ is defined as $\{f, g\} = \omega(X_f, X_g)$.

Locally, the Poisson Bracket takes on the form $\{f, g\} = \frac{\partial f}{\partial p}\frac{\partial g}{\partial q} - \frac{\partial g}{\partial p}\frac{\partial f}{\partial q}$.

The Poisson Bracket is a way of measuring how one observable changes along the flow of another. For example $\{f, H\} = 0$ implies that f is invariant over time, since flow of H denotes the time evolution of the system. Or stated more formally,

$$\{f, H\} = \frac{\partial f}{\partial p}\frac{\partial H}{\partial q} - \frac{\partial H}{\partial p}\frac{\partial f}{\partial q} = \frac{\partial f}{\partial p}\frac{dp}{dt} + \frac{\partial f}{\partial q}\frac{dq}{dt} = \frac{df}{dt}$$

We now have the foundations of Hamiltonian formalism of classical mechanics. We can define a classical system with the space of states M, the Hamiltonian H which is the variable linked to the time evolution of the system, the form ω which gives the isomorphism between TM and T^*M. In short, $\mathcal{C} = (M, H, \omega)$.

Relationship between the Lagrangian and the Hamiltonian (Optional). This section is devoted to demystifying the geometrical relationship between the Lagrangian and Hamiltonian formalisms.

Instead of T^*Q, we will work with TQ, and use the observable *Lagrangian* $L \in C^\infty(TQ)$.

Lagrangian formalism works by looking at all possible paths $q(t) = (q^i(t))$ in Q between fixed x_i and x_f, and choosing the path with the least action.

Definition 2.12. Given a path $q(t) : [t_i, t_f] \to Q$ with $q(t_i) = q_i$ and $q(t_f) = q_f$, its *action* is given by

$$I[q(t)] = \int_{t_i}^{t_f} L(q(t), \dot{q}(t)) dt$$

[2]See [9] for the proof.

Hamilton's Principle states that the paths $x(t)$ such that $\delta I = 0$ are the trajectories the system will follow. What is meant by $\delta I = 0$ can be explained mathematically. I is a function on the space of paths $\Omega = \{q(t)|q(t_i) = q_i \text{ and } q(t_f) = q_f\}$. We would like to find the $q(t)$ for which $X(I) = 0$ for every vector $X \in T_{q(t)}\Omega$. In more intuitive terms, it means that changing the path $q(t)$ infinitesimally while keeping the endpoints fixed does not change the action.

Using calculus of variations[3], this problem is equivalent to solving *Lagrange's Equations*

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}^i} - \frac{\partial L}{\partial q^i} = 0 \tag{2.13}$$

Again, we would like to find a coordinate free formulation of Lagrange's Equations (2.13) similar to what we did with Hamilton's Equations (2.5).

We will construct a map from T_qQ to $T_{(q,v)}(TQ)$. Let $u \in T_qQ$. Map this u to $F(u)$, the tangent vector to the curve $(q, v + tu)$ as $t \to 0$.

Let θ_L, a one form on TQ be defined by $\theta_L(X) = dL(F_{\pi_* X})$, where $\pi : TQ \to Q$ is the projection map.

Now let $\omega_L = d\theta_L$ and define the observable H on TQ by $H(q, v) = dL(F(v)) - L$.

We look at $\omega_L(X, \cdot) + dH$ where X is arbitrary and in local coordinates, $X = q^i \frac{\partial}{\partial q^i} + \dot{q}^i \frac{\partial}{\partial \dot{q}^i}$. After calculations[4] in local coordinates, we find

$$\omega_L(X, \cdot) + dH = (\frac{d}{dt}\frac{\partial L}{\partial \dot{q}^i} - \frac{\partial L}{\partial q^i})dq^i$$

So if we require $\omega_L(X, \cdot) + dH = 0$, it is a coordinate free way of expressing Lagrange's Equations (2.13).

We now make the transition to T^*Q with the *Legendre Transformation* $\rho : TQ \to T^*Q \quad (q, v) \mapsto (q, p)$, where p is determined by $p(u) = dL(F(u))$ for all $u \in T_qQ$.

3. Quantum Mechanics

Quantum mechanics is the physics of the very small, by very small it is meant it takes effects near the scale of $h = 1.0545718 * 10^{-34} m^2 kg/s$. To simplify the notation, we will take $\hbar = 1$.

Postulates of Quantum Mechanics are:

1. **States**. The states of a quantum mechanical system are rays $|\psi\rangle$ in an infinite dimensional seperable complex Hilbert Space $\mathcal{H}$.
2. **Observables**. An observable $\mathcal{O}$ is a linear Hermitian operator on $\mathcal{H}$. One may observe only one of the eigenvalues of $\mathcal{O}$ and the state will assume the corresponding eigenstate after measurement.
3. **Dynamics**. Time evolution of a system is given by $\hat{H}|\psi\rangle = i\frac{\partial}{\partial t}|\psi\rangle$.

Given the observable $\mathcal{O}$ with eigenvalues $a_1, ...$ and eigenstates $|a_1\rangle, ...$, we may expand any state in this eigenbasis: $|\psi\rangle = \langle a_1|\psi\rangle |a_1\rangle + ...$ and we have the probability that we will observe the value a_1 as $\langle a_1|\psi\rangle^2$, which will cause it to "collapse" to the state $|a_1\rangle$ after the measurement.

[3]For the intermediary steps, refer to [9]

[4]The details can be found in [9]

A Quantum System therefore consists of the Hilbert space $\mathcal{H}$ (modulo scalars) of states and the Hamiltonian $\hat{H}$ which is responsible for the time evolution, i.e. $\mathcal{Q} = (\mathcal{H}, \hat{H})$.

4. Quantization

The goal of quantization is to find a quantum mechanical system $(\mathcal{H}, \hat{H})$ given a classical system $\mathcal{C} = (M, H, \omega)$. We have no way of differentiating H from other observables when not given a specific system, so it is better if we could find a mapping from classical observables to quantum mechanical observables $f \mapsto \hat{f}$ which would not only help us determine the Hamiltonian in the Quantum System, but also provide us with all the observables we could ask for from that system.

The properties such a mapping should satisfy are determined so that they preserve the *Poisson Algebra* structure of classical observables(3): If $\{f, g\} = h$, then $[\hat{f}, \hat{g}] = \hat{h}$ (modulo constants). Furthermore, we want such a mapping to behave well, so we would ask it to be linear (1) and map constant classical observables to constant quantum observables (2).

The conditions such that this structure is preserved are **Dirac Quantization Conditions**:

(1) The map $f \mapsto \hat{f}$ is $\mathbb{R}$-linear
(2) f is constant $\implies$ $\hat{f}$ is the corresponding multiplication operator.
(3) $\{f_1, f_2\} = f_3 \implies [\hat{f}_1, \hat{f}_2] = -i\hat{f}_3$

Remark 4.1. The appearance of $-i$ in condition (3) might seem odd. However, notice $[\hat{q}, \hat{p}]^\dagger = -[\hat{q}, \hat{p}]$ so $[\hat{q}, \hat{p}]$ should be purely imaginary. Both i and $-i$ are valid and choosing $-i$ is merely a matter of convention.

First Attempt. Our first attempt is to use square integrable functions on the manifold with respect to $\epsilon = \frac{1}{(2\pi)^n}\omega \wedge ... \wedge \omega$ as the states. The inner product to render this space a Hilbert space would be

$$\langle \psi, \phi \rangle = \int_M \bar{\psi}\phi\epsilon$$

Our first candidate as the mapping of observables is $f \mapsto -iX_f$ with conditions (1) and (3) in mind. However, it fails to satisfy condition (2) since

$$f = c \implies \hat{f} = -iX_f = 0 \neq c\mathbb{I}$$

Second Attempt. To mend the shortcoming on condition (2), we may try a new map $f \mapsto -iX_f + f$. This time, it fails to satisfy condition (3):

$$[\hat{f}_1, \hat{f}_2] = -X_{f_1}X_{f_2} + X_{f_2}X_{f_1} - 2iX_{f_1}f_2 + 2iX_{f_2}f_1$$

$$= -X_{\{f_1,f_2\}} - 2i\{f_1, f_2\} = -X_{f_3} - 2if_3 = -i\hat{f}_3 - if_3 \neq -i\hat{f}_3$$

Third Attempt. Now we assume, for the sake of argument that $\omega = d\theta$, where θ is called the symplectic potential (ω may not be always globally exact, but we will deal with this later). Then we try the following map:

$$f \mapsto -iX_f + f - \theta(X_f) \tag{4.2}$$

We check if it satisfies the conditions:

(1) $cf \mapsto -iX_cf + cf - \theta(X_cf) = c\hat{f}$
(2) $f = c \implies -iX_c + c\mathbb{I} - \theta(X_cf) = c\mathbb{I}$

(3) $[\hat{f}_1, \hat{f}_2] = -X_{\{f_1,f_2\}} - 2i\{f_1, f_2\} + i(X_{f_1}(\theta(X_{f_2})) - X_{f_2}(\theta(X_{f_1})))$
using the identity $X_{f_1}(\theta(X_{f_2}))-X_{f_2}(\theta(X_{f_1})) = \theta([X_{f_1}, X_{f_2}])+d\theta(X_{f_1}, X_{f_2})$
$[\hat{f}_1, \hat{f}_2] = -X_{\{f_1,f_2\}} - 2i\{f_1, f_2\} + i\theta(X_{\{f_1,f_2\}}) + i\{f_1, f_2\} = -i(-iX_{f_3} + f_3 + \theta(X_{f_3})) = -i\hat{f}_3$

We have found a map that satisfies all the conditions, however, we have problems. Firstly, the map is only locally defined (if ω is not globally exact). Secondly, the map depends on the choice of θ since we could as well choose $\theta' = \theta + d\phi$ with arbitrary ϕ and possibly get a different $\hat{f}$.

The first way out is to cover M with contractible open sets $\{U_i\}$ and keep track of all local θ_i's and the corresponding ϕ_{ij}'s for $U_i \cap U_j \neq \emptyset$. Whenever local symplectic potential changes by $d\phi_{ij}$ on $U_i \cap U_j$ change the wave functions as $\psi_j = e^{i\phi_{ij}}\psi_i$. We do this so that the operators act on wavefunctions in a well-defined manner:

$$\hat{f}^{(j)}\psi_j = e^{i\phi_{ij}}(\hat{f}^{(j)}\psi_i + d\phi_{ij}(X_f)\psi_i) = e^{i\phi_{ij}}\hat{f}^{(i)}\psi_i$$

so we have

$$\hat{f}^{(j)} = e^{i\phi_{ij}} \circ \hat{f}^{(i)} \circ e^{-i\phi_{ij}}$$

We have an observable $\hat{f}$ as a collection of locally defined $\hat{f} = \{\hat{f}_i\}$ operators depending on the θ_i and wavefunctions as collection of functions $\{\psi_i\}$ defined on open sets U_i with transitions between them as $e^{i\phi_{ij}}$. This structure resembles that of a complex line bundle and its transition functions between local trivializations, which we introduce in the next section.

5. Prequantization

Definition 5.1. A *Complex Line Bundle* L over a manifold M together with the projection $\pi : L \to M$ is a triple (L, π, M) such that the following hold:

(1) For all points $m \in M$, $\pi^{-1}(m)$ is a one dimensional complex vector space.
(2) There exists an open covering of M $\{U_i\}$ together with diffeomorphisms $\tau_i : U_i \times \mathbb{C} \to \pi^{-1}(U_i)$ mapping $(m, z) \mapsto zs_i(m)$ where $s_i : U_i \to \pi^{-1}(U_i)$ are *unit sections*. Such pairs are called *local trivializations* $\{(U_i, \tau_i)\}$.

Definition 5.2. A *section* $s : M \to L$ of a complex line bundle L is a map such that $\pi \circ s = Id$.

Remark 5.3. Notice that $\tau_i \circ \pi : U_i \times \mathbb{C} \to U_i$ is a projection since s_i is a section.

It is useful to think of a Complex Line Bundle as attaching a vector space V isomorphic to the complex line $\mathbb{C}$ to every point m on M. Since the attached object is a vector field and not the complex line itself, there is no apparent choice for the basis. That's why local trivializations are useful. They specify an isomorphism between $\mathbb{C}$ and V and therefore the basis is determined, $s_i(m)$.

Definition 5.4. A *Transition Function* $\tau_{ij} : U_i \cap U_j \times \mathbb{C} \to U_i \cap U_j \times \mathbb{C}$ is a map which gives the transition from the local trivialization τ_j to τ_i on $U_i \cap U_j$ with

$$\tau_{ij} = \tau_i^{-1} \circ \tau_j$$
$$(m, z) \mapsto (m, g_{ij}(m)z)$$

where $g_{ij} : M \to \mathbb{C}$ is determined by $s_j(m) = g_{ij}(m)s_i(m)$

The g_{ij} satisfy the following *cocycle conditions*:

(1) $g_{ii} = Id$
(2) $g_{ij}g_{ji} = Id$
(3) $g_{ij}g_{jk}g_{ki} = Id$

Apparently, we need sections of a line bundle as wavefunctions to explain the phase differences while transitioning between different U_i's. We require transition functions to coincide with our earlier findings:

$$g_{ij} = e^{i\phi_{ij}}$$

If we rewrite the conditions in terms of ϕ_{ij}'s:

(1) $\phi_{ii} = 0$
(2) $\phi_{ij} + \phi_{ji} = 0$
(3) $\frac{1}{2\pi}(\phi_{ij} + \phi_{jk} + \phi_{ki}) \in \mathbb{Z}$

The third condition can be framed in Čech Cohomology, which in turn can be rephrased in deRham Cohomology[5] because of the isomorphism between the two, and the finally, the condition becomes

$$[\frac{\omega}{2\pi}] \in H_{dR}(M, \mathbb{Z})$$

which is also known as the Integrality Condition.

If this condition is met, we can have a line bundle which has the desired transition functions, and the states would be the space of sections $\Gamma(L)$. The space of states need to be a Hilbert Space, which means our next task is to find an inner product for $\Gamma(L)$.

Before we can do that, we realize that we needed another structure on the bundle.

Definition 5.5. A *Hermitian Structure* $H : L \times L \to \mathbb{C}$ is an inner product satisfying

(1) $H(x, x) \geq 0$
(2) $H(x, y) = H(\bar{y}, x)$
(3) $H(x, \alpha y + z) = \alpha H(x, y) + H(x, z)$
(4) $H(x, x) = 0 \iff x = 0$

Every complex line bundle can be given a Hermitian Structure. With this in mind, we define the inner product for $\Gamma(L)$

$$\langle \psi, \phi \rangle = \int_M H(\psi, \phi)\epsilon$$

Now that we have an inner product, let's only use the square integrable sections $\mathcal{H}$ to avoid having sections with divergent norm.

So we have the space of wavefunctions $\mathcal{H}$ with the inner product H.

Now for the observables we had

$$f \mapsto -iX_f + f - \theta(X_f)$$

This bears another resemblance to a structure we can put on a line bundle, and which most sources use as the main motivation for prequantization as opposed to using transition functions as we've discussed in this paper.

Definition 5.6. A *Connection* $\nabla : \Gamma(TM) \times \Gamma(L) \to \Gamma(L)$ is a way of differentiating sections along vector fields, satisfying the following rules:

[5]See [9] for the details.

(1) $\nabla_X(s+t) = \nabla_X(s) + \nabla_X(t)$
(2) $\nabla_{X+Y}(s) = \nabla_X(s) + \nabla_Y(s)$
(3) $\nabla_X(fs) = f\nabla_X(s) + X(f)s$
(4) $\nabla_{fX}s = f\nabla_X s$

where $s, t \in \Gamma(L)$ and $X, Y \in \Gamma(TM)$.

It should be noted that the Lie derivative and a connection are different. They are generalizations of taking a directional derivative, but the Lie derivative doesn't necessarily satisfy condition (4).

Why connections were introduced will be more apparent when we investigate their relationship to one-forms. If s is the unit section for some U, the connection one form θ is characterized by

$$\nabla s = -i\theta s$$

Such an association with a one form is appropriate since ∇s acts on TM linearly. The appearance of $-i$ in the definition is conventional to clear up notation.

If $s' = \psi s$ is any section on U, we have

$$\nabla_X s' = (X\psi - i\theta(X)\psi)s$$

This bears resemblance to the observables map (4.2) we had constructed from local symplectic potentials. Let's use a connection ∇ with one-form θ as in $\omega = d\theta$ (the existence of such a connection is guaranteed by the integrality condition[6]), and the map becomes

$$f \mapsto -i\nabla_{X_f} + f$$

Remark 5.7. So far, we've been motivated by analogy and Dirac's Conditions. It is natural to seek a more geometric understanding of this mapping of observables. Indeed, this map's Lie Derivative

$$\mathcal{L}_{X_f}(\theta) = \omega(X_f,) + d(\theta(X_f)) = df + d(\theta(X_f))$$
$$d\hat{f} = df + d(\theta(X_f))$$

We've constructed a Hilbert space $\mathcal{H} = (\Gamma(L), H)$ and a mapping for the observables $f \mapsto \hat{f}$ from the following ingredients:

(1) A Symplectic Manifold (M, ω) such that $[\frac{\omega}{2\pi}] \in H_{dR}(M, \mathbb{Z})$
(2) A Line Bundle $\mathcal{B} = (L, \pi, M)$
(3) A Hermitian structure H on the line bundle
(4) A Connection ∇ on the line bundle

So we may define the object we've created by the prequantization process.

Definition 5.8. A *Prequantization* of a symplectic manifold (M, ω) is a triple $(\mathcal{B}, H, \nabla)$.

Remark 5.9. The reader might complain that we've lost ω by excluding it from the definition. However, notice $\operatorname{curv} \nabla = \frac{\omega}{2\pi}$.

[6]See [9] for the proof.

So far so good, however, this isn't the whole story. As we know from quantum mechanics, we shouldn't get functions, or more appropriately sections, on a 2n-dimensional manifold, but we should be getting either functions of position $\psi(x)$ or functions of momentum $\tilde{\psi}(p)$. What condition enforces this, and with what structure we solve this problem follows in the next section.

6. Polarization

6.1. **Problem with Prequantization.** Mathematically, prequantization sure looks more elegant than what we're about to do. But we will have to somehow reduce this space to a smaller space because that's what we observe in quantum mechanics. There is a fourth condition in addition to Dirac Quantization Conditions which calls for *Polarization*:

(4) If $\{f_i\}$ are a complete set of observables, then $\{\hat{f}_i\}$ are a complete set of quantum observables.

Definition 6.1. $\{f_i\}$ are a *complete set of classical observables* if and only if $\{g, f_i\} = 0$ for all i implies g is a constant observable.

Definition 6.2. $\{\hat{f}_i\}$ are a *complete set of quantum observables* if and only if $[\hat{g}, \hat{f}_i] = 0$ for all i implies $\hat{g} = c\mathbb{I}$ is a constant observable.

There is an equivalent formulation by the following theorem

Theorem 6.3. *$\{\hat{f}_i\}$ is a complete set of quantum observables $\implies$ $\mathcal{H}$ is irreducible under the action of $\{\hat{f}_i\}$, meaning a subset $\mathcal{F} \subset \mathcal{H}$ invariant under the action of $\{\hat{f}_i\}$ is either $\{0\}$ or $\mathcal{H}$.*

Proof. Suppose $\{\hat{f}_i\}$ are a complete set of quantum observables and $\mathcal{F}$ a subset invariant under its action. Let P be the projection operator to $\mathcal{F}$. P commutes with $\{\hat{f}_i\}$. Indeed if one has $\psi = \psi_F + \psi_{F^\perp}$,

$$\begin{aligned}[P, \hat{f}_i]\psi &= P\hat{f}_i\psi_F - \hat{f}_iP\psi_F + P\hat{f}_i\psi_{F^\perp} - \hat{f}_iP\psi_{F^\perp} \\ &= \hat{f}_i\psi_F - \hat{f}_i\psi_F + 0 - 0 = 0\end{aligned}$$

since $\mathcal{F}$ is invariant under $\{f_i\}$, $\mathcal{F}^\perp$ is invariant, and $P\phi = 0$ for $\phi \in \mathcal{F}^\perp$. □

Example 6.4. We can see our current space does not satisfy this condition. Choose $M = \mathbb{R}^{2n}$ as the manifold M, the canonical symplectic form $\omega = dq^i \wedge dp_i$, and observables as generalized coordinates$f_i = q^i$ for $i \leq n$ and generalized momenta $f_i = p_i$ for $i > n$. We know q^i and p_i form a complete set of classical observables, now let's see if $\mathcal{H}$ is irreducible or not.

$$\hat{q}_i = -i\frac{\partial}{\partial p_i}$$
$$\hat{p}_i = p_i + i\frac{\partial}{\partial q_i}$$

Now take the set $\mathcal{F}$ of functions of p_i and constant on all q_i. For $\psi \in \mathcal{F}$, we have

$$\hat{q}_i\psi = -i\frac{\partial}{\partial p_i}\psi \in \mathcal{F}$$
$$\hat{p}_i\psi = p_i\psi + i\frac{\partial}{\partial q_i}\psi = p_i\psi \in \mathcal{F}$$

So $\mathcal{F}$ is a nontrivial invariant subspace of $\mathcal{H}$ under the action of a complete set of observables. Therefore it is apparent we will need some way of shrinking the space if we are to satisfy condition (4).

6.2. **Distribution and Polarization.** Our aim is intuitively clear. We should somehow choose which of the n directions the sections should depend on, and stay constant on others (or vice versa). We shall naturally choose n bases in each tangent space, and have them vary smoothly on the manifold.

Definition 6.5. A *Complex Distribution* P on a manifold M is a subbundle of the complex tangent bundle $TM^{\mathbb{C}} = TM \otimes \mathbb{C}$ such that the subspace $P_m \subset T_m M^{\mathbb{C}}$ varies smoothly with m.

Why we used complex tangent bundle instead of the regular one will be apparent when we consider Kähler Polarizations.

We require the sections to satisfy $\nabla_X s = 0$ for $X \in P$. So we've required s to stay constant relative to the directions in P.

For any other section $s' = \psi s$, we would require

$$\nabla_X s' = 0$$
$$= (X(f) + i\theta(X)f)s = 0$$

for all X in P. Now we impose a condition on the polarization. It should be *adapted to the connection* ∇, i.e. for every point in M there exists a local symplectic potential such that $\theta(X) = 0$ for all $X \in P$ such that $d\theta = \omega$ given by the curvature of ∇. With this condition in mind, we have

$$X(f) = 0$$

We require this to be *integrible*: around every point m in M there is a coordinate system (x_i) such that for surfaces Λ of constant $x_{n+1}, ..., x_2 n$ we have $P_m = T_m \Lambda$. By Frobenius Theorem, this condition is equivalent to being *involutive*: for X, Y in P, $[X, Y]$ is in P.

We also have

$$0 = [\nabla_X, \nabla_Y]s = (\nabla_{[X,Y]} + i\omega(X, Y))s$$

By previous condition $\nabla_{[X,Y]} s = 0$ for X, Y in P. Therefore we are left with

$$\omega(X, Y) = 0$$

which means that the distribution is *Lagrangian*: $\omega(X, Y) = 0$ for X, Y in P and $\dim P_m = n$.

Therefore we have the conditions to impose on a distribution to get a *Polarization*.

Definition 6.6. A *Polarization* P on a symplectic manifold (M, ω) is a distribution in $TM^{\mathbb{C}}$ such that it is

(1) **Lagrangian**.
   (a) For all X, Y in P, $\omega(X, Y) = 0$
   (b) For every point $m \in M$, $\dim P_m = n$
(2) **Involutive** (or integrable). For all X, Y in P, $[X, Y]$ is in P.

(3) **Constant Real Directions**. The real part of P, which is called *isotropic distribution* of P, $D = P \cap \bar{P} \cap TM$ has constant dimension $\dim D_m = k$ for all m.

For admissibility, we will assume an additional property on the polarization, but before we can do that, we have to investigate the foliation of M by distributions.

Definition 6.7. Given a distribution P, an *integral manifold* Λ of P is such that $T_m\Lambda = P_m$ where Λ is defined. A maximal integral manifold of P is called a *leaf* of the distribution. The space of leaves is denoted M/P.

Now we define the requirement for admissibility

Definition 6.8. A polarization P is called a *strongly integrable* polarization if and only if

(1) The *coisotropic distribution* of P, $E = (P + \bar{P} \cap TM) = D^\perp$ is involutive.
(2) $\mathcal{D} = M/D$ and $\mathcal{E} = M/E$ are differentiable manifolds.
(3) The projection $\pi : \mathcal{D} \to \mathcal{E}$ is a submersion.

Theorem 6.9. [7] *If P is a strongly integrable polarization, then it is admissible.*

We will assume all polarizations are strongly integrable from now on.

Now that we have polarizations defined, we are ready to use them on our prequantized manifold. Given a polarization, define the space of *polarized states* as the sections of L that do not depend on the directions in P.

$$\Gamma_P(L) = \{s \in \Gamma(L) | \nabla_X s = 0 \quad \forall X \in P\} \tag{6.10}$$

and only choose the observables which map polarized states to polarized states

$$\mathcal{O}_P = \{\hat{f} | \forall s \in \Gamma_P(L) \quad \hat{f}s \in \Gamma_P(L)\} \tag{6.11}$$

It looks like this is the end of the story, but there is a problem we've swept under the rug.

There may be no square integrable sections in $\Gamma_P(L)$. Indeed, if the polarization has real directions, i.e. D is nonempty, then the polarized sections are constant on leaves of D by construction. Therefore, the integral $\int_M (s, s)\epsilon$ diverges if leaves are complete and not compact, and it can be shown there are no polarized sections for a polarization whose leaves are compact. To avoid this problem, we will introduce Kähler Polarizations which have no real directions, but still we will encounter another problem.

Before we delve into Kähler Polarizations, let's look at a trivial example.

Example 6.12. Take the cotangent bundle $T^*\mathbb{R}^{2n}$ with the canonical symplectic form $\omega = dq^i \wedge dp_i$. Take the polarization P spanned by $\{\frac{\partial}{\partial_{p^i}}\}$. The set of polarized sections are such that $\frac{\partial \psi}{\partial p^i} = 0$, which means wavefunctions $\psi(q^i)$ depend only on q^i.

The symplectic potential is $\theta = p_i dq^i$, therefore the map for the observables is $f \mapsto -i\frac{\partial f}{\partial p_i}\frac{\partial}{\partial q^i} - \frac{\partial f}{\partial q^i}\frac{\partial}{\partial p_i} + f - p_i dq^i(\frac{\partial f}{\partial p_i}\frac{\partial}{\partial q^i} - \frac{\partial f}{\partial q^i}\frac{\partial}{\partial p_i})$ and we get

$$\hat{q^i} = q^i$$

[7] See [9] for the proof.

$$\hat{p_i} = -i\frac{\partial}{\partial q^i}$$

Unfortunately, the usual hamiltonian $H = \frac{1}{2}(q^2 + p^2)$ is not quantizable in this example since it does not preserve polarizations. But it is quantizable in the Kähler case as we will show.

6.3. **Kähler Polarizations.** Since we believe it is the real directions in P that's causing the trouble, we will remove them and investigate a special case.

Definition 6.13. A polarization P is a *Kähler* if $P \cap \bar{P} = 0$.

Remark 6.14. Notice that $D = 0$ and $TM^{\mathbb{C}} = P + \bar{P}$ since P is lagrangian.

Definition 6.15. A Kähler Manifold is a triple $(M, \omega, \mathcal{J})$ such that ω is a symplectic form, $\mathcal{J}$ is a complex structure ($\mathcal{J}_m : T_mM \to T_mM$ such that $\mathcal{J}_m^2 = -Id$), and for every X and Y, $\omega(\mathcal{J}X, \mathcal{J}Y) = \omega(X, Y)$.

We now show one can obtain a Kähler Manifold from a Kähler Polarization, and vice versa.

Given a Kähler Manifold, define

$$P = \{X \in TM^{\mathbb{C}} | \mathcal{J}_m X_m = iX_m\}$$

Suppose we have a local coordinate system (q^i, p_i) such that $\mathcal{J}_m \frac{\partial}{\partial q^i} = \frac{\partial}{\partial p_i}$ $\mathcal{J}_m \frac{\partial}{\partial p_i} = -\frac{\partial}{\partial q^i}$. Define *holomorphic coordinates* as $z_i = \frac{1}{2}(q^i + ip_i)$ which gives a nice basis (together with the conjugate) for the complexified tangent space

$$\frac{\partial}{\partial z_i} = \frac{1}{2}(\frac{\partial}{\partial q_i} - i\frac{\partial}{\partial p^i})$$

so that $\mathcal{J}_m \frac{\partial}{\partial z_i} = -\frac{\partial}{\partial z_i}$ and $P_m = span\frac{\partial}{\partial z_i}$.

Now suppose we are given a Kähler polarization P. We will construct a complex structure $\mathcal{J}$ compatible with ω.

Since we have $TM^{\mathbb{C}} = P + \bar{P}$, given X, we can decompose it.

$$X = X_P + X_{\bar{P}}$$

Define $\mathcal{J}$ such that $\mathcal{J}X = iX_P - iX_{\bar{P}}$.

A nice property about Kähler Manifolds is that closed 2-forms can be represented locally by a smooth function. Before we show this, we introduce new notation.

$$\partial = \frac{\partial}{\partial z_i} dz_i$$

$$\bar{\partial} = \frac{\partial}{\partial \bar{z_i}} d\bar{z_i}$$

Note that we have $d = \partial + \bar{\partial}$.

Theorem 6.16. *A 2-form ω is closed if and only if there is a smooth function $\mathcal{K}$ such that $\omega = \partial\bar{\partial}\mathcal{K}$*

Proof. See [8, p. 85] □

We define the function for the symplectic form, also known as the Kähler form, with an extra factor of i.

$$\omega = i\partial\bar{\partial}\mathcal{K}$$

We choose symplectic potentials adapted to P and $\bar{P}$.

$$\theta_{\bar{P}} = i\partial\mathcal{K}$$

$$\theta_P = -i\bar{\partial}\mathcal{K}$$

If we choose the polarization P, we will have polarized sections ψ such that locally $\psi_i = f_i s_i$ where s_i is the unit section and f_i satisfies $\partial f_i = 0$.

Now we take $\mathcal{H}_P = \mathcal{H} \cap \Gamma_P(L)$ as the space of polarized square integrable sections as our space of states. This space is complete and therefore a hilbert space.

Mathematically, it seems as if we're done, but let's quantize an example using Kähler Polarizations to show how there is still a discrepancy.

Example 6.17. Let $M = \mathbb{R}^{2n}$ with $\omega = dq^i \wedge dp_i$. Endow it with a complex structure $\mathcal{J}$ such that

$$\mathcal{J}\frac{\partial}{\partial q^i} = \frac{\partial}{\partial p_i}$$

$$\mathcal{J}\frac{\partial}{\partial p_i} = -\frac{\partial}{\partial q^i}$$

and denote holomorphic coordinates by z_i as usual. Hamiltonian and symplectic form in this coordinates are given by

$$H(m) = \frac{1}{2}\sum (q^i)^2 + p_i^2 = \frac{1}{2} z_i \bar{z}_i$$

$$\omega = \frac{i}{2} d\bar{z}_i \wedge dz_i$$

We choose the polarization spanned by $\{\frac{\partial}{\partial \bar{z}_i}\}$ so that as the space of wavefunctions we have functions $\psi : \mathbb{R}^{2n} \to \mathbb{C}$ depending only on z_i, i.e. $\frac{\partial}{\partial \bar{z}_i}\psi = 0$.

The symplectic potential admissible for our polarization is $\theta_P = -i\bar{\partial}\mathcal{K}$. We quantize the hamiltonian.

$$\hat{H} = z_i \frac{\partial}{\partial z_i}$$

Solving the eigenvalue problem, we find

$$z_i \frac{\partial}{\partial z_i}\psi = E\psi$$

Therefore eigenfunctions have form $\psi(z) = \alpha z_1^{n_1} \dots z_k^{n_k}$ and the eigenvalues are $N = n_1 + \ldots + n_k$. However, we know this is incorrect since it doesn't take account of the vacuum energy. Metaplectic correction gives us the correct eigenvalues of $N + \frac{1}{2}$.

7. Half-form Correction

Motivation. The main problem is that for polarizations with real directions, the inner product is constant along the leaves of D, therefore the integral diverges. To remedy this, we will define an inner product on M/D. However, there is no natural choice of a measure in M/D, therefore we will pair the wavefunctions with "half forms" to get $\psi \otimes \nu$ and $\psi' \otimes \nu'$ and so that (ν, ν') defines a measure on M/D and we integrate (ψ, ψ') with respect to this measure.

Density. The object of integration is usually an n-form, but then for integral to be defined, the manifold would have to be oriented. This is because n-forms change with the determinant of the Jacobian when we have a change of coordinates, whereas the integral changes by the absolute value of the determinant. We introduce densities which change by the absolute value of the Jacobian.

Definition 7.1.
A *density* μ on a vector space V is a function such that $\mu : V \times \cdots \times V \to \mathbb{R}$ such that

$$\mu(Av_1, \ldots, Av_n) = |\det A| \mu(v_1, \ldots, v_n)$$

A density μ on a manifold M is such that μ_m is a density on $T_m M$.

Notice that densities are integrable the same way n-forms are, and every n-form ω defines a density $|\omega|$ by $|\omega|(a) = |\omega(a)|$.

Consider the bundle of frames $\mathcal{B}P$ for the polarization P. This is a bundle such that if $\pi : \mathcal{B}P \to M$ and $\pi(a) = m$, then $a = \{v_1, \ldots, v_n\}$ is a frame for P_m. If two frames $a, b \in \mathcal{B}P$ project down to the same point m, they are related by a linear transformation $g \in GL_n(\mathbb{C})$ by $b = ga$. We call such bundles *principal* $GL_n(\mathbb{C})$*-bundles.*

Metalinear Group. We have $\mu_m(b) = |\det g| \mu_m(a)$. We would like to construct functions ν, χ such that $\nu_m(b) = \chi(g)\nu_m(a)$ with $\chi(g)^2 = |\det g|$ where ν is a function such that if paired with another ν', would give μ, and χ is a square root of $|\det g|$. However, this requires us to choose a specific square root of $|\det g|$ for every g. Instead, we will construct a double cover of $GL_n(\mathbb{C})$ such that the two elements projecting to the same $g \in GL_n(\mathbb{C})$ will represent the two roots of $|\det g|$.

Consider the surjective map $p : \mathbb{C} \times SL_n(\mathbb{C}) \to GL_n(\mathbb{C})$ given by $(z, A) \mapsto e^z A$. We find the kernel of p.

$$p(z, A) = e^z A = I \implies A = e^{-z} I$$
$$\det A = 1 = \det e^{-z} I = e^{-nz}$$
$$z = \frac{2\pi i k}{n} \quad k \in \mathbb{Z}$$
$$\ker p = \{(z, A) | z = \frac{2\pi i k}{n} \quad A = e^{-z} I \quad k \in \mathbb{Z}\}$$

So that we have $\frac{\mathbb{C} \times SL_n(\mathbb{C})}{\ker p} \simeq GL_n(\mathbb{C})$. To obtain a double cover, we "half" $\ker p$.

$$2 \ker p = \{(z, A) | z = \frac{4\pi i k}{n} \quad A = e^{-z} I \quad k \in \mathbb{Z}\}$$

and we define the *Metalinear Group* by

$$ML_n(\mathbb{C}) = \frac{\mathbb{C} \times SL_n(\mathbb{C})}{2 \ker p}$$

and denote the quotient map by $q : \mathbb{C} \times SL_n(\mathbb{C}) \to ML_n(\mathbb{C})$. Given $\bar{g} \in ML_n(\mathbb{C})$, we represent it by the equivalence class $\bar{g} = \overline{(z, A)} = \{(z + \frac{4\pi ik}{n}, e^{-\frac{4\pi ik}{n}} A) | k \in \mathbb{Z}\}$.

There is a natural projection $\rho : ML_n(\mathbb{C}) \to GL_n(\mathbb{C})$ by $\overline{(z, A)} \mapsto p(z, A)$ such that the following diagram commutes.

$$\begin{array}{ccc} & \mathbb{C} \times SL_n & \\ {\scriptstyle q}\swarrow & & \searrow{\scriptstyle p} \\ ML_n(\mathbb{C}) & \xrightarrow{\quad\rho\quad} & GL_n(\mathbb{C}) \end{array}$$

We now define the function χ.

$$\chi : ML_n(\mathbb{C}) \to \mathbb{C}$$
$$\overline{(z, A)} \mapsto e^{\frac{nz}{2}}$$

Observe that χ is well defined since if $\overline{(z, A)} = \overline{(w, B)}$ with $w = z + \frac{4\pi ik}{n}$, then $\chi\overline{(w, B)} = e^{\frac{nz}{2}} e^{2\pi ik} = e^{\frac{nz}{2}}$.

Remark 7.2. χ is not well defined on $GL_n(\mathbb{C})$. If $p(z, A) = p(w, B)$ with $w = z + \frac{2\pi ik}{n}$, then $\chi\overline{(w, B)} = e^{\frac{nz}{2}} e^{\pi ik} = \pm e^{\frac{nz}{2}}$.

And we have that $\chi^2 = \det \circ \rho$ since for $\bar{g} = \overline{(z, A)} \in ML_n(\mathbb{C})$

$$\chi(\bar{g})^2 = (e^{\frac{nz}{2}})^2 = e^{nz}$$
$$\det(\rho(\bar{g})) = det(p(z, A)) = e^{nz} \det A = e^{nz}$$

Metalinear Bundle. Given a principal GL_n-bundle, we would like to have a principal ML_n-bundle which interacts nicely with the principal GL_n-bundle. Generally, a vector bundle admits a metalinear bundle if and only if the square of its first Stiefel-Whitney class is zero [Guillemin]. We assume a metalinear bundle $\bar{\pi} : \bar{\mathcal{B}}P \to M$ exists with a projection $\bar{\rho} : \bar{\mathcal{B}}P \to \mathcal{B}P$ such that the following diagram commutes

$$\begin{array}{ccccc} ML_n(\mathbb{C}) \times \bar{\mathcal{B}}P & \xrightarrow{\cdot} & \bar{\mathcal{B}}P & & \\ \downarrow{\scriptstyle \rho \times \bar{\rho}} & & \downarrow{\scriptstyle \bar{\rho}} & \overset{\bar{\pi}}{\searrow} & \\ & & & & M \\ GL_n(\mathbb{C}) \times \mathcal{B}P & \xrightarrow{\cdot} & \mathcal{B}P & \underset{\pi}{\nearrow} & \end{array}$$

where $\cdot$ denotes group action.

Since if we are given frames $a, b \in \mathcal{B}P$ related by a linear transformation $b = ga$, we have $\mu_m(b) = |\det g| \mu_m(a)$. So we may as well define μ_m as equivalence relationships in $\mathcal{B}P \times \mathbb{C}$ such that $(a, z) \sim (b, w)$ if $p(a) = p(b)$ with $b = ga$ and $w = |\det g| z$. Notice this is only a pairing of elements of μ_m's domain and range. Denote this space $\mathrm{Vol}(P) = {}^{\mathcal{B}P \times \mathbb{C}}/_{\sim}$. Observe $\mathrm{Vol}(P)$ is a bundle over M. Therefore a density is a section of $\mathrm{Vol}(P)$.

Similarly we have similar constructions for the metalinear bundle. Given $\bar{a} \in \bar{\mathcal{B}}P$ related by an element of metalinear group by $\bar{b} = \bar{g}\bar{a}$, we would like to construct the set of functions ν such that $\nu_m(\bar{b}) = \chi(\bar{g}) \nu_m(\bar{a})$. Therefore we take the equivalence classes in $\bar{\mathcal{B}}P \times \mathbb{C}$ such that $(\bar{a}, z) \sim (\bar{b}, w)$ if $\bar{p}(\bar{a}) = \bar{p}(\bar{b})$, $\bar{b} = \bar{a}\bar{g}$, and $z = \chi(\bar{g}) w$.

Denote the set of equivalence classes as $\delta(P) = {}^{\bar{\mathcal{B}}P\times\mathbb{C}}/_{\sim}$. The sections of this set are called *half-forms*.

Theorem 7.3. *Let ν,$\nu' : M \to \delta(P)$ be half-forms. Then $\mu = \nu\bar{\nu'} \circ \bar{\rho}^{-1}$ is a density on M.*

Proof. First we show that $\nu\bar{\nu'} \circ \bar{\rho}^{-1}$ is well defined. Suppose $\bar{\rho}(\bar{a}) = \bar{\rho}(\bar{b}) = a$ for $\bar{a}, \bar{b} \in \bar{\mathcal{B}}P$. Then there exists $\bar{g} \in \ker\rho$ such that $\bar{b} = \bar{g}\bar{a}$.

$$\nu\bar{\nu'}(\bar{b}) = \nu\bar{\nu'}(\bar{g}\bar{a}) = \chi(\bar{g})\bar{\chi}(\bar{g})\nu\bar{\nu'}(\bar{a}) = |\det\rho(\bar{g})|\nu\bar{\nu'}(\bar{a})$$

Where $\rho : ML_n(\mathbb{C}) \to GL_n(\mathbb{C})$ is the projection. Therefore we have $|\det\rho(\bar{g})| = 1$ and the expression $\nu\bar{\nu'} \circ \bar{\rho}^{-1}$ is well defined.

To show it is a density, suppose $\pi(a) = \pi(b) = m$. Then there exists $g \in GL_n(\mathbb{C})$ such that $b = ga$. Since group action and projection commutes, there exists $\bar{a}, \bar{g}$ such that $\bar{\rho}(\bar{a}) = a$ and $\bar{\rho}(\bar{g}) = g$ with $\bar{\rho}(\bar{g}\bar{a}) = b$.

$$\mu(b) = \nu\bar{\nu'} \circ \bar{\rho}^{-1}(ga) = \nu\bar{\nu'}(\bar{g}\bar{a}) = \chi(\bar{g})\bar{\chi}(\bar{g})\nu\bar{\nu'}(\bar{a}) = |\det g|\nu\bar{\nu'}(\bar{a}) = |\det g|\mu(a)$$

Thus $\mu(b) = |\det g|\mu(a)$.

□

Denote the density arising from the pairing of half forms by $\mu = (\nu, \nu')$.

Density on M/D. Our main goal was to obtain an object of integration on $\mathcal{D}$. We will show that if half-forms ν, ν' are polarized, i.e. constant in directions of P, then (ν, ν') induces a density on $\mathcal{D}$.

First we define the Lie Derivative for densities and half-forms. It is useful to think of density μ as a complex function $\tilde{\mu} : \mathcal{B}P \to \mathbb{C}$ assigning a complex number to every frame. So if X is a vector field on $\mathcal{B}P$, the Lie Derivative can be defined the usual way it is defined on complex functions $\mathcal{L}_X\tilde{\mu}(a) := X_a\mu$. We will use μ instead of $\tilde{\mu}$ to clear up notation. We can make a similar construction for half-forms using a vector field $\bar{X}$ on $\bar{\mathcal{B}}P$ and the induced function $\tilde{\nu} : \bar{\mathcal{B}}P \to \mathbb{C}$ to get $\mathcal{L}_{\bar{X}}\tilde{\nu}$.

Definition 7.4. If G is a group, a vector field X is G-invariant if $X_{ga} = (\ell_g)_*X_a$ for all $g \in G$ and $a \in M$ where $\ell_g : a \mapsto ga$.

Proposition 7.5. *Let X be a vector field on $\mathcal{B}P$ and μ a density on M. $\mathcal{L}_X\mu$ is a density if and only if X is $GL_n(\mathbb{C})$-invariant.*

Proof. If X is $GL_n(\mathbb{C})$-invariant, then for all $a \in \mathcal{B}P$ and for all $g \in GL_n(\mathbb{C})$ we have $X_{ga} = (\ell_g)_*X_a$ where $\ell_g : a \mapsto ga$.

$$\mathcal{L}_X\mu(ag) = X_{ga}\mu = (\ell_g)_*X_a(\mu) = X_a(\mu \circ \ell_g) = X_a(|\det g|\mu) = |\det g|\mathcal{L}_X\mu(a)$$

Converse can be proved with the same equations. □

Corollary 7.6. *Let $\bar{X}$ be a vector field on $\bar{\mathcal{B}}P$ and ν a half density on M. $\mathcal{L}_{\bar{X}}\nu$ is a half-form if and only if $\bar{X}$ is $ML_n(\mathbb{C})$-invariant.*

Proposition 7.7. *Let ν, ν' be half-forms and X be a $GL_n(\mathbb{C})$-invariant vector field on $\mathcal{B}P$. Then there exists a $ML_n(\mathbb{C})$-invariant lift $\bar{X}$ on $\bar{\mathcal{B}}P$ of X such that*

$$\mathcal{L}_X(\nu, \nu') = (\bar{\nu'}\mathcal{L}_{\bar{X}}\nu + \nu\mathcal{L}_{\bar{X}}\bar{\nu'}) \circ \bar{\rho}^{-1}$$

Proof. Since $\bar{\rho}$ is locally a diffeomorphism, we have a lift $\bar{X}$ of X. Suppose we have $\bar{a} \in \bar{\mathcal{B}}P$ and $\bar{g} \in ML_n(\mathbb{C})$, then

$$X_{ga} = \bar{\rho}_* \bar{X}_{\bar{g}\bar{a}}$$
$$(\ell_g)_* X_a = (\ell_g)_* \bar{\rho}_* X_{\bar{a}} = (\ell_g \circ \bar{\rho})_* X_{\bar{a}}$$

Since $\bar{\rho}$ and group action commutes

$$(\ell_g \circ \bar{\rho})_* X_{\bar{a}} = (\bar{\rho} \circ \ell_{\bar{g}})_* X_{\bar{a}}$$

So we have $X_{\bar{g}\bar{a}} = (\ell_{\bar{g}})_* X_{\bar{a}}$. Therefore the lift $\bar{X}$ is $ML_n(\mathbb{C})$-invariant.

Now we prove the identity. Suppose $a \in \mathcal{B}P$.

$$\mathcal{L}_X(\nu, \nu')(a) = X_a(\nu, \nu') = X_a(\nu\bar{\nu'} \circ \bar{\rho}^{-1}) = (\bar{\rho})_* \bar{X}_{\bar{a}}(\nu\bar{\nu'} \circ \bar{\rho}^{-1})$$

$$= \bar{X}_{\bar{a}}(\nu\bar{\nu'} \circ \bar{\rho}^{-1} \circ \bar{\rho}) = \bar{X}_{\bar{a}}(\nu\bar{\nu'}) = \bar{\nu'}(\bar{a})\bar{X}_{\bar{a}}\nu + \nu(\bar{a})\bar{X}_{\bar{a}}\bar{\nu'} = (\bar{\nu'}\mathcal{L}_{\bar{X}}\nu + \nu\mathcal{L}_{\bar{X}}\bar{\nu'})(\bar{a})$$

$$= (\bar{\nu'}\mathcal{L}_{\bar{X}}\nu + \nu\mathcal{L}_{\bar{X}}\bar{\nu'})(\bar{\rho}^{-1}(a))$$

□

We can differentiate densities and half-forms over vector fields on $\mathcal{B}P$. However, we would like to differentiate them over vector fields on M, since they are sections on M.

Suppose we have X as a vector field on M involutive with P, i.e. for all $Y \in P$ $[X, Y] \in P$. Consider the flow $\rho_t^X : M \to M$ of X. The flow can be lifted as $(\rho_t^X)_* : TM^{\mathbb{C}} \to TM^{\mathbb{C}}$. Since X is involutive with P, $(\rho_t^X)_*$ maps vectors in P to P. Therefore there is an induced flow in $\mathcal{B}P$, for which there is an associated vector field $\tilde{X}$ on $\mathcal{B}P$. Now we are allowed to define $\mathcal{L}_X\mu := \mathcal{L}_{\tilde{X}}\mu$ for $X \in TM$ involutive with P. Furthermore, $\tilde{X}$ is $GL_n(\mathbb{C})$-invariant since

$$(\rho_t^X)_*(ga) = g(\rho_t^X)_*(a)$$

We lift $\tilde{X}$ to $\bar{X}$ on $\bar{\mathcal{B}}P$ to define $\mathcal{L}_X\nu := \mathcal{L}_{\bar{X}}\nu$ for half-forms. $\bar{X}$ is similarly $ML_n(\mathbb{C})$-invariant.

Theorem 7.8. *Let ν, ν' be half-forms on M. If for every vector field X in $D = P \cap \bar{P} \cap TM$ the Lie Derivatives of half-forms vanish $\mathcal{L}_{\bar{X}}\nu = \mathcal{L}_{\bar{X}}\bar{\nu}' = 0$, then (ν, ν') induces a density on M/D.*

Proof. We will first show that if μ is a density on M and $\mathcal{L}_X\mu = 0$ for all $X \in D$, then there exists a density $\bar{\mu}$ on M/D induced by μ.

Let $\{y_1, \ldots, y_n\}$ be local coordinates on M/D. Let $x_i = \phi^* y_i$ where $\phi : M \to M/D$ is the projection to the foliation.

Since $\mathcal{L}_X\mu = 0$ for $X \in D$, μ is constant on the leaves of M/D. Therefore we can define a density on the basis vectors as $\bar{\mu}(\{\frac{\partial}{\partial y_i}\}) = \mu(\{X_{x_i}\})$.

Now since

$$\mathcal{L}_X(\nu, \nu') = \bar{\nu'}\mathcal{L}_{\bar{X}}\nu \circ \bar{\rho}^{-1} + \nu\mathcal{L}_{\bar{X}}\bar{\nu'} \circ \bar{\rho}^{-1} = 0$$

we have the desired result.

□

Space of States and Mapping Observables. As states, we will use $\psi \otimes \nu$ such that $\psi \in \Gamma_P(L)$ and ν is a half-form such that $\mathcal{L}_X \nu = 0$ for all $X \in D$. Define the inner product of states as

$$(\psi \otimes \nu, \psi' \otimes \nu') = \int_{M/D} (\psi, \psi')(\nu, \nu')$$

Define the map of observables $\mathcal{Q}$ as

$$\mathcal{Q}f(\psi \otimes \nu) = \hat{f}\psi \otimes \nu + i\psi \otimes \mathcal{L}_{X_f}\nu$$

where $\mathcal{Q}$ maps only observables which map polarized states to polarized states as in section 6.

Theorem 7.9. *The map $\mathcal{Q}$ satisfies all Dirac Quantization Conditions.*

We end the paper with the example quantization of simple harmonic oscillator.

Example 7.10. Take the one dimensional simple harmonic oscillator, with phase space as $\mathbb{R}^2$. Again choose the polarization spanned by $\{\frac{\partial}{\partial \bar{z}}\}$ as in Example (6.17). Take the half form $\sqrt{dz}$ as the trivializing section. We calculate the Hamiltonian.

$$\mathcal{Q}H(\psi \otimes \sqrt{dz}) = z\frac{\partial}{\partial z}\psi \otimes \sqrt{dz} + i\psi \otimes \mathcal{L}_{X_H}(\sqrt{dz})$$

We have $\mathcal{L}_{X_H} dz = idz$ so we have $\mathcal{L}_{X_H}\sqrt{dz} = \frac{i}{2}\sqrt{dz}$ which accounts for the missing $\frac{1}{2}$ term in the eigenvalues of $\hat{H}$.

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