AN INTRODUCTION TO THE BRAID GROUPS AND THEIR INTEGRAL COHOMOLOGY

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Abstract. This paper is intended to provide the reader with an introduction to the braid groups, with an emphasis on the topology of these groups. First, a brief topological background will be given through the definitions of homotopy, the Fundamental Group, and the configuration space. Next the braid group will be introduced through three equivalent definitions, with the final definition being that of the Fundamental Group of the configuration space. Next, the notions of homology, cohomology, and fibre bundles will be introduced, completing the background information necessary to reach the primary result of this paper: a proof of Vladimir Arnold's computation of the cohomology of the braid group.

Contents

1. Introduction

Mathematical braids are inspired by their physical counterparts, and thus represent n strings moving through $\mathbb{R}^3$, intertwining with one another. Although this paper will introduce several equivalent definitions of the braid group, the focus of this paper will be the realization of the braid group as the fundamental group of the configuration space. More explicitly, this is the group of loops beginning and ending at n ordered points in the complex plane. Under this definition of the braid group, it is possible to compute the cohomology of the braid group, a result originally achieved by Vladimir Arnold in 1969.

In proving Arnold's result, we rely heavily on the Serre spectral sequence, which is more rigorously presented in Jenny Wilson's paper, *The Geometry and Topology of Braid Groups*. This sequence gives us a fibration for which we compute the cohomology groups. Altogether these cohomology groups give us the cohomology ring for the braid group, the primary result of this paper.

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2. The Topological Background

This section will give some of the topological prerequisites necessary in understanding the equivalent definitions of the braid group.

Definition 2.1. A *homotopy* is a family of maps $f_t : X \to Y$, $t \in I$ (where I is the unit interval, $[0,1]$) such that the associated map $F : X \times I \to Y$ given by $f(x,t) = f_t(x)$ is continuous. Two maps f_0, $f_1 : X \to Y$ are *homotopic* if there exists a homotopy connecting them. In this case we say $f_0 \simeq f_1$. Finally, we say that two spaces, X and Y, are homotopy equivalent if there exist continuous maps $f : X \to Y$ and $g : Y \to X$ such that the composition $f \circ g$ is homotopic to the identity $\mathbb{1}_X$ on X and $g \circ f$ is homotopic to the identity on Y, $\mathbb{1}_Y$.

Definition 2.2. A *path* in a space X is a continuous map $f : I \to X$. A *loop* in X is a path f with the added condition that $f(0) = f(1) = x_0 \in X$. X is *path connected* if there exists a path between any two points in X.

Definition 2.3. The homotopy classes of loops $f : I \to X$ beginning and ending at a basepoint x_0 form a group under concatenation as a binary operation, called the *fundamental group of X at the basepoint* x_0, and denoted $\pi_1(X, x_0)$.

Proposition 2.4. *In a path connected space, X, the choice of basepoint is arbitrary.*

Proof. For $x_0, x_1 \in X$, loops in $\pi_1(X, x_1)$ can have loops in $\pi_1(X, x_0)$ associated to them. We first let $h : I \to X$ be a path from x_0 to x_1, and let h^{-1} be the inverse path from x_1 to x_0, given by $h^{-1}(x) = h(1-x)$. Given a loop $f \in \pi_1(X, x_1)$, we are able to transform this into a loop in $\pi_1(X, x_0)$ through the associated loop $h \cdot f \cdot h^{-1}$. The map

$$\begin{aligned} \beta_h : \pi_1(X, x_0) &\to \pi_1(X, x_1) \\ f &\mapsto h \cdot f \cdot h^{-1} \end{aligned}$$

gives an isomorphism between the two groups. □

This tells us that the choice of basepoint is irrelevant in defining the fundamental group up to isomorphism. Thus for path connected spaces, we simply denote the fundamental group as $\pi_1(X)$.

3. Some Equivalent Definitions for the Braid Group

Our first definition is motivated by the geometric notion of n strands in $\mathbb{R}^3$ anchored at the top and bottom by n distinct points in $\mathbb{R}^2$. The strands move downward through space twisting around, but not passing through each other. A braid is given by an equivalence class of such strands. Together these braids form a group under concatenation, with the identity given by the braid with no twists, and inverse elements given by taking the braid going from bottom to top rather than top to bottom. Formally, we have the following:

Definition 3.1. For fixed n, let $p_1, \dots, p_n \in \mathbb{R}^2$ be given, along with an n-tuple of functions, $f = (f_1, \dots, f_n)$ with $f_i : [0,1] \mapsto \mathbb{R}^2$ satisfying

$$\begin{aligned} &f_i(0) = p_i \\ &f_i(1) = p_j \text{ for some } j \in \{1, \dots, n\} \end{aligned}$$

and such that the corresponding n paths

$$[0,1] \to [0,1] \times \mathbb{R}^2 \text{ with } t \mapsto (t, f_i(t))$$

have disjoint images. Thus, each function f_i represent a *strand*. Altogether, these n strands form a *braid*, with the *braid group on n strands,* B_n being the group of homotopy classes of such braids. These braids can be represented through the use of braid diagrams, which come courtesy of Wilson's paper [4].

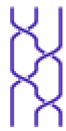

FIGURE 1. A Braid on Three Strands

This forms a group under the following binary operation for two braids f and g:

$$(f \circ g)_i(t) = \begin{cases} f_i(2t) & 0 \leq t \leq 1/2 \\ g_i(2t) & 1/2 \leq t \leq 1 \end{cases}$$

Represented by braid diagrams, this gives the binary operation of concatenation.

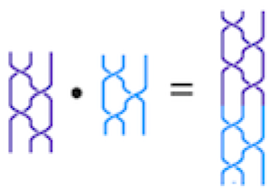

FIGURE 2. Two Braids Under Concatenation

The inverse of a braid f is given by:

$$f^{-1}(t) = f(1-t)$$

Definition 3.2. The *pure braid group on n strands,* PB_n is defined in the same way with the added condition that $f_i(0) = p_i = f_i(1)$ for any strand f_i of a given braid.

FIGURE 3. A Pure Braid on Three Strands

We next give an algebraically motivated definition of the (pure) braid groups, using Artin's presentations.

Definition 3.3. The *Artin braid group on n strands* is given by the generators $\sigma_1, \sigma_2, \ldots, \sigma_{n-1}$ with the following relations:

$$\sigma_i\sigma_j = \sigma_j\sigma_i \text{ for } |i-j| \geq 2$$
$$\sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1} \text{ for } i = 1, 2, \ldots, n-2$$

Under this representation, each σ_i represents a half twist of a strand i around the strand $i+1$. The first relation states that two twists can be ordered arbitrarily if they involve different strands (in the geometric sense, since the twists involve different strands, they can be moved up or down relative to each other homotopically). The second relation is less obvious, but also follows from homotopy in the geometric picture.

Definition 3.4. The *Artin pure braid group on n strands* has generators $T_{i,j}$ for $1 \leq i < j \leq n$ with the following relations:

$$[T_{p,q}, T_{r,s}] = 1 \text{ for } p < q < r < s$$
$$[T_{p,s}, T_{q,r}] = 1 \text{ for } p < q < r < s$$
$$T_{p,r}T_{q,r}T_{p,q} = T_{q,r}T_{p,q}T_{p,r} = T_{p,q}T_{p,r}T_{q,r} \text{ for } p < q < r$$
$$[T_{r,s}T_{p,r}, T_{r,s}^{-1}T_{q,s}] = 1 \text{ for } p < q < r < s$$

In this case, the generators $T_{i,j}$ correspond to the strand i completing a full twist around the strand j, crossing in front of each of the intermediate strands as it goes towards and returns from the strand j. That these relations hold for our geometric definition above is not the focus of this paper, and becomes relatively clear with the use of braid diagrams, and thus will not be proven here. Furthermore, a rigorous proof that these algebraic definitions for the (pure) braid group can be used to obtain any geometric braid, and thus the proof of the equivalence of these definitions, can be found in Joshua Lieber's *Introduction to Braid Groups*.

Our final definition for the braid group will use the notion of the configuration space of a topological space.

Definition 3.5. For a topological space, M, we define the *unordered configuration space of M on n points* to be:

$$\mathrm{UConf}_n(M) = \{\{m_1, m_2, ..., m_n\} \subset M \mid m_i \neq m_j \text{ for } i \neq j\}$$

The unordered configuration space is thus composed of n-element subsets of M. Similarly, the *ordered configuration space of M on n points* is given by

$$\mathrm{PConf}_n(M) = \{(m_1, m_2, ..., m_n) \in M^n \mid m_i \neq m_j \text{ for } i \neq j\}$$

Definition 3.6. We now define *the braid group via configuration spaces* to be the fundamental group of unordered configuration space of the complex plane

$$\mathrm{B}_n = \pi_1(\mathrm{UConf}_n(\mathbb{C}))$$

and *the pure braid group via configuration spaces* to be the fundamental group of the ordered configuration space of the complex plane.

$$\mathrm{PB}_n = \pi_1(\mathrm{PConf}_n(\mathbb{C}))$$

This definition realizes the braid group as loops beginning and ending at n points in the complex plane, with each point in the configuration space giving a path to another point in the configuration space (or in the case of the pure braid group, to itself). These paths don't intersect as this would mean that at some point in the loop $m_i = m_j$ for $i \neq j$ in contradiction of the definition of the braid group. The paths can however twist around one another, giving us a picture very similar to that of the geometric braid group defined initially.

4. Fiber Bundles and the Braid Group

Definition 4.1. A *fiber bundle* is a structure (E, B, π, F) such that E, B, and F are topological spaces and $\pi : E \to B$ is a continuous surjection with the added condition that for every $x \in E$ there exists an open neighborhood $U \subset B$ of $\pi(x)$ such that the following diagram commutes:

$$\begin{tikzcd} \pi^{-1}(U) \arrow[r, "\cong"] \arrow[d, "\pi"'] & U \times F \arrow[dl, "\mathrm{proj}_1"] \\ U & \end{tikzcd}$$

We call E the *total space*, B the *base space*, and F the *fiber*.

Definition 4.2. A *fibration* is a continuous mapping $p : E \to B$ satisfying the homotopy lifting property for any space. The *fibers* of this fibration are the subsets of E given by the inverse images of point b in B. The notion of a fibration can be thought to generalize that of a fiber bundle, requiring only that the fibers be equivalent up to homotopy. In the case that B is path connected, the fibers of two different points in B must be homotopy equivalent, and thus we speak simply of *the fiber*, F. Finally we denote a fibration using the following *fibration sequence*:

$$F \to E \to B$$

where the first map is an inclusion from the fiber to the total space E and the second map is a projection onto the base space B.

Here we define a projection:

$$\begin{aligned} \rho_{n+1} : \mathrm{PConf}_{n+1}(\mathbb{C}) &\to \mathrm{PConf}_n(\mathbb{C}) \\ (z_1 \ldots, z_n, z_{n+1}) &\mapsto (z_1, \ldots, z_n) \end{aligned}$$

For an element b in $\mathrm{PConf}_n(\mathbb{C})$, its preimage is the n points of that b, along with any other distinct point in the complex plane. Thus, the fiber of the fibration above is homeomorphic to the complex plane with n points removed, $\mathbb{C} \setminus \{z_1, \ldots, z_n\}$, which itself is homotopic to wedge of n 1-spheres, $\vee^n S^1$. This gives us the fibration sequence:

$$\bigvee^n S^1 \cong \mathbb{C} \setminus \{z_1, \ldots, z_n\} \to \mathrm{PConf}_{n+1}(\mathbb{C}) \to \mathrm{PConf}_n(\mathbb{C})$$

We also give a section of this fibration, given by:

$$\begin{aligned} \iota_{n+1} : \mathrm{PConf}_n(\mathbb{C}) &\to \mathrm{PConf}_{n+1}(\mathbb{C}) \\ (z_1, \ldots, z_n) &\mapsto (z_1, \ldots, z_n, \max_i \{|z_i| + 1\}) \end{aligned}$$

Definition 4.3. An *exact sequence* is a chain of homomorphisms

$$G_1 \xrightarrow{f_1} G_2 \xrightarrow{f_2} \cdots \xrightarrow{f_{n-1}} G_n$$

such that for all $k \in \{1, \ldots, n-1\}$, one has $\mathrm{Im}(f_k) = \mathrm{Ker}(f_{k+1})$. Further, we define the special case of a *short exact sequence* to be of the form

$$0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$$

where 0 is the trivial group. Because the image of the first homomorphism is trivial, so must be $\mathrm{Ker}(f)$ so f must be injective. And because the kernel of the final homomorphism is the whole space C, $\mathrm{Im}(g)$ must also be all of C, making g a surjection.

Lemma 4.4. *Given a short exact sequence of finitely generated abelian groups, $0 \to A \to B \to C \to 0$, if A and C are free abelian, then $B \cong A \oplus C$, with B also being free.*

Proof. Since A is free abelian, any element of A is a $\mathbb{Z}$-linear combination of its a generators, giving us that $A \cong \mathbb{Z}^a$. Similarly we have that for some c, $C \cong \mathbb{Z}^c$. We also have that because the sequence is exact, $C \cong B/A$. Torsion elements in B are represented by torsion elements in C, which are non-existent. Therefore, B is free of rank $a + c$ and we have

$$B \cong \mathbb{Z}^{a+c} \cong A \oplus C$$

□

5. The (Co)homological Background

In studying the topology of higher dimensional spaces, the fundamental group, $\pi_1(X)$ alone does not provide the desired precision for these spaces, which necessitates the use of analogous higher dimension homotopy groups, $\pi_i(X)$. However these groups are difficult to compute, which motivates the definition of homology groups, which are easier to compute and are often quite similar to the homotopy groups, but whose function is less immediately obvious from its definition.

Definition 5.1. Because the desired result of this paper does not require an explicit chain level definition of homology, we motivate the notion of homology as a way to define holes in a manifold. Homology thus examines closed submanifolds known as *cycles* and the boundaries of those cycles, referred to as *boundaries* which themselves are submanifolds as well. From these we can construct *homology classes*, which are equivalence classes of cycles modulo their boundaries. The i^{th} homology class of a manifold X is denoted $H_i(X)$.

There are several theories that can be used to calculate these homology groups, and more detailed chain level definitions for these theories can be found in Allen Hatcher's *Algebraic Topology.*

Definition 5.2. Without giving an explicit formula for the computation of the n^{th} *cohomology group*, $H^n(X)$ of a space X, we non-rigorously introduce it as an abelian group, arising from dualizing the construction of homology.

Again, a more rigorous definition for the cohomology groups can be found in Hatcher's *Algebraic Topology.*

6. The Cohomology of the Braid Group

In this section we will state and prove the main result of this paper, a theorem from Vladimir Arnold in 1969 that gives a computation of the integral cohomology of the braid group. We begin by introducing the following forms on $\mathrm{PConf}_n(\mathbb{C})$:

$$\omega_{i,j} := \frac{1}{2\pi i}\left(\frac{dz_i - dz_j}{z_i - z_j}\right) \text{ where } i \neq j \text{ and } i,j \in \{1,2,\dots,n\}$$

Proposition 6.1. *The forms $\omega_{i,j}$ satisfy the identity*

$$\omega_{i,j} \wedge \omega_{j,k} + \omega_{j,k} \wedge \omega_{k,i} + \omega_{k,i} \wedge \omega_{i,j} = 0 \textit{ for } i,j,k \textit{ distinct}$$

Proof. Consider the 2-form

$$D = dz_i \wedge dz_j + dz_j \wedge dz_k + dz_k \wedge dz_i.$$

By wedging the forms $\omega_{i,j}$ and $\omega_{j,k}$, we obtain

$$\begin{aligned} \omega_{i,j} \wedge \omega_{j,k} &= \left(\frac{1}{2\pi i}\right)^2 \left(\frac{1}{z_i - z_j}\right)\left(\frac{1}{z_j - z_k}\right)(dz_i \wedge dz_j - dz_i \wedge dz_k + dz_j \wedge dz_k) \\ &= \frac{-1}{4\pi^2}\left(\frac{z_k - z_i}{(z_i - z_j)(z_j - z_k)(z_k - z_i)}\right) D. \end{aligned}$$

Then, summing over these three equations we obtain:

$$\begin{aligned} \omega_{i,j} \wedge \omega_{j,k} + \omega_{j,k} \wedge \omega_{k,i} + \omega_{k,i} \wedge \omega_{i,j} &= \frac{-1}{4\pi^2}\left(\frac{(z_k - z_i) + (z_i - z_j) + (z_j - z_k)}{(z_i - z_j)(z_j - z_k)(z_k - z_i)}\right) D \\ &= 0 \end{aligned}$$

□

Intuitively, $\omega_{i,j}$ represents the winding number of a loop around the deleted hyperplane $z_i = z_j$, or in the context of braids, it represents the number of full twists of the strand i around the strand j for an element of the pure braid group.

Before we state the theorem from Arnold, we use the split fibration $\rho_{n+1} : \mathrm{PConf}_{n+1}(\mathbb{C}) \to \mathrm{PConf}_n(\mathbb{C})$ to make an observation about the induced map on the cohomology $\rho_{n+1}^* : H^*(\mathrm{PConf}_n(\mathbb{C})) \to H^*(\mathrm{PConf}_{n+1}(\mathbb{C}))$. Looking at the composition of the maps ρ_{n+1} and ι_{n+1}, we get

$$\mathrm{PConf}_n(\mathbb{C}) \xrightarrow{\iota_{n+1}} \mathrm{PConf}_{n+1}(\mathbb{C}) \xrightarrow{\rho_{n+1}} \mathrm{PConf}_n(\mathbb{C})$$

which gives the identity, with ι_{n+1} being a section. Furthermore, the map induced on the cohomology

$$H^*(\mathrm{PConf}_n(\mathbb{C})) \xrightarrow{\rho_{n+1}^*} H^*(\mathrm{PConf}_{n+1}(\mathbb{C})) \xrightarrow{\iota_{n+1}^*} H^*(\mathrm{PConf}_n(\mathbb{C}))$$

which also gives the identity map on the cohomology. Because this composition gives the identity, $\rho_{n+1}^* : H^*(\mathrm{PConf}_n(\mathbb{C})) \hookrightarrow H^*(\mathrm{PConf}_{n+1}(\mathbb{C}))$ must be injective and $\iota_{n+1}^* : H^*(\mathrm{PConf}_{n+1}(\mathbb{C})) \twoheadrightarrow H^*(\mathrm{PConf}_n(\mathbb{C}))$ must be surjective.

Theorem 6.2. *The cohomology algebra $H^*(PConf_n(\mathbb{C}))$ is the exterior graded algebra generated by the $\binom{n}{2}$ forms $\omega_{i,j}$, which are subject to the $\binom{n}{3}$ relations of (5.1).*

$$H^*(PConf_n(\mathbb{C})) \cong \frac{\wedge_{\mathbb{Z}}^* \omega_{i,j}}{\langle \omega_{q,r} \wedge \omega_{r,s} + \omega_{r,s} \wedge \omega_{s,q} + \omega_{s,q} \wedge \omega_{q,r} \rangle}$$

where $i,j,q,r,s \in \{1,2,...,n\}$, i,j distinct, and q,r,s are distinct.

Before we give the proof of this theorem, we give background on and state several results of a structure called the cohomology Serre spectral sequence, which is introduced with greater detail in Wilson's paper. Cohomology spectral sequences in general are sequences of bigraded abelian groups called pages, with the form $E_r = \oplus_{p,q} E_r^{p,q}$.

Important for our purposes a theorem about the Serre spectral sequence which states that for a fibration $F \to E \to B$, the associated Serre spectral sequence has E_2 page given by

$$E_2^{p,q} = H^p(B; H^q(F)) \tag{6.3}$$

Further, in the case of our fibration

$$\bigvee^n S^1 \cong \mathbb{C} \setminus \{z_1, \ldots, z_n\} \to \mathrm{PConf}_{n+1}(\mathbb{C}) \to \mathrm{PConf}_n(\mathbb{C})$$

these E_2 pages give a short exact sequence of free abelian groups

$$0 \to E_2^{k,0} \to H^k(\mathrm{PConf}_n(\mathbb{C})) \to E_2^{k-1,1} \to 0 \tag{6.4}$$

These two facts will be material in our proof of Arnold's Theorem.

Proof. We begin by computing the groups $E_2^{p,q}$ using (6.3) for the fibration above,

$$E_2^{p,q} = H^p(\mathrm{PConf}_n(\mathbb{C}); H^q(\vee^n S^1))$$

Now we wish to compute $H^q(\vee^n S^1)$ for a given q. Realizing $\vee^n S^1$ as a cell complex, we have that

$$C_1(\vee^n S^1) = \mathbb{Z}[s_1, \ldots, s_n] \cong \mathbb{Z}^n$$
$$C_0(\vee^n S^1) = \mathbb{Z}[p] \cong \mathbb{Z}$$

where p is the 0-cell at which the n 1-cells $s_1, \ldots, s_n$ are attached, and that $C_i(\vee^n S^1) = 0$ for $i \geq 2$. From this we conclude that

$$H^1(\vee^n S^1) \cong \mathbb{Z}^n$$
$$H^0(\vee^n S^1) \cong \mathbb{Z}$$

and that for $q \geq 2$, $H^q(\vee^n S^1) = 0$. Altogether we have that

$$E_2^{p,q} = H^p(\mathrm{PConf}_n(\mathbb{C}); H^q(\vee^n S^1)) = \begin{cases} H^p(\mathrm{PConf}_n(\mathbb{C})) \otimes \mathbb{Z} & q = 0 \\ H^p(\mathrm{PConf}_n(\mathbb{C})) \otimes \mathbb{Z}^n & q = 1 \\ 0 & q \geq 2 \end{cases} \tag{6.5}$$

By applying Lemma 3.4 and (6.5) to (6.4), we obtain

$$\begin{aligned} H^k(\mathrm{PConf}_{n+1}(\mathbb{C})) &\cong E_2^{k,0} \oplus E_2^{k-1,1} \\ &\cong H^k(\mathrm{PConf}_n(\mathbb{C})) \oplus (H^{k-1}(\mathrm{PConf}_n(\mathbb{C})) \otimes \mathbb{Z}^n) \end{aligned}$$

This gives

$$H^1(\mathrm{PConf}_{n+1}(\mathbb{C})) = H^1(\mathrm{PConf}_n(\mathbb{C})) \oplus H^1(\vee^n S^1)$$

We have that

$$H^1(\mathrm{PConf}_n(\mathbb{C})) \cong \mathbb{Z}[\{\omega_{i,j} \mid 1 \leq i < j \leq n\}]$$
$$H^1(\mathrm{PConf}_{n+1}(\mathbb{C})) \cong \mathbb{Z}[\{\omega_{i,j} \mid 1 \leq i < j \leq n+1\}].$$

Since we have shown that the first group injects into the latter, the remaining generators $\{\omega_{i,n+1} \mid i \in \{1, \ldots, n\}\}$ of $H^1(\mathrm{PConf}_{n+1}(\mathbb{C}))$ are identified with the n generators of $H^1(\vee^n S_1)$.

Now beginning with $\mathrm{PConf}_1(\mathbb{C})$, we show by induction on n and k that the groups $H^k(\mathrm{PConf}_n(\mathbb{C}))$ each have an additive basis in

$$\omega_{i_1,j_1} \wedge \omega_{i_2,j_2} \wedge \cdots \wedge \omega_{i_k,j_k} \text{ where } i_s < j_s \text{ and } j_1 < j_2 < \cdots < j_k$$

For $\mathrm{PConf}_1(\mathbb{C}) = \mathbb{C}$ we have that $H^0(\mathrm{PConf}_1(\mathbb{C})) \cong \mathbb{Z}$ since $\mathbb{C}$ is homotopic to a point. In fact, for all n, $H^0(\mathrm{PConf}_n(\mathbb{C})) \cong \mathbb{Z}$. We have shown that for all n we have $H^1(\mathrm{PConf}_n(\mathbb{C})) = \langle \omega_{i,j} \mid 1 \leq i \leq j \leq n \rangle$. We now compute $H^2(\mathrm{PConf}_n(\mathbb{C}))$ through induction on n.

In the base case $n = 1$, we again have $\mathrm{PConf}_1(\mathbb{C})$ homotopic to a point and thus containing no cells of degree greater than 1, implying that $H^2(\mathrm{PConf}_1(\mathbb{C})) = 0$.

For $n = 2$, we compute

$$\begin{aligned} H^2(\mathrm{PConf}_2(\mathbb{C})) &= H^2(\mathrm{PConf}_1(\mathbb{C})) \oplus (H^1(\mathrm{PConf}_n(\mathbb{C})) \otimes \mathbb{Z}) \\ &= 0 \oplus (0 \otimes \mathbb{Z}) = 0 \end{aligned}$$

Our first non-trivial case is $n = 3$, for which we have

$$\begin{aligned} H^2(\mathrm{PConf}_3(\mathbb{C})) &= H^2(\mathrm{PConf}_2(\mathbb{C})) \oplus (H^1(\mathrm{PConf}_2(\mathbb{C})) \otimes H^1(\vee^2 S^1)) \\ &= 0 \oplus \langle \omega_{1,2} \otimes \{\omega_{1,3}, \omega_{2,3}\} \rangle \end{aligned}$$

Here the tensor product indicates a wedging of the elements being tensored giving

$$H^2(\mathrm{PConf}_3(\mathbb{C})) = \langle \omega_{1,2} \wedge \omega_{1,3}, \omega_{1,2} \wedge_{2,3} \rangle$$

Inductively, we assume

$$H^2(\mathrm{PConf}_n(\mathbb{C})) = \langle \omega_{i_1,j_1} \wedge \omega_{i_2,j_2} \mid i_s < j_s, j_1 < j_2 \leq n \rangle$$

and compute, taking always $i_s < j_s$,

$$\begin{aligned} H^2(\mathrm{PConf}_{n+1}(\mathbb{C})) &= H^2(\mathrm{PConf}_n(\mathbb{C})) \oplus (H^1(\mathrm{PConf}_n(\mathbb{C})) \otimes H^1(\vee^n S^1)) \\ &= \langle \omega_{i_1,j_1} \wedge \omega_{i_2,j_2} \mid j_1 < j_2 \leq n \rangle \oplus (\langle \omega_{i_1,j_1} \mid j_1 \leq n \rangle \otimes \langle \omega_{i,n+1} \mid i \leq n \rangle) \\ &= \langle \omega_{i_1,j_1} \wedge \omega_{i_2,j_2} \mid j_1 < j_2 \leq n \rangle \oplus \langle \omega_{i_1,j_1} \wedge \omega_{i_2,n+1} \mid j_1 \leq n \rangle \\ &= \langle \omega_{i_1,j_1} \wedge \omega_{i_2,j_2} \mid j_1 < j_2 \leq n+1 \rangle \end{aligned}$$

This provides the base case for our induction on k. We now assume that

$$H^k(\mathrm{PConf}_n(\mathbb{C})) = \langle \omega_{i_1,j_1} \wedge ... \wedge \omega_{i_k,j_k} \mid i_s < j_s \text{ and } j_1 < \cdots < j_k \rangle$$

and proceed to compute the inductive step

$$\begin{aligned} H^{k+1}(\mathrm{PConf}_n(\mathbb{C})) &= H^{k+1}(\mathrm{PConf}_{n-1}(\mathbb{C})) \oplus (H^k(\mathrm{PConf}_{n-1}(\mathbb{C})) \otimes H^1(\vee^{n-1} S^1)) \\ &= H^{k+1}(\mathrm{PConf}_{n-1}(\mathbb{C})) \oplus (\langle \omega_{i_1,j_1} \wedge \cdots \wedge \omega_{i_k,j_k} \mid j_1 < \cdots < j_k \rangle \otimes \langle \omega_{i,n} \mid i \leq n-1 \rangle) \\ &= H^{k+1}(\mathrm{PConf}_{n-1}(\mathbb{C})) \oplus \langle \omega_{i_1,j_1} \wedge \cdots \wedge \omega_{i_k,j_k} \wedge \omega_{i_{k+1},n} \mid j_1 < \cdots < j_k \leq n-1 \rangle \end{aligned}$$

Inductively on n, this shows that for all $m \leq n$ we have

$$H^{k+1}(\mathrm{PConf}_m(\mathbb{C})) = H^{k+1}(\mathrm{PConf}_{m-1}(\mathbb{C})) \oplus \langle \omega_{i_1,j_1} \wedge \cdots \wedge \omega_{i_k,j_k} \wedge \omega_{i_{k+1},m} \mid j_1 < \cdots < j_k \leq m-1 \rangle$$

And therefore, we conclude

$$\begin{aligned} H^{k+1}(\mathrm{PConf}_n(\mathbb{C})) &= \bigoplus_{m \leq n} \langle \omega_{i_1,j_1} \wedge \cdots \wedge \omega_{i_k,j_k} \wedge \omega_{i_{k+1},m} \mid j_1 < \cdots < j_k \leq m-1 \rangle \\ &= \langle \omega_{i_1,j_1} \wedge \cdots \wedge \omega_{i_k,j_k} \wedge \omega_{i_{k+1},j_{k+1}} \mid i_s < j_s \text{ and } j_1 < j_2 < \cdots < j_{k+1} \rangle \end{aligned}$$

as desired.

By taking the product of the cohomologies and quotienting over the forms satisfying the identity in Proposition 5.1, we are able to compute the cohomology ring of $\mathrm{PConf}_n(\mathbb{C})$, thus deducing the result from Arnold

$$\begin{aligned} H^*(\mathrm{PConf}_n(\mathbb{C})) &= \bigoplus_i H^i(\mathrm{PConf}_n(\mathbb{C})) \\ &\cong \frac{\wedge^*_{\mathbb{Z}}\omega_{i,j}}{\langle \omega_{q,r}\wedge\omega_{r,s}+\omega_{r,s}\wedge\omega_{s,q}+\omega_{s,q}\wedge\omega_{q,r}\rangle} \end{aligned}$$

where $i, j, q, r, s \in \{1, 2, ..., n\}$, i, j are distinct, and q, r, s are distinct.

That these forms give an additive basis for the cohomology ring, and thus define a surjection to $H^*(\mathrm{PConf}_n(\mathbb{C}))$ is shown. That the forms go to independent elements in the cohomology ring and thus define an injection is a result of the identity in Proposition 6.1. This allows us to express any wedge of forms $\omega_{k,l}$ as a sum of wedges with the added conditions that $k_s < l_s < l_{s+1}$. Thus these forms give us a ring isomorphism for $H^*(\mathrm{PConf}_n(\mathbb{C}))$. □

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