COEFFICIENT RINGS OF C_2-EQUIVARIANT EILENBERG-MACLANE SPECTRA

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ABSTRACT. Let C_2 be the cyclic group of order two. We construct a spectral sequence to determine the coefficient rings of the C_2-equivariant spectra $H\underline{\mathbb{Z}/2^p}$ and $H\underline{\mathbb{Z}_2}$, the Eilenberg-Maclane spectra corresponding to the corresponding constant Mackey functors.

CONTENTS

1. INTRODUCTION

Equivariant homotopy theory can be described as the study of spaces with continuous group actions from a topological group G and continuous maps which preserve the group action. The study of such spaces involves an interesting interplay between representation theory and homotopy theory. The theory of such G-spaces is best understood when the group G is finite or more generally a compact Lie group, where powerful tools are gained from representation theory.

The main results of this paper are the computations of the equivariant analogue of the homotopy groups of various Eilenberg-Maclane spectra. The computations can also be thought of as computing the cohomology ring of representation spheres, graded over the representation ring, or the coefficient ring of the corresponding Eilenberg-Maclane spectrum. Very little theory at the level of spaces will be needed for the main results of this paper. An exposition of equivariant cohomology as well as many of the space-level constructions can be found in [1] and [2] is a standard resource for many of the constructions involved in this paper.

The first part of this paper is devoted to the key definitions needed, both to serve as a reminder and to fix notation. From there, we specialize to when the group action is C_2, the cyclic group of order two. Using the computation of the coefficient ring of $H\underline{\mathbb{F}_2}$ found as proposition 6.1 in [3], a certain spectral sequence closely related to the classical Bockstein spectral sequence will be used to compute the coefficient

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ring of the Eilenberg-Maclane spectra $H\underline{\mathbb{Z}_2}$ and $H\underline{\mathbb{Z}/2^p}$. These computations can be viewed either as the homotopy rings of these spectra or as the cohomology of a point with coefficients lying in the corresponding Mackey functors.

2. Preliminaries

This section gives an overview of the main definitions needed. Throughout this section, proofs will often be omitted for brevity, with a reference on where to find them. We first need some basic definitions from the level of spaces. Throughout this paper, the term G-space refers to a topological space equipped with continuous group action from the topological group G. A pointed G-space is a G-space with a specified basepoint, which we require to be fixed under the action of G. For G-spaces X and Y, a continuous map $f: X \to Y$ is said to be a G-map if for any $g \in G$ and $x \in X$, we have that $f(gx) = gf(x)$ and a pointed G-map is a G-map between pointed spaces which preserves the basepoint. We now recall the relevant notion of representation. We will only consider real representations rather than over a general field.

Definition 2.1. A *representation* of G is a real inner product space equipped with a smooth action by G through linear isometries.

For a finite dimensional representation V of G, the *representation sphere* S^V is the one-point compactification of V, with the point at infinity receiving trivial G-action.

The finite dimensional representations of a group have additional structure given by direct sum and tensor product as representations. The Grothendieck construction for this object is called the representation ring and denoted $RO(G)$.

The representations of C_2 have a nice structure, which we adopt special notation for. The following is a standard result; consult any textbook on representation theory for a proof.

Proposition 2.2. *Any finite dimensional representation of C_2 can be decomposed into a direct sum of trivial representations and copies of $\mathbb{R}$ with action $x \mapsto -x$, known as the sign representation σ. Furthermore, the representation ring $RO(C_2) \cong \frac{\mathbb{Z}[\sigma]}{(\sigma^2-1)}$.*

The representation sphere of C_2 with underlying representation $a - b + b\sigma$ is denoted $S^{a,b}$. The numbers a and b are referred to as the *total dimension* and *equivariant twisting* respectively.

We next define the notion of universe, which we use to index the equivariant notion of spectra.

Definition 2.3. A universe U is a representation of G which can be decomposed as a direct sum $\oplus_{i \in I} V_i$ such that

1. I is a countably infinite index set
2. Each V_i is a finite dimensional representation of G
3. At least one of the V_i is the trivial representation of G
4. Each finite dimensional subrepresentation of U occurs infinitely often

There are (up to isomorphism) only two universes when the group is C_2, namely $\mathbb{R}^\infty$ with trivial G-action and the direct sum of infinitely many copies of the trivial representation and the sign representation. These are referred to as the *trivial* and *complete* universes. Unless otherwise stated, we will always be using the complete universe. We can now define the notion of prespectra and spectra. For $V \subseteq W$ both representations, write $W - V$ as the orthogonal complement of V in W.

Definition 2.4. A *prespectrum* E indexed over a universe U is a sequence of based G-spaces EV for each finite dimensional representation $V \subset U$ equipped with structural maps $\sigma_{V,W} \colon \Sigma^{W-V}EV \to EW$ for each $V \subseteq W$ such that

(1) $\sigma_{V,V}$ is the identity
(2) the following diagram commutes for all $V \subseteq W \subseteq Z$

$$\begin{CD}
\Sigma^{Z-W}\Sigma^{W-V}EV @>{\sigma_{V,W}}>> \Sigma^{Z-W}EW \\
@V{\cong}VV @VV{\sigma_{W,Z}}V \\
\Sigma^{Z-V}EV @>{\sigma_{V,Z}}>> EZ
\end{CD}$$

A map f between prespectra E and F indexed over a common universe U is a sequence of maps $f_V \colon EV \to FV$ for each finite dimensional $V \subseteq U$ commuting with the structural maps. That is, the following diagram commutes for each finite dimensional $V \subset W \subset U$ with V and W finite dimensional representations.

$$\begin{CD}
EV @>{\sigma_{V,W}}>> EW \\
@V{f_V}VV @VV{f_W}V \\
FV @>>{\sigma_{V,W}}> FW
\end{CD}$$

The notion of spectra and their relationship with prespectra is identical to the non-equivariant theory.

Definition 2.5. A *spectrum* indexed over a universe U is a prespectrum such that the adjoints of the structural maps are homeomorphisms.

The forgetful functor from spectra to presepectra has a left adjoint which we call spectrification and write as L.

We now define some particularly important spectra. The first example of spectra will be the notion of the spectrum corresponding to a space.

Example 2.6. For a based G-space X, write $\Sigma^\infty X$ as the spectrification of the prespectrum $\{X \wedge S^V\}$ for each $V \subset U$ with structural maps the identity.

This specific example is actually a special case of a general construction.

Example 2.7. For a based G-space X and representation V, write $\Sigma^\infty_V X$ as the spectrification of the prespectrum

$$(\Sigma^\infty_V X)(W) = \begin{cases} \Sigma^{W-V}X & V \subseteq W \\ * & \text{otherwise} \end{cases}$$

Remark 2.8. Σ^∞_V is left adjoint to the functor Ω^∞_V taking a spectrum E to the space EV.

We will need the notion of homotopy of maps between spectra, which is defined analogously to the usual case. The smash product between a spectrum E and a space X in the following definition is the spectrification of the prespectrum obtained by smashing with X levelwise.

Definition 2.9. A homotopy of maps between spectra E and E' as a map $E \wedge I_+ \to E'$ (I is, as usual, the unit interval with trivial G-action).

Definition 2.10. $[E, E']$ is the set of homotopy classes of maps from E to E'.

We will work in the homotopy category of spectra, where a morphism between spectra is a homotopy class of maps between spectra. We first define an external smash product.

Definition 2.11. For spectra E and E' indexed over universes U and U', we define $E \wedge E'$ is the spectrum indexed over $U \oplus U'$ with $(E \oplus E')(V \oplus W) = EV \wedge EW$ with structural maps the smash product of those coming from E and E'.

This smash product can be internalized. Details can be found in of chapter XII in [2]. From this point on, the smash product will always be understood to be internal.

We need a notion of sphere spectra. First, for any representation V, we have the spectra $\Sigma^\infty S^V$, which we will abuse notation and denote as S^V. We define S^{-V} to be $\Sigma^\infty_V S^0$. It is possible to check that $\left(\Sigma^\infty S^V\right) \wedge \left(\Sigma^\infty_V S^0\right)$ is homotopy equivalent to S^0, motivating the notation used. We therefore have spectra S^V for every element of the representation ring. Similarly to the space level, we will write $S^{a,b}$ as the sphere with dimension a and twisting b. We now have a C_2-equivariant spectrum for each bigraded index.

Suspension and desuspension of spectra occur identically to the non-equivariant version, although we can now suspend and desuspend by any finite dimensional representations. For a spectrum E, the suspension of E by the representation V, denoted $\Sigma^V E$, is the smash product $S^V \wedge E$. The desuspension of the spectrum E by V, denoted $\Sigma^{-V} E$, is the product $S^{-V} \wedge E$.

There are two notions of homotopy groups we would like to consider. The first will be Abelian group valued and the second will be Mackey functor valued.

Definition 2.12. A Mackey functor is a contravariant functor from the full subcategory of spectra consisting of the suspension of orbits $\Sigma^\infty G/H_+$ to the category of Abelian groups.

There is a more algebraic definition of Mackey functor which is often used; see page 250 of [2] for a proof of their equivalence. The two notions of homotopy groups will serve different roles. Mackey functors are an Abelian category, with relevant structure defined termwise. The group valued homotopy groups will be the focus of the major computations later in this paper. The Mackey functor valued homotopy groups are needed to specify Equivariant Eilenberg-Maclane spectra. We will not, however, be computing any of the Mackey functor valued homotopy groups.

Both the group valued and the Mackey functor valued theories are graded on the representation ring.

Definition 2.13. For a spectrum E, representation (real or virtual) V, and subgroup $H \subseteq G$, the group-valued homotopy groups are defined as

$$\pi_V(E) = [S^V, E].$$

The Mackey-functor-valued homotopy groups are defined as

$$\underline{\pi}_V(E)(G/H) = [G/H_+ \wedge S^V, E].$$

The connection between the group valued homotopy groups and cohomology is that for any spectrum E the functor taking a based space X to the collection of abelian groups graded on $RO(G)$ defined by $[\Sigma^\infty X, \Sigma^V E]$ satisfies an anologue of the Eilenberg-Steenrod axioms for reduced cohomology. We can view the homotopy groups of a spectrum as the cohomology of a point in the cohomology theory defined by the spectrum. This theory on the representation ring agrees with the usual notion, for example as presented in [1] or [2]. The group valued homotopy groups can be seen as the image of G/G under the corresponding homotopy group Mackey functor. The only Mackey functors we will use here are constant Mackey functors.

Proposition 2.14. *Suppose G is finite and A is an Abelian group. There is a unique Mackey functor $\underline{A}$, known as the constant Mackey functor for A, such that the image of $\underline{A}$ is constant at A and the homomorphism $A \to A$ induced by the transfer homomorphism $G/K_+ \to G/H_+$ associated to the inclusion $H \subseteq K$ is multiplication by $|K/H|$.*

Proof. See Theorem 5.6 in [1]. For a generalization to compact Lie groups, see Proposition 4.3 on page 101 of [2]. □

We have an analogue of Eilenberg-Maclane spectra in the equivariant world.

Proposition 2.15. *There is a covariant functor H, defined up to weak equivalence, from the category of Mackey functors to the homotopy category of G-spectra such that the following properties are satisfied.*

(1) For all Mackey functors M, $\underline{\pi}_0(HM) = M$.
(2) For all Mackey functors M, $\underline{\pi}_n(HM) = 0$ for all $n \neq 0$.

Proof. See page 163 of [2]. □

Notation 2.16. For a Mackey functor M, the spectrum HM is called the G-equivariant Eilenberg-Maclane spectrum for M.

There are two properties we will need about the functor H. The first states that the functor H is symmetric monoidal and the second states that H carries short exact sequences to cofibrations. We first recall the relevant definitions.

Definition 2.17. For M and M' Mackey functors, we define
$$M \otimes M' = \underline{\pi}_0(HM \wedge HM').$$

The notions of fibration and cofibrations depend only on the homotopy lifting and extension property and therefore are defined for spectra. As with non-equivariant spectra, fibrations and cofibrations are the same. There is also a spectrum version of the long exact sequence of homotopy groups for fibrations, using the fact that exact sequences are preserved under colimits. Recalling that Mackey functors are an Abelian category with products, kernels, and cokernels defined termwise, the notion of a short exact sequence is the same as usual.

Proposition 2.18.

(1) For all Mackey functors M, M', and M'' and any map from $M \otimes M'$ to M'', there is a canonical map from $HM \wedge HM'$ to HM'' such that for every spectrum E, the induced map
$$[E, HM \wedge HM'] \to [E, HM'']$$
is an isomorphism.

(2) H carries 3-term exact sequences to (co)fibrations. That is, for any short exact sequence

$$0 \longrightarrow M \longrightarrow M' \longrightarrow M'' \longrightarrow 0\ ,$$

the induced map

$$HM \longrightarrow HM' \longrightarrow HM''$$

is a (co)fibration.

Proof. For the first part, see page 168 of [2]. The proof of the second part is the same as the nonequivariant case. The construction of a homotopy fiber is identical, now using equivariant maps, which can be used to turn any map into a fibration. A surjective map $f\colon M' \twoheadrightarrow M''$ induces a map $Hf\colon HM' \to HM''$. Taking the corresponding long exact sequence yields that the homotopy fiber is HM. □

The first part of the preceding proposition tells us that a map $M \otimes M$ to M giving a ring structure gives a ring spectrum structure on HM. In particular, for the constant Mackey functor of a commutative ring R, the Eilenberg-Maclane spectrum $H\underline{R}$ has a ring spectrum structure induced by multiplication in R. This ring spectrum structure induces a bigraded ring structure on homotopy groups through a composition

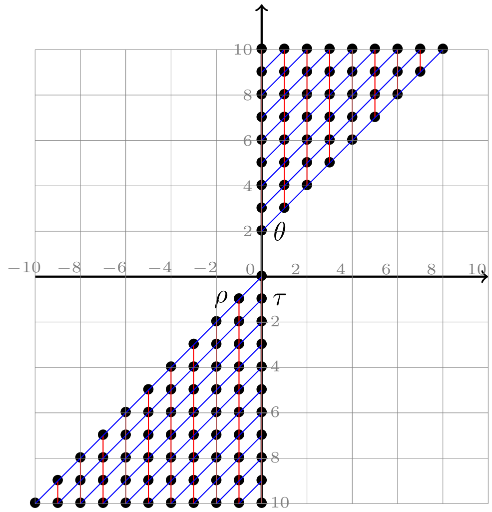
$$[\,S^{a,b},\, H\underline{R}\,] \times [\,S^{c,d},\, H\underline{R}\,] \to [S^{a+c,b+d}, H\underline{R} \wedge H\underline{R}] \to [\,S^{a+c,b+d},\, H\underline{R}\,].$$

This is called the coefficient ring of the spectrum $H\underline{R}$. Note that this is in the homotopy groups, not the homotopy group Mackey functors.

The definition of Eilenberg-Maclane spectra tells us immediately what the homotopy group Mackey functors of HM are for trivial representations, but tells us nothing about the corresponding homotopy groups are for nontrivial representations. As is often the case with equivariant homotopy theory, the homotopy group Mackey functors for nontrivial representations are difficult to compute. Much of the rest of this paper will be devoted to computing the coefficient rings for the Eilenberg-Maclane spectra $H\underline{\mathbb{F}_2}$, $H\underline{\mathbb{Z}_2}$, and $H\underline{\mathbb{Z}/2^p}$. We once again observe that these computations can be viewed as computing the cohomology of a point.

3. Homotopy groups of $H\underline{\mathbb{F}_2}$

The computation of the coefficient ring of the C_2 equivariant Eilenberg-Maclane spectrum $H\underline{\mathbb{F}_2}$ is the statement of Proposition 6.2 in [3] and a proof can be found there for the negative cone and the ring structure. The end result is pictured below, where a dot represents a copy of $\mathbb{F}_2$, red lines are multiplication by τ, and blue lines are multiplication by ρ (multiplication by θ is always 0 expect for the sphere $S^{0,0}$ and is therefore not shown).



The node marked ρ is of bidegree $(-1,-1)$, the node marked τ is of bidegree $(0,-1)$, and the node marked θ is of bidegree $(0,2)$. The notation used of θ is suggestive of the ring structure, and in general we will write $\frac{\theta}{\tau^{j-1}\rho^i}$ as the generator of the group at position $(i, i+j+1)$.

The ring structure is defined by

$$\begin{cases} \rho^i\tau^j \text{is the generator of the corresponding } \mathbb{F}_2 \\ (\frac{\theta}{\tau^j\rho^i})(\tau^n\rho^m) = \frac{\theta}{\tau^{j-n}\rho^{i-m}} \text{ if } i > n \\ \text{all other multiplications are 0} \end{cases}.$$

A nice way of expressing this ring structure can be found in [4]. The bottom cone is the polynomial ring $\mathbb{F}_2[\rho,\tau]$ and the top cone has an element $\theta \in \pi_{0,2}H\underline{\mathbb{F}_2}$ which is infinitely ρ and τ divisible and satisfies $\theta^2 = 0$.

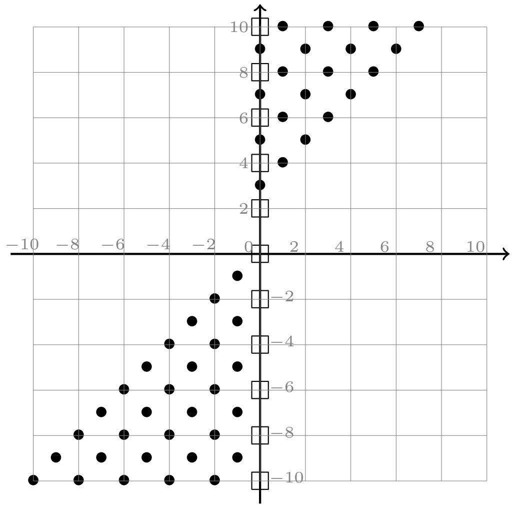
4. A Bockstein spectral sequence: from $H\underline{\mathbb{F}_2}$ to $H\underline{\mathbb{Z}_2}$

Through the use of a spectral sequence, we can use the computation of the homotopy groups of $H\underline{\mathbb{F}_2}$ to obtain the homotopy groups of $H\underline{\mathbb{Z}_2}$. The result will be the following theorem.

Theorem 4.1.

$$\pi_{a,b}(H\underline{\mathbb{Z}_2}) \approx \begin{cases} \mathbb{Z}_2 & a = 0 \textit{ and } b = 2m+2 \textit{ or } b = -2m \textit{ for some } m \geq 0 \\ \mathbb{Z}/2 & a > 0 \textit{ and } b = a + 2\ell + 1 \textit{ for some } \ell \geq 1 \\ & \textit{or } a < 0 \textit{ and } b = a - 2\ell \textit{ for some } \ell \geq 0 \\ & \textit{or } a = 0, b = 2m \textit{ for some } m > 0 \\ 0 & \textit{otherwise} \end{cases}$$

In pictoral format, we obtain the following picture, where a dot is $\mathbb{F}_2$ and a square is $\mathbb{Z}_2$ with the same scale as the picture for $\mathbb{F}_2$.



The construction of the spectral sequence is obtained through a composition of functors beginning with the commutative diagram shown below.

$$\begin{array}{ccccccccc}
 & & \vdots & & \vdots & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \to & \mathbb{Z} & \xrightarrow{2} & \mathbb{Z} & \to & \mathbb{F}_2 & \to & 0 \\
 & & \downarrow{\scriptstyle 2} & & \downarrow{\scriptstyle 2} & & & & \\
0 & \to & \mathbb{Z} & \xrightarrow{2} & \mathbb{Z} & \to & \mathbb{F}_2 & \to & 0 \\
 & & \downarrow{\scriptstyle 2} & & \downarrow{\scriptstyle 2} & & & & \\
0 & \to & \mathbb{Z} & \xrightarrow{2} & \mathbb{Z} & \to & \mathbb{F}_2 & \to & 0 \\
 & & \downarrow{\scriptstyle 2} & & \downarrow{\scriptstyle 2} & & & & \\
0 & \to & \mathbb{Z} & \xrightarrow{2} & \mathbb{Z} & \to & \mathbb{F}_2 & \to & 0.
\end{array}$$

Effectively, we are filtering $\mathbb{Z}$ by the descending filtration $2^i\mathbb{Z}$.

Applying the Eilenberg-Maclane functor H, we obtain a diagram

$$\begin{array}{ccccc}
\vdots & & \vdots & & \\
\downarrow & & \downarrow & & \\
H\underline{\mathbb{Z}} & \xrightarrow{2} & H\underline{\mathbb{Z}} & \to & H\underline{\mathbb{F}_2} \\
\downarrow{\scriptstyle 2} & & \downarrow{\scriptstyle 2} & & \\
H\underline{\mathbb{Z}} & \xrightarrow{2} & H\underline{\mathbb{Z}} & \to & H\underline{\mathbb{F}_2} \\
\downarrow{\scriptstyle 2} & & \downarrow{\scriptstyle 2} & & \\
H\underline{\mathbb{Z}} & \xrightarrow{2} & H\underline{\mathbb{Z}} & \to & H\underline{\mathbb{F}_2} \\
\downarrow{\scriptstyle 2} & & \downarrow{\scriptstyle 2} & & \\
H\underline{\mathbb{Z}} & \xrightarrow{2} & H\underline{\mathbb{Z}} & \to & H\underline{\mathbb{F}_2},
\end{array}$$

with each row a (co)fibration since the rows were exact in the preceding diagram.

Applying the functors $\pi_{*,*}$ and the equivariant stable analogue of the long exact sequence of homotopy groups, we obtain a staircase diagram of the form

$$
\begin{array}{ccccccccccc}
 & & \vdots & & & & \vdots & & & & \vdots \\
 & & \downarrow & & & & \downarrow & & & & \downarrow \\
\dots & \to & \pi_{a+1,b}H\underline{\mathbb{Z}} & \to & \pi_{a+1,b}H\underline{\mathbb{F}_2} & \to & \pi_{a,b}H\underline{\mathbb{Z}} & \to & \pi_{a,b}H\underline{\mathbb{F}_2} & \to & \pi_{a-1,b}H\underline{\mathbb{Z}} & \to & \dots \\
 & & \downarrow & & & & \downarrow & & & & \downarrow \\
\dots & \to & \pi_{a+1,b}H\underline{\mathbb{Z}} & \to & \pi_{a+1,b}H\underline{\mathbb{F}_2} & \to & \pi_{a,b}H\underline{\mathbb{Z}} & \to & \pi_{a,b}H\underline{\mathbb{F}_2} & \to & \pi_{a-1,b}H\underline{\mathbb{Z}} & \to & \dots \\
 & & \downarrow & & & & \downarrow & & & & \downarrow \\
\dots & \to & \pi_{a+1,b}H\underline{\mathbb{Z}} & \to & \pi_{a+1,b}H\underline{\mathbb{F}_2} & \to & \pi_{a,b}H\underline{\mathbb{Z}} & \to & \pi_{a,b}H\underline{\mathbb{F}_2} & \to & \pi_{a-1,b}H\underline{\mathbb{Z}} & \to & \dots \\
 & & \downarrow & & & & \downarrow & & & & \downarrow \\
 & & 0 & & & & \pi_{a,b}H\underline{\mathbb{Z}} & \to & \pi_{a,b}H\underline{\mathbb{F}_2} & \to & \pi_{a-1,b}H\underline{\mathbb{Z}} & \to & \dots \\
 & & & & & & \downarrow & & & & \downarrow \\
 & & & & & & 0 & & & & \pi_{a-1,b}H\underline{\mathbb{Z}} & \to & \dots \\
 & & & & & & & & & & \downarrow \\
 & & & & & & & & & & 0
\end{array}
.
$$

Remark 4.2. Using Brown representability, we could turn this spectral sequence into one involving the cohomology of representation spheres. This transition would give us essentially the equivariant analogue of the spectral sequence described in [5], which is itself a generalization of the Bockstein spectral sequence by filtering the chain complexes in a manner analogous to what has been done here. This is why the section title specifies that this is a Bockstein spectral sequence.

The resulting spectral sequence can be used to compute $\pi_{*,*}H\underline{\mathbb{Z}_2}$, where $\mathbb{Z}_2$ is the completion of $\mathbb{Z}$ with respect to the filtration. The spectral sequence is trigraded, with $E_1^{p,t,q} = \pi_{p,q}(H\underline{\mathbb{F}_2})$ for $t \geq 0$ and differential d_r of tridegree $(-1, r, 0)$.

An important part of our computation will be the fact that we have a product structure on E_r. For E_1, the product structure $E_r^{p,t,q} \times E_r^{u,s,v} \to E_r^{p+u,t+s,q+v}$ is the product induced by the ring spectrum structure on $H\underline{\mathbb{F}_2}$. We also have that for any xy that $d_1(xy) = xd_1(y) + d_1(x)y$. This follows from the fact that the following diagram commutes, which can be viewed as naturality of equivariant Steenrod squares with respect to the ring structure. The map marked as $\bullet$ is the ring map on $H\underline{\mathbb{F}_2}$

$$
\begin{array}{ccc}
\pi_{*,*}H\underline{\mathbb{F}_2} \times \pi_{*,*}H\underline{\mathbb{F}_2} & \xrightarrow{\substack{d_1 \times \mathbb{1} \\ \oplus \\ \mathbb{1} \times d_1}} & \substack{\pi_{*,*}H\underline{\mathbb{F}_2} \times \pi_{*,*}H\underline{\mathbb{F}_2} \\ \oplus \\ \pi_{*,*}H\underline{\mathbb{F}_2} \times \pi_{*,*}H\underline{\mathbb{F}_2}} \\
\downarrow \wedge & & \downarrow \wedge \oplus \wedge \\
\pi_{*,*}(H\underline{\mathbb{F}_2} \wedge H\underline{\mathbb{F}_2}) & & \substack{\pi_{*,*}H\underline{\mathbb{F}_2} \wedge H\underline{\mathbb{F}_2} \\ \oplus \\ \pi_{*,*}H\underline{\mathbb{F}_2} \wedge H\underline{\mathbb{F}_2}} \\
\downarrow \bullet & & \downarrow \bullet + \bullet \\
\pi_{*,*}H\underline{\mathbb{F}_2} & \xrightarrow{d_1} & \pi_{*,*}H\underline{\mathbb{F}_2}.
\end{array}
$$

The product structure on E_1 with the differential structure of d_1 induces a product on d_r for each r by a standard argument holding for all spectral sequences with such a multiplicative structure.

From now on, we will use the letter t as an indicator of the filtration; it denotes the copy of $\mathbb{F}_2$ in position $(0,1,0)$, so multiplication by t is shifting the second index up by one. For degree reasons, $d_1(\rho) = d_1(t) = 0$. An important note about this spectral sequence that we can use to compute $d_1(\tau)$ is that the differentials d_1 come from the connecting homomorphism from homotopy groups, which can also be thought of as the Steenrod squares Sq^1 for the cohomology of a point. As a consequence of Proposition 6.6 in [3], where the full Steenrod algebra is computed, we have that $d_1(\tau) = Sq^1(\tau) = \rho$. The remaining differentials can be computed from the product structure, as indicated in the following proposition.

Proposition 4.3. *The only nonzero differentials on the E^1 page are $d_1(\rho^i\tau^j t^k)$ for j odd and $d_1(t^k\frac{\theta}{\tau^{j+1}\rho^i})$ for j odd*

Proof. For degree reasons, $d_1(\rho) = d_1(t) = d_1(\frac{\theta}{\tau^i}) = 0$. Using that the differentials are multiplicative, we have that

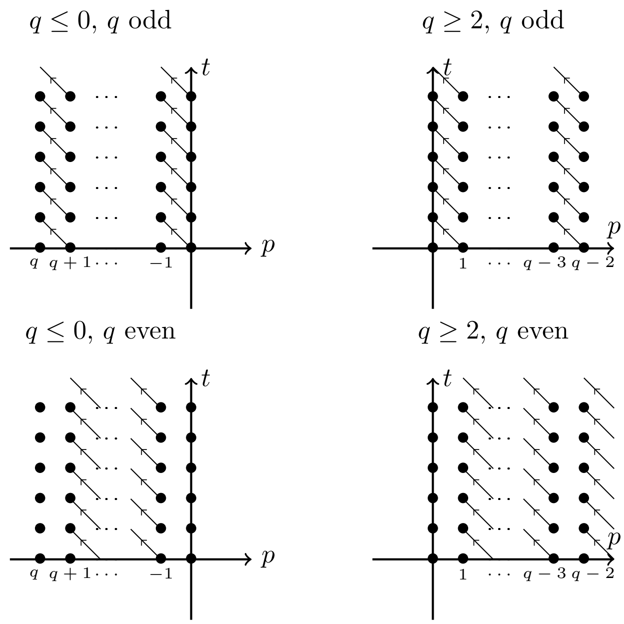
$$d_1(\rho^i\tau^j t^k) = \rho^i\tau^j d_1(t^k) + \rho^i t^k d_1(\tau^j) + \tau^j t^k d_i(\rho^j) = j\rho^{i+1}t^{k+1}\tau^{j-1},$$

which is zero precisely when j is even. The differentials on the top cone are also computed from the multiplicative structure. Recall that $t^k(\frac{\theta}{\tau^{j-1}\rho^i})\tau^j = 0$, so the differential must have trivial image. Therefore,

$$0 = d_1(t^k(\frac{\theta}{\tau^{j-1}\rho^i})\tau^j) = t^k\tau^j d_1(\frac{\theta}{\rho^i}) + t^k\frac{\theta}{\tau^{j-1}\rho^i}d_1\tau^j = t^k\tau^j d_1(\frac{\theta}{\tau^{j-1}\rho^i}) + jt^{k+1}\frac{\theta}{\rho^{i-1}\tau^{j-2}}.$$

The only way to have this equality hold is if $t^k\tau^j d_1(\frac{\theta}{\tau^{j-1}\rho^i})$ is zero when j is even and nonzero when j is odd, completing the proof. □

The following diagram depicts the E_1 page, where a dot represents a copy of $\mathbb{F}_2$ and an arrow represents a nontrivial differential. Since the sequence is trigraded, the diagrams represent the four possible pictures depending on the last coordinate. Since the differential d_1 has tridegree $(-1,1,0)$, we may fix q and look at the bigraded slices of the first page, with the differentials having sources and targets in the same slice. How many dots there are and in which posititions depends on q, as the diagram of $\pi_{*,*}H\underline{\mathbb{F}_2}$ indicates. Therefore, the p axis is labeled coordinates in terms of q, for example when $q \leq 0$, there are $q+1$ dots in positions q through 0 corresponding to the fact that $\pi_{p,q}H\underline{\mathbb{F}_2}$ is nonzero for this q precisely when $q \leq p \leq 0$.



This gives us all the information we need to compute the E_2 page. From there, we can compute the remainder of the spectral sequence.

Proposition 4.4. *The spectral sequence stabilizes after the E_2 page. That is, $E_2 = E_3 = ... = E_\infty$ and $d_r = 0$ for all $r \geq 2$.*

Proof. The only differentials which we cannot immediately conclude are trivial from degree considerations are those in positions $(1, 0, 2k)$ for $k \geq 2$. We will abuse notation and label generators of groups with groups in the corresponding positions on the E_1 page. Standard arguments for multiplicative spectral sequences reveal that the product structure on the E_1 page induces a product structure on subsequent pages. From here, the remaining differentials can be computed as

$$d_r\left(\frac{\theta}{\rho\tau^{2k+1}}\tau^{4k}\right) = \tau^{4k} d_1\left(\frac{\theta}{\rho\tau^{2k+1}}\right),$$

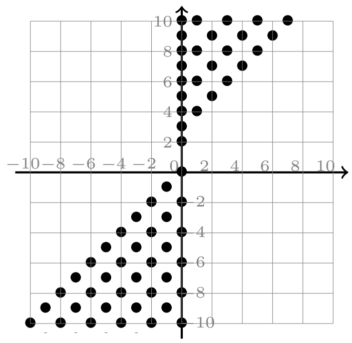
meaning $d_1(\frac{\theta}{\rho\tau^{2k+1}}) = 0$, which was what we wanted. □

Taking both upper and lower cones into account, we have a descending filtration

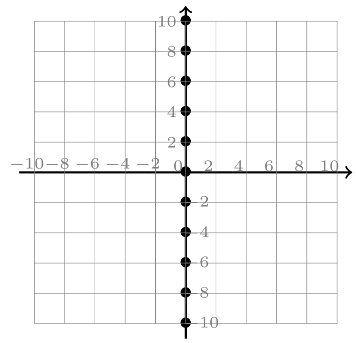
$$\pi_{*,*} H\underline{\mathbb{Z}_2} = F_0 \supseteq F_1 \supseteq F_2 \supseteq ...$$

with successive quotients indicated in the diagram below.

F_0/F_1



$F_i/F_{i+1}, i \neq 0$



From the setup of the spectral sequence, the map from one filtration to the next is multiplication by two. The outcome is the following.

$$\pi_{a,b}(H\underline{\mathbb{Z}_2}) \cong \begin{cases} \mathbb{Z}_2 & a = 0 \text{ and } b = 2m+2 \text{ or } b = -2m \text{ for some } m \geq 0 \\ \mathbb{Z}/2 & a > 0 \text{ and } b = a + 2\ell + 1 \text{ for some } \ell \geq 1 \\ & \text{or } a < 0 \text{ and } b = a - 2\ell \text{ for some } \ell \geq 0 \\ & \text{or } a = 0, b = 2m \text{ for some } m > 0 \\ 0 & \text{otherwise} \end{cases}.$$

As mentioned the Preliminaries section, the product structure on $\mathbb{Z}_2$ induces a product on the homotopy groups of $H\mathbb{Z}_2$. This product can, to some extent, be computed from the spectral sequence. The product structure on $H\underline{\mathbb{Z}_2}$ does restrict to maps on the filtration that coincide with multiplication induced on the E^∞ page. We do, however, run into extension issues. For example, for α the generator of $\pi_{0,2}H\underline{\mathbb{Z}_2}$ and β the generator of $\pi_{0,4}H\underline{\mathbb{Z}_2}$ we would find that $\alpha^2 = 2\beta$, which cannot be detected from the spectral sequence. The argument for this relies on the fact that, forgetting the group action, θ and τ are the degree one and two maps in $H^2(S^2)$. Therefore, θ/τ^2 would be a degree two map and θ^2 would be degree four (in particular, it is nonzero). See [4] for the geometric interpretation of these maps. However, we do obtain that the product of two of the groups $\mathbb{F}_2$ agrees with their product in $\pi_{*,*}H\underline{\mathbb{F}_2}$, as this is the product on E^∞.

5. From $H\underline{\mathbb{Z}_2}$ to $H\underline{\mathbb{Z}/2^p}$

The spectral sequence used in the preceding section can also be used to compute the homotopy groups of $H\underline{\mathbb{Z}/2^p}$ for each p. The diagram used for constructing the sequence can be modified to give the following.

$$\begin{array}{ccccccccc}
0 & \longrightarrow & 0 & \longrightarrow & \mathbb{F}_2 & \longrightarrow & \mathbb{F}_2 & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow{\scriptstyle 2} & & & & \\
0 & \longrightarrow & \mathbb{F}_2 & \xrightarrow{2} & \mathbb{Z}/4\mathbb{Z} & \longrightarrow & \mathbb{F}_2 & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle 2} & & \downarrow{\scriptstyle 2} & & & & \\
0 & \longrightarrow & \mathbb{Z}/4\mathbb{Z} & \xrightarrow{2} & \mathbb{Z}/8\mathbb{Z} & \longrightarrow & \mathbb{F}_2 & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle 2} & & \downarrow{\scriptstyle 2} & & & & \\
 & & \vdots & & \vdots & & & & \\
 & & \downarrow{\scriptstyle 2} & & \downarrow{\scriptstyle 2} & & & & \\
0 & \longrightarrow & \mathbb{Z}_{2^{p-2}} & \xrightarrow{2} & \mathbb{Z}/2^{p-1}\mathbb{Z} & \longrightarrow & \mathbb{F}_2 & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle 2} & & \downarrow{\scriptstyle 2} & & & & \\
0 & \longrightarrow & \mathbb{Z}/2^{p-1}\mathbb{Z} & \xrightarrow{2} & \mathbb{Z}/2^p & \longrightarrow & \mathbb{F}_2 & \longrightarrow & 0.
\end{array}$$

The composition with the functors H and $\pi_{*,*}$ now gives the staircase diagram

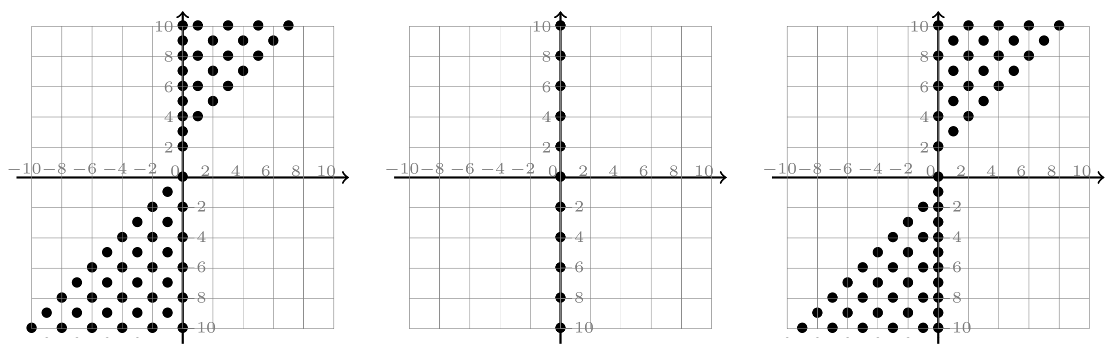
$$\begin{array}{ccccccccccc}
\dots & \longrightarrow & 0 & & & & & & & & \\
 & & \downarrow & & & & & & & & \\
\dots & \longrightarrow & \pi_{a+1,b}H\underline{\mathbb{F}_2} & \longrightarrow & \pi_{a+1,b}H\underline{\mathbb{F}_2} & \longrightarrow & 0 & & & & \\
 & & \downarrow & & & & \downarrow & & & & \\
\dots & \longrightarrow & \pi_{a+1,b}H\underline{\mathbb{Z}/4\mathbb{Z}} & \longrightarrow & \pi_{a+1,b}H\underline{\mathbb{F}_2} & \longrightarrow & \pi_{a,b}H\underline{\mathbb{F}_2} & \longrightarrow & \pi_{a,b}H\underline{\mathbb{F}_2} & \longrightarrow & 0 \\
 & & \downarrow & & & & \downarrow & & & & \downarrow \\
 & & \vdots & & & & \vdots & & & & \vdots \\
 & & \downarrow & & & & \downarrow & & & & \downarrow \\
\dots & \longrightarrow & \pi_{a+1,b}H\underline{\mathbb{Z}/2^{p-1}\mathbb{Z}} & \longrightarrow & \pi_{a+1,b}H\underline{\mathbb{F}_2} & \longrightarrow & \pi_{a,b}H\underline{\mathbb{Z}/2^{p-2}} & \longrightarrow & \pi_{a,b}H\underline{\mathbb{F}_2} & \longrightarrow & \pi_{a-1,b}H\underline{\mathbb{Z}/2^{p-3}} \longrightarrow \dots \\
 & & \downarrow & & & & \downarrow & & & & \downarrow \\
\dots & \longrightarrow & \pi_{a+1,b}H\underline{\mathbb{Z}/2^p} & \longrightarrow & \pi_{a+1,b}H\underline{\mathbb{F}_2} & \longrightarrow & \pi_{a,b}H\underline{\mathbb{Z}/2^{p-1}\mathbb{Z}} & \longrightarrow & \pi_{a,b}H\underline{\mathbb{F}_2} & \longrightarrow & \pi_{a-1,b}H\underline{\mathbb{Z}}_{2^{p-2}} \longrightarrow \dots \\
 & & \downarrow & & & & \downarrow & & & & \downarrow \\
 & & 0 & & & & \pi_{a,b}H\underline{\mathbb{Z}/2^p} & \longrightarrow & \pi_{a,b}H\underline{\mathbb{F}_2} & \longrightarrow & \pi_{a-1,b}H\underline{\mathbb{Z}/2^{p-1}\mathbb{Z}} \longrightarrow \dots \\
 & & & & & & \downarrow & & & & \downarrow \\
 & & & & & & 0 & & & & \pi_{a-1,b}H\underline{\mathbb{Z}/2^p} \longrightarrow \dots \\
 & & & & & & & & & & \downarrow \\
 & & & & & & & & & & 0.
\end{array}$$

The result is a trigraded spectral sequence converging to $\pi_{*,*}H\underline{\mathbb{Z}/2^p}$. Furthermore, for each r, we will have that the differential d_r agrees with the differential d_r from the previous spectral sequence so long as the filtration index is between 0 and $r-p$ and the other differentials will be trivial (as their target is trivial). In particular, we will have that $E^2 = E^3 = ... = E^\infty$. We will still have a copy of $\mathbb{F}_2$ for each position in the computation of $\mathbb{Z}_2$, although we will now have an additional copy of $\mathbb{F}_2$ in filtration p for each group which is not the target of a nontrivial differential. The result is a filtration

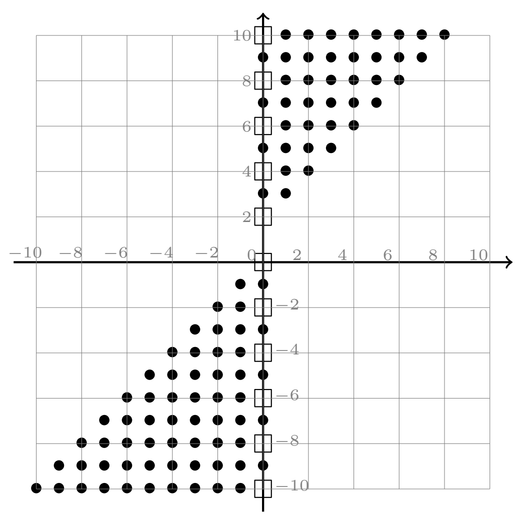
$$\pi_{*,*}H\underline{\mathbb{Z}/2^p} = F_0 \supseteq F_1 \supseteq F_2 \supseteq ... \supseteq F_p$$

with successive quotients as indicated in the diagrams below.

F_0/F_1 $\qquad$ $F_i/F_{i+1}, 1 < i < p$ $\qquad$ F_{p-1}/F_p



Once again, extension issues can be resolved by observing that successive terms in the filtration are obtained through multiplication by two. The groups $\pi_{*,*}H\underline{\mathbb{Z}/2^p}$ are pictured below, with a dot for $\mathbb{F}_2$ and a square for a $\mathbb{Z}/2^p$.



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