PHYSICAL APPLICATIONS OF FIXED POINT METHODS IN DIFFERENTIAL EQUATIONS

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Abstract. This paper presents methods for proving the existence of a solution to certain differential equations by using fixed point theorems. Furthermore, we present selected applications for each method. We start by proving the Stampacchia and Lax-Milgram theorems with the Banach fixed point theorem. Next, we use Brouwer's, Schauder's, and Schaefer's fixed point theorems to examine non-linear PDEs. Following each section of theory, we prove a solution's existence for notable problems and examine how our results apply to the Schrödinger equation.

Contents

1. Introduction

Fixed point theorems offer a powerful method for guaranteeing the existence of a solution to partial differential equations. Under certain conditions, they become useful at handling non-linear cases of common PDEs. To lay a foundation, we will first encounter theorems in functional analysis that are results of fixed point theorems and also prove useful in providing the existence of PDE and ODE solutions.

Definition 1.1. Let X be a topological space and $F : X \to X$. We say that $x \in X$ is a *fixed point of* F if $F(x) = x$.

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The theorems covered in this paper provide the existence of a fixed point when certain criteria are met for a given map, which we will construct for each application to yield a solution. Once we have the right map, the problem reduces to showing that it satisfies the requirements imposed by our fixed point theorems.

A basic knowledge of functional analysis and familiarity with Hilbert and Sobolev spaces is assumed. For additional details on definitions omitted from the text, one should check the appendix. Readers interested in further background information should consult chapters four, eight, and nine of Brezis' *Functional Analysis, Sobolev Spaces, and Partial Differential Equations* [1] and Irvii's *Partial Differential Equations* [4].

2. Fixed Point Methods and Functional Analysis

We begin this section by using the Banach fixed point theorem to prove the Stampacchia and Lax-Milgram theorems, two important results in functional analysis. These theorems will allow for the recovery of a solution to a boundary value problem from a weaker definition of a solution.

This guarantees solutions to the Sturm-Liouville problem, of which the time independent Schrödinger equation is a special case.

2.1. **The Stampacchia and Lax-Milgram Theorems.**

Theorem 2.1 (Riesz-Fréchet Representation Theorem)**.** *Let H be a Hilbert space with dual H^*. Given $\phi \in H^*$, there is a unique $f \in H$ such that for any $u \in H$,*

$$\langle \phi, u \rangle = (f, u).$$

Moreover, when taking the norm defined by the scalar product of H, i.e. $|u| = (u,u)^{\frac{1}{2}}$,

$$|f| = \|\phi\|_{H^*}.$$

Proof. Define a map $T : H \to H^*$ such that given $f \in H$,

$$\langle T(f), u \rangle = (f, u)$$

for all $u \in H$. Since the scalar product $(\cdot,\cdot)$ is a bilinear form, we have that for fixed f, the map $u \mapsto (f,u)$ is a continuous linear functional. We see that

$$\begin{aligned} \|T(f)\|_{H^*} &= \sup_{|u|=1} \langle T(f), u \rangle \\ &= \sup_{|u|=1} (f, u) \\ &\leq \sup_{|u|=1} (f,f)^{\frac{1}{2}}(u,u)^{\frac{1}{2}} = |f| \end{aligned}$$

by the Cauchy-Schwarz inequality. On the other hand, since $\|T(f)\|_{H^*}$ is a supremum for $|u| = 1$, consider $u = \frac{f}{|f|}$. Then,

$$\|T(f)\|_{H^*} = \sup_{|u|=1} (f,u) \geq \left(f, \frac{f}{|f|}\right) = |f|.$$

Thus we have that $\|T(f)\|_{H^*} = |f|$

We now have that T is a linear isometry from H into a closed subspace $T(H) \subseteq H^*$. To show that $T(H)$ is dense in H^*, consider $v \in (H^*)^*$ such that $\langle T(f), v \rangle = 0$ for all $T(f) \in T(H)$. Since H is a Hilbert Space, $H = (H^*)^*$. Thus, $(f, v) = 0$ for all $f \in H$, so $v = 0$.

Now for any $\phi \in H^*$, there exists a sequence $\{f_n\} \subset H$ such that $T(f_n) \to \phi$. By continuity of $(\cdot, \cdot)$, there exists f such that $f_n \to f$ and $T(f) = \phi$.

□

Definition 2.2. For a metric space X, a *contraction* is a map $S : X \to X$ such that there exists a constant $k \in (0, 1)$ satisfying

$$d(S(x), S(y))_X \leq kd(x, y)_X$$

for all $x, y \in X$.

Remark 1. It immediately follows that a contraction is uniformly continuous.

Theorem 2.3 (Banach Fixed Point Theorem)**.** *Let X be a complete metric space. If $S : X \to X$ is a contraction, then there exists a unique fixed point, i.e. there is a unique $x \in X$ such that $S(x) = x$.*

Proof. Consider a sequence $(x_n) \subset X$ where $x_{n+1} = Sx_n$. Then for some $k \in (0, 1)$, we have that

$$d(x_{n+1}, x_n) \;=\; d(Sx_n, Sx_{n-1}) \;\leq\; kd(x_n, x_{n-1}).$$

Thus,

$$d(x_{n+1}, x_n) \;\leq\; k^n d(x_1, x_0)$$

by induction. We can now show that (x_n) is indeed a Cauchy sequence. By the triangle inequality,

$$\begin{aligned} d(x_{n+m}, x_n) &\leq \sum_{i=0}^{m-1} d(x_{(n+1)+i}, x_{n+i}) \\ &\leq \sum_{i=0}^{m-1} k^{n+i} d(x_0, x_1) \\ &= k^n \frac{1-k^m}{1-k} d(x_0, x_1) \\ &\leq \frac{k^n}{1-k} d(x_0, x_1), \end{aligned}$$

and since $k \in (0, 1)$ by definition, we have that (x_n) is Cauchy and it converges to some $x \in X$ by completeness.

By continuity of S,

$$Sx = S \lim_{n\to\infty} x_n = \lim_{n\to\infty} Sx_n = \lim_{n\to\infty} x_{n+1} = x.$$

We now argue uniqueness by contradiction. Assume that $x, y \in X$ are two distinct fixed points under S. Then for any $k \in (0, 1)$,

$$d(S(x), S(y)) = d(x, y) > kd(x, y),$$

so S is not a contraction. □

Definition 2.4. Let H be a normed vector space. We say a bilinear form $b : H \times H \to \mathbb{R}$ is

(*i*) *symmetric* if for all $u, v \in H$,

$$b(u, v) = b(v, u).$$

(*ii*) *continuous* if there exists a constant $C > 0$ such that for all $u, v \in H$,

$$|b(u, v)| \leq C\|u\|\|v\|.$$

(*iii*) *coercive* if there exists a constant $\alpha > 0$ such that for all $u \in H$,

$$|b(u, u)| \geq \alpha\|u\|^2.$$

Definition 2.5. For a closed convex set K in a Hilbert space H, we define the *projection* of $f \in H$ onto K to be the unique element $u \in K$ such that

$$(f - u, v - u) \leq 0 \tag{1}$$

for all $v \in K$. We denote such a u as $u = P_K f$.

We can interpret the projection of some $f \in H$ as the point in K that minimizes the distance to f. Although this an intuitive concept, one can verify that it does indeed admit a unique solution by consulting chapter 5.1 of Brezis [1].

Lemma 2.6. *Let $K \subset H$ be a closed convex set. Then P_K does not increase distance.*

Proof. For $f_1, f_2 \in H$, let $u_1 = P_K f_1$, $u_2 = P_K f_2$. Then since $u_1, u_2 \in K$,

$$\begin{aligned}
(f_1 - u_1, u_2 - u_1) + (f_2 - u_2, u_1 - u_2) &\leq 0 \\
-(f_1 - u_1, u_1 - u_2) + (f_2 - u_2, u_1 - u_2) &\leq 0 \\
(f_2 - u_2 - f_1 + u_1, u_1 - u_2) &\leq 0 \\
-(f_1 - f_2, u_1 - u_2) + |u_1 - u_2|^2 &\leq 0.
\end{aligned}$$

By rearranging and applying the Cauchy-Schwartz inequality,

$$|u_1 - u_2|^2 \leq (f_1 - f_2, u_1 - u_2) \leq |u_1 - u_2||f_1 - f_2| \tag{2}$$

Thus,

$$|u_1 - u_2| \leq |f_1 - f_2|.$$

□

Theorem 2.7 (Stampacchia)**.** *Let H be a Hilbert space, with a closed, convex subset $K \subset H$. If b is a continuous, coercive bilinear form on H, then for any $\phi \in H^*$, there exists a unique $u \in K$ such that*

$$b(u, v - u) \geq \langle \phi, v - u \rangle \tag{3}$$

for all $v \in K$. If b is also symmetric, then

$$\frac{1}{2}b(u, u) - \langle \phi, u \rangle = \min_{v \in K}\{\frac{1}{2}b(v, v) - \langle \phi, v \rangle\}. \tag{4}$$

Proof. Fix $\phi \in H^*$. By the Riesz-Fréchet representation theorem we have a unique $f \in H$ such that

$$\langle \phi, v \rangle = (f, v) \qquad \text{for any } v \in H.$$

Since the map $v \mapsto b(u, v)$ is a continuous linear functional for fixed $u \in H$, we can also use Riesz-Fréchet to guarantee $Bu \in H$ such that $b(u, v) = (Bu, v)$ for any $v \in H$.

By continuity and coerciveness of b, there exist positive constants C, α such that

$$|Bu| \leq C|u|$$

$$(Bu, u) \geq \alpha|u|^2.$$

Define a positive constant $\rho = \frac{\alpha}{C^2}$. Define a function $S : K \to K$ by

$$Sv = P_K(\rho f - \rho Bv + v).$$

We will prove (3) by showing that S has a fixed point. By lemma 2.6, for any $v_1, v_2 \in K$ we have

$$|Sv_1 - Sv_2| \leq |(v_1 - v_2) - \rho(Bv_1 - Bv_2)|$$

$$|Sv_1 - Sv_2|^2 \leq |v_1 - v_2|^2 - 2\rho(Bv_1 - Bv_2, v_1 - v_2) + \rho^2|Bv_1 - Bv_2|^2$$

$$|Sv_1 - Sv_2|^2 \leq |v_1 - v_2|^2(1 - 2\rho\alpha + \rho^2C^2)$$

$$|Sv_1 - Sv_2|^2 \leq |v_1 - v_2|^2(1 - \alpha\rho).$$

Thus S is a contraction, and since a closed subset of a complete metric space is complete, the Banach fixed point theorem gives us $u \in K$ such that

$$u = P_K(\rho f - \rho Bu + u).$$

By definition, for all $v \in K$ we have that

$$0 \geq (\rho f - \rho Bu + u - u, v - u),$$

so

$$0 \geq \rho(f - Bu, v - u)$$

$$(Bu, v - u) \geq (f, v - u).$$

Thus,

$$b(u, u - v) \geq \langle \phi, v - u \rangle$$

for all $v \in K$.

If the bilinear form b is symmetric, then $b(u, v)$ satisfies the criteria of a scalar product and produces a corresponding norm $\sqrt{b(u, u)}$ that is equivalent to $|\cdot|$.

The space H is then also a Hilbert space with respect to $b(u, v)$ as a scalar product. By the Riesz-Fréchet theorem, there exists unique $g \in H$ such that

$$\langle \phi, v \rangle = b(g, v)$$

for any $v \in H$.

Following similar steps as before, we can project g onto K using this new scalar product to get $w = P_K(g)$, characterized by

$$b(g - w, v - w) \leq 0$$

for any $v \in K$.

From (3), we have that for all $v \in K$,

$$b(u, v - u) \geq \langle \phi, v - u \rangle = b(g, v - u)$$

$$b(g - u, v - u) \leq 0.$$

Therefore by (1) and the uniqueness of a projection, $u = w$. Since u minimizes the distance from g to K,

$$b(g - u, g - u) = \min_{v \in K} b(g - v, g - v).$$

We expand this with the identity

$$b(g - v, g - v) = b(v, v) - 2\langle g, v \rangle + b(g, g).$$

and subtract off the the constant $b(g,g)$ term from both sides. Thus, we can characterize this projection u by

$$\frac{1}{2}b(u,u) - \langle \phi, u\rangle = \min_{v\in K}\{\frac{1}{2}b(v,v) - \langle \phi, v\rangle\}.$$

□

Corollary 2.8 (Lax-Milgram)**.** *Let b be a continuous, coercive bilinear form on a Hilbert space H. Given any $\phi \in H^*$, there exists a unique $u \in H$ such that*

$$b(u, v-u) = \langle \phi, v-u\rangle \tag{5}$$

for all $v \in H$. If b is also symmetric, then

$$\frac{1}{2}b(u,u) - \langle \phi, u\rangle = \min_{v\in H}\{\frac{1}{2}b(v,v) - \langle \phi, v\rangle\}. \tag{6}$$

Proof. We repeat the proof for Stampacchia's theorem with $H = K$. After securing a fixed point for the constructed projection, we have that for all $v \in K$,

$$0 \geq (\rho f - \rho Bu + u - u, v-u)$$
$$0 \geq \rho(f - Bu, v-u)$$
$$(Bu, v-u) \geq (f, v-u).$$

Since we are now considering the entire Hilbert space, the above must also hold for $v_1 = 2u - v$. Thus,

$$0 \geq \rho(f - Bu, v_1 - u) = \rho\big(f - Bu, -(v-u)\big) = -\rho(f - Bu, v-u).$$

This forces equality in (5) to hold, with the remainder of the proof following as in Stampacchia's.

□

2.2. **The Sturm-Liouville Problem.** Let $I = (a,b)$. Recall that the Sobolev space $W^{1,p}(I)$ is all $u \in L^p(I)$ such that there exists $g \in L^p(I)$ satisfying

$$-\int_I g\phi = \int_I u\phi'$$

for all $\phi \in C_C^\infty$. We denote $u' = g$.

It is also equipped with norm

$$\|u\|_{W^{1,p}} = \|u\|_{L^p} + \|u'\|_{L^p}$$

and the Hilbert space $H^1 = W^{1,2}$ is equipped with the scalar product

$$(u,v)_{H^1} = (u,v)_{L^2} + (u',v')_{L^2} = \int_a^b (uv + u'v').$$

Also recall $W_0^{1,p}(I)$ is defined as the closure of $C_C^1(I)$ in $W^{1,p}(I)$. Note that $W_0^{1,p}$ has the norm of $W_0^{1,p}$, and H_0^1 has the scalar product of H^1.

Finally, we also define the space $H^2(I)$ as

$$H^2(I) = \{u \in H^1 : u' \in H^1\}$$

with scalar product $(u,v)_{H^2} = (u,v)_{L^2} + \sum_{n=1}^2 (D^n u, D^n v)_{L^2}$.

Theorem 2.9 (Poincaré's Inequality)**.** *Consider the interval $I = (a, b)$. There exists a constant C_I such that*

$$\|u\|_{W^{1,p}(I)} \leq C_I \|u'\|_{L^p(I)}$$

for all $u \in W_0^{1,p}(I)$.

Sketch of Proof. We know that for any $u \in W_0^{1,p}(I)$, $u = 0$ on ∂I.

First we show that $\|u\|_{L^\infty(I)} \leq \|u'\|_{L^1(I)}$ since

$$|u(x)| = |u(x) - u(a)| = \Big|\int_a^x u'(t)dt\Big| \leq \|u'\|_{L^1}.$$

Poincaré's inequality then follows from Hölder's inequality. □

Definition 2.10. Given a boundary value problem

$$\begin{cases} u'' + u = f \\ u(0) = u(1) = 0, \end{cases} \tag{7}$$

on $I = [0, 1]$, we say that u is

(i) a *strong solution* if it satisfies (7).

(ii) a *weak solution* if when we multiply (7) by a test function $\phi \in C^1(I)$ and integrate by parts,

$$\int_a^b u'\phi' + \int_a^b u\phi = \int_a^b f\phi$$

for all $\phi \in C^1(I)$ such that $\phi(a) = \phi(b) = 0$.

Now we shall consider the Sturm-Liouville problem. That is, we desire a solution to the boundary value problem

$$\begin{cases} -(pu')' + qu = f & \text{on } I = (0, 1) \\ u(0) = u(1) = 0 \end{cases} \tag{8}$$

where $p \in C^1(\bar{I})$, $q \in C(\bar{I})$, and $f \in L^2(I)$,and $p(x) \geq k > 0$ for all $x \in I$.

Proposition 2.11. *There exists a strong solution to BVP (8).*

Proof. Our method will be to use the Lax-Milgram theorem to generate a weak solution, then show it is C^2 to recover a strong solution from it.

Define a symmetric bilinear form $b : H_0^1(I) \times H_0^1(I) \to \mathbb{R}$ by

$$b(u, v) = \int_I pu'v' + \int_I quv. \tag{9}$$

Then b is continuous by the Cauchy-Schwarz inequality. Since I is bounded, we can apply Poincaré's inequality to show that b is coercive if $q \geq 0$ on I. First, Poincaré's inequality gives

$$\frac{\|v\|_{W_0^1}^2}{C_I^2} \leq \|v'\|_{W_0^1}^2.$$

Choosing $u = v$ in (9) and assuming q is non-negative on I,

$$b(v, v) = \int_I p(v')^2 + \int_I qv^2 \geq k\|v'\|^2 + \int_I qv^2 \geq \frac{k}{C_I^2}\|v\|^2.$$

By applying the Lax-Milgram Theorem to the continuous linear functional $u \mapsto (f,u)_{H_0^1}$, there exists a unique $u \in H_0^1$ such that for all $v \in H_0^1(I)$,

$$\int_I pu'v' + \int_I quv = \int_I fv, \tag{10}$$

characterized by

$$\frac{1}{2}\int_I (p(u')^2 + qu^2) - \int_I fu = \min_{v \in H_0^1(I)} \Big\{ \frac{1}{2}\int_I (p(v')^2 + qv^2) - \int_I fv \Big\}.$$

Thus, we have obtained a unique weak solution to (8). We observe that $pu' \in H^1$, therefore $u \in H^2$. Similarly, if $f \in C(\bar{I})$, then $pu' \in C^1(\bar{I})$, so $u \in C^2(\bar{I})$.

Using the above observations, we can integrate (10) by parts to get

$$\int_I (-(pu')' + qu - f)v = 0$$

for all $v \in H_0^1(I)$. Thus,

$$-(pu')' + qu = f$$

almost everywhere on I. Since $u \in C^2$, we have that equality holds everywhere on I, therefore u is a strong solution to (8). □

2.3. **The Time Independent Schrödinger Equation.**

Example 2.12 (Time Independent Schrödinger Equation)**.** Starting with the complete Schrödinger equation for a particle of mass m in a spherically symmetric potential, that is, a potential V determined only by the distance x from a point, we have that the probability of a particle's position Ψ is described by

$$i\hbar \frac{\partial \Psi(x,t)}{\partial t} = -\frac{\hbar^2}{2m}\frac{\partial^2 \Psi(x,t)}{\partial x^2} + V(x)\Psi(x,t), \tag{11}$$

where $\hbar = \frac{h}{2\pi}$ is the reduced Plank's constant.

We will use separation of variables to reduce (11) to the time independent case. Depending on how nasty the potential V is, this still may be a difficult problem to solve. However, we will see that the time independent case becomes a Sturm-Liouville problem, which by the Lax-Milgram theorem will reduce to a minimization problem.

To derive the time independent Schrödinger Equation, we first assume that $\Psi = \Psi(x,t)$ can separated into parts $\Psi = \psi(x)\phi(t)$. Plugging this into (11) and dividing by Ψ yields terms

$$i\hbar \frac{1}{\phi}\frac{\partial \phi}{\partial t} = -\frac{\hbar^2}{2m}\frac{1}{\psi}\frac{\partial^2 \psi}{\partial x^2} + V(x),$$

which must equal a constant. We call this constant E, the energy level of the system. We rearrange the right hand side to get the time independent Schrödinger Equation, which describes the standing waves produced by Ψ.

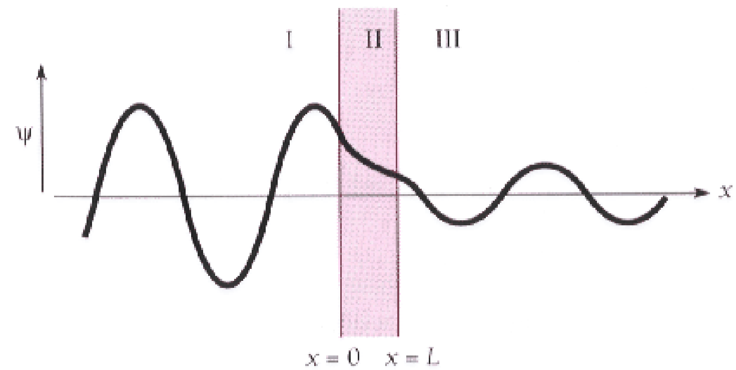
$$E\psi = -\frac{\hbar^2}{2m}\frac{\partial^2 \psi}{\partial x^2} + V(x)\psi. \tag{12}$$

Thus if the particle is in a region where the potential is greater than the energy level , i.e. for all $x \in I = (a,b)$ we have $V(x) - E \geq 0$ in I, then proposition 2.11 gives a unique strong solution to (8) characterized by

$$\frac{1}{2}\int_I (\frac{\hbar^2}{2m}(u')^2 + (V(x)-E)u^2) = \min_{v \in H_0^1(I)} \Big\{ \frac{1}{2}\int_I (\frac{\hbar^2}{2m}(v')^2 + (V(x)-E)v^2) \Big\}.$$

This solution, in which $V(x) - E \geq 0$ in I, can be interpreted as describing the particle's probability of tunneling through a barrier and how much amplitude it retains after passing.

Quantum tunneling through barrier region II. Image Source: University of the Witwatersrand [2]



3. Results in Non-Linear Analysis

In this section we seek to apply fixed point methods to non-linear PDEs. This will amount to proving the existence of semilinear second order differential equations as well as the existence of a periodic wave solution to the Schrödinger equation in a discrete spacial system. The former scenario is common in physical applications, such as waves being dampened non-linearly.

Definition 3.1. We say that a differential equation is

(*i*) *semilinear* if it is linear in all of its highest ordered terms.
(*ii*) *quasilinear* if all highest order terms can be factored into the form

$$f(x, t, u)\frac{\partial^n u}{\partial x_i^n},$$

where $(x, t) = (x_1, ..., x_N)$, and $u = u(x, t)$ is the dependant variable of the n-th order differential equation.

Example 3.2. A wave travelling in an inhomogeneous medium for a given density distribution can be described by the semilinear PDE

$$u_{tt} - b(\vec{x}, t, u)\nabla u = \Delta u,$$

where $b(\vec{x}, t, u)$ depends on the compressibility of the fluid as the system mixes.

Example 3.3. In hydraulics, the Navier-Stokes equations use a quasilinear PDE to describe the velocity u of an incompressible fluid (e.g. water):

$$\rho u_t + \rho(u \cdot \nabla)u - \nabla\sigma(u, p) = f,$$

where ρ is fluid density, p measures local pressure, $\sigma(u, p)$ is stress on some volume of the fluid, and f is an external force.

3.1. **The Brouwer Fixed Point Theorem.**

Definition 3.4. A continuous map $r : A \subset \mathbb{R}^n \to A$ is a *retraction* if for a subset $B \subset A$,

$$r(x) = x$$

for all $x \in B$.

Lemma 3.5. *There is no retraction from a closed ball in $\mathbb{R}^n$ to its boundary.*

Sketch of Proof. Let B_n be the unit ball in $\mathbb{R}_n$. Assume there exists a contraction $r = (r_1, ..., r_n) : \overline{B^n} \to S^{n-1}$. Using Stokes' theorem on some differential form admitted by r, we obtain

$$\int_{S^{n-1}} dr_1 \wedge ... \wedge r_i \wedge ... \wedge dr_n = \int_{\overline{B^n}} dr_1 \wedge ... \wedge dr_n = \int_{\overline{B^n}} det(J_r(x))dx, \tag{13}$$

where J_r is the Jacobian matrix of r. Since $r(x) \in S^{n-1}$, it has constant length 1 for any choice of x, thus forcing $J_r(x)$ to have a zero eigenvalue everywhere. If it did not, it would have an inverse mapping to a ball around $r(x)$ containing points outside the $(n-1)$-sphere.

Observing that $r(x) = id_x$ on S_{n-1}, we apply Stokes' theorem again and plug in the above result for (13) to get

$$\begin{aligned} 1 = \int_{\overline{B^n}} dx_1 \wedge ... \wedge dx_n &= \int_{S^{n-1}} d(id_{x_1}) \wedge ... \wedge id_{x_i} \wedge ... \wedge d(idx_n) \\ &= \int_{S^{n-1}} dr_1 \wedge ... \wedge r_i \wedge ... \wedge dr_n \\ &= \int_{\overline{B^n}} det(J_r(x))dx \\ &= 0. \end{aligned}$$

Note that this sketch only covers C^1 retractions. A complete proof for C^0 retractions can be found in Torres' chapter 1.2 [6]. □

Theorem 3.6 (Brouwer Fixed Point Theorem)**.** *Every continuous function from a closed ball in $\mathbb{R}^n$ to itself has a fixed point.*

Proof. Without loss of generality, consider the closed unit ball $B = \overline{B_1^n(0)} \subset \mathbb{R}^n$. Assume that there exists a continuous $T : B \to B$ without any fixed points.

Then there exists a line $L(x)$ connecting x and $T(x)$ characterized by

$$L(x) = \{\lambda T(x) - (1-\lambda)x : \lambda \in \mathbb{R}, x \in B\} \tag{14}$$

This line always intersects S^{n-1} at two points, once when $\lambda \geq 0$ and once when $\lambda < 0$. Define a retraction $r : B \to S^{n-1}$ by

$$r(x) = y \in S^{n-1} \cap L(x) \;\; \text{where } \lambda \geq 0, \text{ for } \lambda \text{ as defined in (14)},$$

with continuity following from continuity of T. Thus the existence of such a $T : B \to B$ without a fixed point contradicts Lemma 3.5. □

Corollary 3.7. *Let $K \subset \mathbb{R}^n$ and $T : K \to K$ be continuous. If K is homeomorphic to the closed ball, then T has a fixed point.*

Proof. Let $\phi : K \to \overline{B^n}$ be a homeomorphism. Then apply Brouwer's fixed point theorem to $T \circ \phi$. □

We shall only use Brouwer's fixed point theorem as a means of proving Schauder's fixed point theorem, from which the non-linear results follow immediately. However, Brouwer's fixed point theorem is also a powerful tool when used with the calculus of variations to prove the existence of solutions to quasilinear PDEs. Since it is only one of many layers in securing a solution, this result has been omitted. For interested readers, Torres' chapter 4 provides several in depth quasilinear examples [6].

3.2. **The Schauder Fixed Point Theorem.**

Definition 3.8. A set S is *precompact* if the closure of S is compact.

Definition 3.9. Let X, Y be Banach spaces. A continuous map $T : M \subset X \to Y$ is a *compact operator* if it maps bounded sets in M to precompact sets in Y.

Definition 3.10. Let X be a Banach space. For a set $M \subseteq X$, the *convex hull* of M is defined as

$$\operatorname{conv}(M) := \big\{ y = \sum_{i=1}^{n} \alpha_i y_i \in X : y_i \in M, \alpha_i \in [0,1], \sum_{i=1}^{n} \alpha_i = 1 \big\}.$$

Lemma 3.11. *Let $K \subset \mathbb{R}^n$ be nonempty, convex, and compact and $T : K \to K$ be continuous. Then T has a fixed point.*

This lemma follows from extending the corollary to Theorem 3.6 to a broader family of sets. The complete process of this generalization can found on pages 19 and 20 of Pata's *Fixed Point Theorems and Applications* [5].

Theorem 3.12 (Schauder's Fixed Point Theorem)**.** *Let X be a Banach space and $K \subseteq X$ be nonempty, convex, and compact. If $T : K \to K$ is continuous, then T has a fixed point.*

Proof. Let (ϵ_n) be a sequence converging to 0. By the continuity of T and the compactness of K, we have that T is uniformly continuous. Thus there exists $\delta_n > 0$ such that

$$\|x - y\| < \delta_n \implies \|Tx - Ty\| < \epsilon_n \qquad \text{for all } x, y \in K.$$

Next, we create a finite open cover of K of δ_n-balls, i.e. $K \subset \bigcup_{i=1}^{M_n} B_{\delta_n}(x_i)$.

Using the centers of the balls in the open cover, define L_{ϵ_n} to be the subspace spanned by $\{Tx_i : 1 \leq i \leq M_n\}$. Thus $K \cap L_{\epsilon_n}$ is compact, convex, and finite-dimensional.

We will now create a partition of unity to distribute T over this open cover. For $i \leq j \leq M_n$, define $\psi_j : X \to \mathbb{R}$ by

$$\psi_j(x) = \begin{cases} 0, & \text{if } \|x - x_j\| \geq \delta_n \\ 1 - \frac{1}{\delta_n}\|x - x_j\|, & \text{if } \|x - x_j\| \leq \delta_n. \end{cases}$$

Then $\sum_{j=1}^{M_n} \psi_j$ is continuous, bounded below, and positive everywhere on $\bigcup_{i=1}^{M_n} B_{\delta_n}(x_i)$.

Let our partition of unity be defined by

$$\phi_i(x) = \frac{\psi_i(x)}{\sum_{j=1}^{M_n} \psi_i(x)} \qquad \text{for all } \ 1 \leq i \leq M_n.$$

Thus $\sum_{i=1}^{M_n} \phi_i(x) = 1$ for any $x \in K$.

Define

$$T_{\epsilon_n} x = \sum_{i=1}^{M_n} \phi_i(x) T(x_i).$$

Finally, we have equicontinuity of the maps from

$$\|T_{\epsilon_n} x - Tx\| \le \sum_{i=1}^{M_n} \phi_i(x) \|Tx_i - Tx\| \le \epsilon_n \sum_{i=1}^{M_n} \phi_i(x) = \epsilon_n.$$

Applying Lemma 3.11 we have that,

$$T_{\epsilon_n} : L_{\epsilon_n} \cap K \to L_{\epsilon_n} \cap K$$

has a fixed point $T_{\epsilon_n} x_n = x_n$. Since K is compact, (x_n) has a convergent subsequent $x_{n_i} \to x'$. Then as $i \to \infty$,

$$\|Tx' - Tx_{n_i}\| \to 0,$$

$$\|Tx_{n_i} - T_{\epsilon_{n_i}} x_{n_i}\| \to 0,$$

$$\|T_{\epsilon_{n_i}} x_{n_i} - x'\| = \|x_{n_i} - x'\| \to 0.$$

Thus by triangle inequality, $\|Tx' - x'\| \to 0$, so x' is a fixed point of T. □

To prove Schaefer's fixed point theorem, we will need a slightly modified version of Schauder's. This will allow us greater freedom in infinite dimensional Banach spaces, where compactness can be more difficult to show. For a method using Mazur's lemma to recover the following corollary from Theorem 3.12, consult pages 25 and 62 of Torres' lecture notes [6].

Corollary 3.13. *Let X be a Banach space and $K \subseteq X$ be nonempty, convex, closed, and bounded. If $T : K \to K$ is a compact operator, then T has a fixed point.*

Theorem 3.14 (Schaefer's Fixed Point Theorem)**.** *Let X be a Banach space. If a compact operator $T : X \to X$ is such that the set*

$$K = \{x \in X : x = \lambda T(x) \text{ for some } \lambda \in [0, 1]\},$$

is bounded, then T has a fixed point.

Proof. Since K is bounded, choose $N > \sup_{x \in K} \|x\|$. Define a map $f : X \to \overline{B_N(0)}$ by

$$f(x) = \begin{cases} T(x) & \text{if } \|T(x)\| \le N \\ \frac{NT(x)}{\|T(x)\|} & \text{if } \|T(x)\| > N \end{cases}$$

By our assumptions on T, we have that f is also a compact operator on X. Applying the corollary of Theorem 3.12 to the set $\overline{B_N(0)}$ yields a fixed point of f in K.

Thus we have two cases, where $x_0 = T(x_0)$ and $x_0 = \frac{NT(x_0)}{\|T(x_0)\|}$. If the first is true, then we are done. Suppose the latter case holds. Then

$$\|x_0\| = \|f(x_0)\| = \left\| \frac{NT(x_0)}{\|T(x_0)\|} \right\| = N,$$

which contradicts our choice of N. □

3.3. **Semilinear Applications.** Suppose $\Omega \in \mathbb{R}^n$ is bounded, open, and has smooth boundary $\partial\Omega$. Consider the semilinear PDE and BVP

$$\begin{cases} -\Delta u + g(u, \nabla u) + \mu u = h & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases} \tag{15}$$

where $\mu \geq 0$, and there exists M_1, M_2 such that

$$g(z,p) \leq M_1 + M_2(|z| + |p|),$$

where $z \in \mathbb{R}, p \in \mathbb{R}^n$, and $h \in L^2(\Omega)$.

Proposition 3.15. *If $M_2 = 0$ or μ is sufficiently large, then a weak solution to (15) exists.*

Proof. Our strategy will be to apply Schaefer's Fixed Point Theorem to the Sobolev space $H_0^1(\Omega)$ and the map $u \mapsto F \circ G(u)$, where $F : L^2(\Omega) \to H_0^1(\Omega)$ is $F(v) = (-\Delta + \mu)^{-1}(v)$ and $G : H_0^1(\Omega) \to L^2(\Omega)$ is $G(u) = -g(u, \nabla u) + h$.

Denoting this composition as $F \circ G = T : H_0^1(\Omega) \to H_0^1(\Omega)$, we see that obtaining a fixed point of T yields

$$-\Delta u + \mu u = (-\Delta + \mu)T(u) = -g(u, \nabla u) + h.$$

The Sobolev embeddings give us a compact inclusion $H_0^1(\Omega) \hookrightarrow L^2(\Omega)$ to take us into $L^2(\Omega)$, thus we must now show compactness arises through the composition with F back into $H_0^1(\Omega)$.

Let $\{v_n\} \subset L^2(\Omega)$ be a bounded sequence with weakly convergent subsequence $v_{n'} \rightharpoonup v$. Consider the sequence $\{u_{n'}\} \subseteq H_0^1(\Omega)$, characterized by $F(v_{n'}) = u_{n'}$. We get the existence of $u_{n'}$ by applying the Lax-Milgram theorem to the bilinear form

$$b(u,v) = \int_\Omega \nabla u \cdot \nabla v + \mu uv$$

on the space $H_0^1(\Omega)$.

We will show that $\{u_{n'}\}$ is convergent in $H_0^1(\Omega)$ in two steps:

Step 1: We claim that $\{u_{n'}\}$ is bounded in $H_0^1(\Omega)$ and converges weakly in $H_0^1(\Omega)$.

As in Definition 2.10, since $u_{n'}$ is a strong solution, and thus also a weak one, to $F(v_{n'}) = u_{n'}$, we can integrate by parts and choose the test function $u_{n'} \in H_0^1(\Omega)$ to get

$$\int_\Omega (\nabla u_{n'})^2 + \mu u_{n'}^2 = \int_\Omega v_{n'} u_{n'}.$$

Since μ is nonnegative,

$$\Big| \int_\Omega (\nabla u_{n'})^2 \Big|^2 \leq \Big| \int_\Omega v_{n'} u_{n'} \Big|^2.$$

Finally, by using the Cauchy-Schwarz inequality on the right hand side and Poincaré's inequality on the left, we get

$$\|u_{n'}\|_{H_0^1(\Omega)}^4 \leq C_\Omega \|\nabla u_{n'}\|_{L^2(\Omega)}^4 \leq C_\Omega \|u_{n'}\|_{L^2(\Omega)}^2 \|v_{n'}\|_{L^2(\Omega)}^2.$$

Therefore $u_{n'}$ is bounded in $H_0^1(\Omega)$, thus we can extract a weakly convergent subsequence from $u_{n'}$. Following a similar method to the proof of Proposition 2.11, we can use the Lax-Milgram theorem with the formation of the weak solution as our bilinear form to get that there is a unique limit to all weakly convergent subsequences. Thus, we write $u_{n'} \rightharpoonup u$.

Step 2: We claim that $\{u_{n'}\} \to u$ strongly in $H_0^1(\Omega)$.

As above, each $u_{n'}$ is a weak solution to $F(v_{n'}) = u_{n'}$.

$$\int_\Omega \nabla u \cdot \nabla \phi + \mu u \phi = \int_\Omega v\phi \tag{16}$$

for any test function $\phi \in H_0^1(\Omega)$.

By compactness of the embedding of H_0^1 into L^2, Step 1 gives us that $\{u_{n'}\}$ converges strongly in L^2. Then we can take the difference of the $n' \to \infty$ limit of (16) with any n' and choose $\phi = u - u_{n'}$ to get

$$\begin{cases} \int_\Omega \nabla u \cdot \nabla(u - u_{n'}) + \mu u(u - u_{n'}) = \int_\Omega v(u - u_{n'}) \\ \int_\Omega \nabla u_{n'} \cdot \nabla(u - u_{n'}) + \mu u_{n'}(u - u_{n'}) = -\int_\Omega v_{n'}(u - u_{n'}) \end{cases}$$

$$\begin{aligned} \int_\Omega |\nabla u - \nabla u_{n'}|^2 + \mu|u - u_{n'}|^2 &= \int_\Omega (v - v_{n'})(u - u_{n'}) \\ &\le \|u - u_{n'}\|_{L^2(\Omega)} \|v - v_{n'}\|_{L^2(\Omega)} \\ &\le C\|u - u_{n'}\|_{L^2(\Omega)} \to 0, \end{aligned}$$

where C is a constant derived from the boundedness of $\{v_n\}$.

To complete the proof we must show that the set

$$K = \{u \in H_0^1(\Omega) : u = \lambda T(u) \text{ for some } \lambda \in [0, 1]\}$$

is bounded.

Suppose $u \in K$. Multiplying $F^{-1}(u) = F^{-1}(\lambda T(u))$ by u and integrating over Ω gives

$$\int_\Omega |\nabla u|^2 + \mu u^2 = \lambda \int_\Omega (-g(u, \nabla u) + h)u.$$

To utilize the bounds on g, separate the right hand side with the Cauchy-Schwarz and the triangle inequality to obtain

$$\begin{aligned} \int_\Omega |\nabla u|^2 + \mu u^2 &\le \lambda \|g(u, \nabla u) + h\|_{L^2(\Omega)} \|u\|_{L^2(\Omega)} \\ &\le \lambda\big(M_1 + M_2\|u\|_{H_0^1(\Omega)} + \|h\|_{L^2(\Omega)}\big)\|u\|_{L^2(\Omega)} \\ &\le \lambda M_2 \|u\|_{H_0^1(\Omega)} \|u\|_{L^2(\Omega)} + \lambda\big(M_1 + \|h\|_{L^2(\Omega)}\big)\|u\|_{L^2(\Omega)}. \end{aligned}$$

Let $\kappa = \frac{1}{C_\Omega}$, where C_Ω is the constant from applying Poincaré's inequality to u. Using the inequality $ab \le \frac{ca^2}{2} + \frac{b^2}{2c}$ on the right hand side with $c_1 = \kappa$ and $c_2 = \frac{\kappa}{2}$, we get

$$\int_\Omega |\nabla u|^2 + \mu u^2 \le \lambda\Big[\frac{\kappa\|u\|^2_{H_0^1(\Omega)}}{2} + A\|u\|^2_{L^2(\Omega)}\Big] + \lambda\Big[\frac{\kappa\|u\|^2_{L^2(\Omega)}}{4} + B\Big], \tag{17}$$

where constants $A = M_2^2/2\kappa$ and $B = (M_1 + \|h\|_{L^2_\Omega})^2/\kappa$ come from applying the previous inequality.

By Poincaré's inequality,

$$\kappa\|u\|^2_{H_0^1(\Omega)} + \mu\|u\|^2_{L^2(\Omega)} \le \int_\Omega |\nabla u|^2 + \mu u^2. \tag{18}$$

Combining (18) with (17) gives

$$\frac{\kappa}{2}\|u\|^2_{H_0^1(\Omega)} + \big(\mu - (A + \frac{\kappa}{4})\big)\|u\|^2_{L^2(\Omega)} \le B.$$

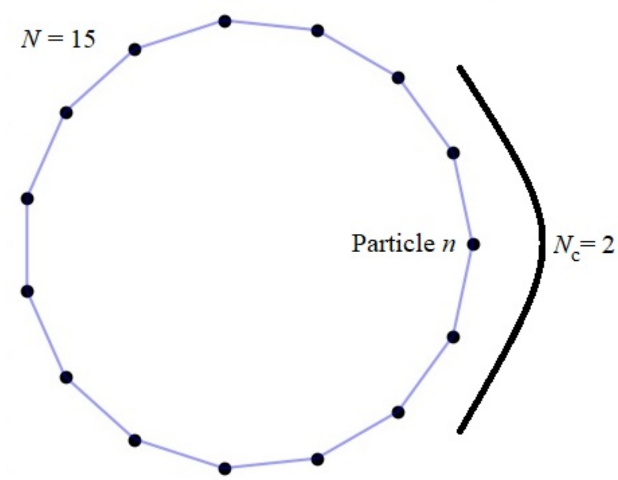
Thus for $\mu \geq A + \frac{\kappa}{4}$, we have that $\|u\|_{H_0^1(\Omega)}$ is bounded for all $u \in K$. □

Example 3.16 (Periodic Solution to a System of Discrete Schrödinger Equations)**.** Consider a finite 1-Dimensional lattice of N particles. Assume that the lattice structure couples the particles such that if one begins to oscillate, a change is immediately measurable in N_c of its neighbors on both sides. That is, the information of the event reaches all lattice points within a distance of N_c points from the source at the same time. The interaction between a particle and its N_c nearest neighbors to the left and right is described for a wave ψ_n by

$$\begin{cases} i\frac{d\psi_n}{dt} = \sum_{j=1}^{N_c} \kappa_j[\psi_{n+j} - 2\psi_n + \psi_{n-j}] + F(|\psi_n|^2)\psi_n, & 1 \leq n \leq N, \\ \psi_{N+m}(t) = \psi_m(t) \end{cases} \tag{19}$$

For a finite system, we can interpret the periodicity condition $\psi_{N+m}(t) = \psi_m(t)$ as the the lattice looping back to its beginning, thus making the n^{th} position arbitrary. This imposes the condition that $N_c \leq \frac{N-1}{2}$ so that a particle cannot have two different interactions with the same neighbor.

A ring-like 1-dimensional lattice of $N = 15$ particles. Periodicity is a consequence of the wave completing a full rotation (created with matplotlib in Python).



We impose the growth condition that there exist constants $a > 0, b > 0$ such that for all $x \geq 0$,

$$|F(x)| < a(1 + x^b). \tag{20}$$

We assume the solution to be a travelling wave, so we expect a solution to be of the form $\psi_n(t) = \Psi(u) = \Psi(kn - \omega t)$, where k, ω are wave number and angular frequency, respectively. Furthermore, we claim we can recover a periodic solution to (19) by starting with a the semilinear differential equation

$$-i\omega\Psi'(u) = \sum_{j=1}^{N_c} \kappa_j \Delta_j \Psi(u) + F(|\Psi(u)|^2)\Psi(u) \tag{21}$$

.

Note that $\Delta_j \Psi(u)$ is not a lapacian, but rather a measure of change around $\Psi(u)$,

$$\Delta_j \Psi(u) = \Psi(u+j) - 2\Psi(u) + \Psi(u-j). \tag{22}$$

A sketch using Schauder's fixed point theorem will be used to show the existence of a solution to (19), with a full proof found in Hennig's paper [3]. This sketch omits almost all calculations. Rather, it seeks to present the spaces and operators one would use to complete the proof.

Proposition 3.17. *Fix $q \in \mathbb{Q} \cap (0,1)$, and let it define wave number k by $k = q\pi$. Suppose (20) holds, angular frequency ω is sufficiently large, and wave power $P = \sum_{n=1}^{N} |\psi_n|^2$ is conserved. Then there exists a solution to* (19)*.*

Sketch of Proof. Decompose $\Psi(kn+\omega t)$ into a complex oscillator and an amplitude,

$$\Psi(u) = \Phi(u) e^{iq(kn-\omega t)}. \tag{23}$$

With this, we can rewrite (21) as

$$-i\omega\Phi'(u) + \omega q \Phi(u) - \sum_{j=1}^{N_c} \kappa_J \overline{\Delta_j} \Phi(u) = F(|\Phi(u)|^2)\Phi(u), \tag{24}$$

where $\overline{\Delta_j}\Phi(u)$ is from plugging (23) into (22).

For our Banach space, consider

$$X^0 = \{\theta \in C(\mathbb{R}, \mathbb{C}) : \theta(x) = \theta(x+2\pi)\}$$

and

$$X^1 = \{\theta \in C^1(\mathbb{R}, \mathbb{C}) : \theta(x) = \theta(x+2\pi)\},$$

with norms

$$\|\theta\|_{X^0} = \max_{x\in[0,2\pi)} |\theta(x)|$$

$$\|\theta\|_{X^1} = \max_{x\in[0,2\pi)} |\theta(x)| + \max_{x\in[0,2\pi)} |\theta'(x)|.$$

Define $M : X^1 \subset X^0 \to X^0$ by

$$M(\theta) = -i\omega\theta' + \omega q \theta - \sum_{j=1}^{N_c} \kappa_J \overline{\Delta_j}\theta.$$

If we apply M to the fourier decomposition of θ, we can show that for sufficiently large ω, $M^{-1} : X^0 \to X^1 \subset X^0$ is always well defined and bounded.

By the bounds in the hypothesis, one can calculate A, B such that any wave solution to (24) satisfies $\|\Phi\|_{X^0} \le A$ and $\|\Phi'\|_{X^0} \le B$.

Now consider the closed, convex subsets of our Banach spaces

$$K^0 = \{\theta \in X^0 : \|\theta\|_{X^0} \le A\}$$

$$K^1 = \{\theta \in X^1 : \|\theta\|_{X^1} \le B\}.$$

We have the linear part of (24) represented by M, so for the non-linear part define a continuous map $N : K^0 \to K^0$ by

$$N(\theta) = F(|\theta|^2)\theta.$$

Using our growth condition (20) and the definition of power P, we have that

$$\|N(\theta)\|_{K^0} = \max_{x\in[0,2)} \left|F(|\theta(x)|^2)\theta(x)\right| \le a(1+P^b)A.$$

Thus, $N(K^0)$ is bounded inside K^0.

Finally, we construct the operator $T : K^0 \to K^1 \subset K^0$ as

$$T = M^{-1} \circ N.$$

We argue the compactness of T by showing a compact embedding of $T(K^0)$ into K^1. Indeed,

$$\|T(\theta)\|_{K^1} = \|M^{-1}\|_{K^1}\|N(\theta)\|_{K^0} \leq (1+B)A.$$

Thus, T maps bounded $K^0 \subset X^0$ to precompact $K^1 \Subset K^0$.

Schauder's fixed point theorem then gives a 2π-periodic C^1 function $\Phi = T(\Phi)$. By applying M to both sides of the equation to recover (24), we guarantee a solution to (19).

□

4. Appendix

Definition 4.1. A *Hilbert space* is a vector space equipped with a scalar product $\langle \cdot, \cdot \rangle$ such that it is a complete metric space with respect to the norm $|x| = \langle x, x\rangle^{\frac{1}{2}}$.

Theorem 4.2. *Hilbert spaces are uniformly convex and reflexive.*

Definition 4.3. Let X be an inner product space. We say that a sequence $\{\phi_n\} \subset X$ *weakly converges* to $\phi \in X$ if for every $\psi \in X$,

$$\langle \phi_n, \psi \rangle \to \langle \phi, \psi \rangle$$

as $n \to \infty$. We denote this by

$$\phi_n \rightharpoonup \phi.$$

Theorem 4.4 (Rellich-Kondrachov Theorem)**.** *Let Ω be bounded and of class C^1. For given p, N, define $p^* = (\frac{1}{p} - \frac{1}{N})^{-1}$.*

(*i*) *If $p < N$, then for all $q \in [1, p^*)$ we have a compact inclusion*

$$T : W^{1,p}(\Omega) \hookrightarrow L^q(\Omega).$$

(*ii*) *If $p = N$, then for all $q \in [p, +\infty)$ we have a compact inclusion*

$$T : W^{1,p}(\Omega) \hookrightarrow L^q(\Omega).$$

(*iii*) *If $p > N$ we have a compact inclusion*

$$T : W^{1,p}(\Omega) \hookrightarrow C(\overline{\Omega}).$$

Remark 2. For any choice of p and N, we have that $W^{1,p}(\Omega)$ compactly embeds into $L^p(\Omega)$.

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