NOTES ON THE 2019 REU PROGRAM

Here are some notes about the program. They will be updated throughout the program as more information comes in. Research faculty are often notably late to prepare talks. (I myself am an extreme offender.) Abstracts for all courses to be given in the first two weeks are included. The program is frontloaded, with the number of talks decreasing as the focus switches to your work on papers.

APPRENTICE PROGRAM
Weeks one through five. Meg Doucette and Isabella Scott (week 1); Daniil Rudenko (weeks 2-5); Laci Babai (Special, Friday June 28).

This is a repeat from 2018, but the content may change!

What is a space? Every few decades the answer to this question undergoes a substantial modification. One of the biggest breakthroughs of the 19th century was the realization of the essential role played by symmetry groups.

It was an exciting time. Almost simultaneously certain objects whose very existence was speculative if not frightening, turned out to be fundamental in mathematics and physics. Among those objects, complex numbers, higher-dimensional spaces and non-Euclidean geometrics were especially important. The goal of this course is to become familiar with these objects and multiple relations existing between them.

We will start our journey in the familiar territory of the Euclidean geometry and will go on to dive into the hyperbolic plane, three-dimensional sphere and higher dimensional Euclidean spaces. Finally, we will find ourselves in the peculiar landscape of projective geometry. Along the way we will develop a major scientific tool provided by linear algebra and group theory. Also, we will touch upon a few related topics in number theory and combinatorics.

It is impossible to achieve comprehension of these concepts without solving plenty of technical exercises as well as challenging problems. Getting this experience will be a large part of the course.

FULL PROGRAM

FINITE SPACES AND LARGER CONTEXTS
Weeks one through eight. Peter May.

This is a repeat abstract from 2018, but the content will change!

What is a topological space? The notion was defined by Hausdorff and, in the still current version, Kuratowski, around a century ago. There is a less general later definition due to Alexandroff. Nearly all mathematicians, regardless of field, know the early definitions. Few know about Alexandroff spaces, but all finite spaces are Alexandroff, and finite spaces are surprisingly interesting. For example there is a

space with 2n+2 points that, to the eyes of algebraic topology, is equivalent to the n-sphere. This is a strange, fascinating, and little-known world of mathematics.

It is a striking, if esoteric, fact that in principle one can do all of algebraic topology using Alexandroff spaces. That is even true equivariantly, when one considers spaces with symmetries given by group actions. Alexandroff spaces in which the topology distinguishes points are the "same thing" as partially ordered sets (posets). This establishes a close connection between algebraic topology and combinatorics. There are many applications, some speculative, others far out. For example, finite spaces have been used to study the role of RNA in evolution.

I am writing a book on this subject, with the same title as the talks, and I will present topics from it and from past REU papers and other papers that have been written over the years. (Maybe some of you might want to contribute.) The intent of the talks will be to introduce both finite spaces and "ordinary" or "usual" spaces, emphasizing how differently the same concepts behave in the two worlds. For example all usual spaces, such as metric spaces, are Hausdorff spaces (as in Hausdoff's original 1914 definition), but finite Hausdorff spaces are discrete. And yet "finite metric spaces" are used extensively in topological data analysis.

HYPERBOLIC GEOMETRY - FROM GRAPHS TO GEOMETRY
Weeks one and two. Danny Calegari.

We start with a combinatorial problem: given a planar graph G, when can we find a collection of round circles in the plane whose interiors are disjoint, and with one circle for every vertex of G, such that two circles are tangent if and only if the corresponding vertices are joined by an edge of G? This will be the starting point that leads to the Riemann mapping theorem, the construction of 3-dimensional hyperbolic polyhedra, and a beautiful family of connections between combinatorics, topology, geometry, and complex analysis.

RANDOM WALK, HEAT EQUATION, AND FOURIER SERIES Weeks one and two. Greg Lawler.

This course will focus on "diffusion", the movement of heat or other random particles. There are many ways to describe this motion: one can use discrete or continuous models and one can use deterministic or random models. We will do all of these. In the discrete we will see random walk, difference equations, and linear algebra. Going to the continuous we get Brownian motion, partial differential equations, and Fourier series.

GEOMETRY OF MEASURES IN $\mathbb{R}^n$
Weeks one through three. Abdalla Dali Nimer (weeks 1-2) and Cornelia Mihaila (week 3).

In the 1920's, Besicovitch proved that the infinitesimal "length" of a set contains information on the general structure and geometry of the set. This marked the beginning of Geometric Measure Theory, a field that studies the interplay between the geometry of sets and the properties of measures supported on them. In this course, we will present some of the basic tools used in Geometric Measure Theory and some important results in the field. Some topics that may be covered : basic measure theory, covering theorems and differentiation of measures, densities, Rademacher's

theorem, Marstrand's density theorem.
Prerequisite: a first year undergraduate analysis course (Continuity, Differentiation, Integrals, Multi-variable derivatives, sequences of functions, etc). A knowledge of Lebesgue theory is not necessary but will make following the course much easier.

INTRODUCTION TO RESOLUTION OF SINGULARITIES
Week one. Antoni Rangachev.

Algebraic geometry is the branch of mathematics that studies the solutions of systems of multivariate polynomial equations. The solutions are called *algebraic varieties,* which can be visualized as geometric objects. Examples include lines, circles, hyperbolas, ellipses, spheres, etc.

There are two main classes of algebraic varieties: *smooth* varieties and *singular* varieties. Smooth varieties are better understood because locally they look like affine spaces. In contrast, singular varieties have points where the local geometry may change abruptly. A fundamental problem in algebraic geometry is to find a smooth model for each singular variety. The model is a smooth algebraic variety which must preserve most of the features of the original singular variety. The process of finding such models is called *resolution of singularities*.

In these lecture series I will give a historical introduction to resolution of singularities. I will illustrate some of the main ideas involved in the resolution process by resolving singular curves following old ideas of Newton and Puiseux. Finally, I will discuss some open problems.

THE EUCLIDEAN PLANE: SOME QUESTIONS IN LOGIC AND COMBINATORICS
Week one. Carolyn Terry.

In these lectures we will discuss a simple but interesting structure definable in the Euclidean plane, namely the unit distance graph. The unit distance graph on the plane is the graph whose vertex set is the real plane, and where two vertices are connected by an edge if and only if they are distance 1 apart. A basic question asks: what is the chromatic number of this graph? Surprisingly, this simple question remains open. In these lectures we will present this problem along with some logic-related questions about this graph, including: How much of Euclidean geometry can be recovered from just the unit distance relation? Is there an algorithmic axiomatization of the unit distance graph? What does the axiom of choice have to do with its chromatic number?

COMPUTABILITY THEORY, REVERSE MATHEMATICS, AND COMBINATORICS
Weeks two through four. Denis Hirschfeldt.

Every mathematician knows that if 2+2=5 then Bertrand Russell is the pope. Russell is credited with having given a proof of that fact in a lecture, though from the point of view of classical logic, no such proof is needed, since a false statement implies every statement. Contrapositively, every statement implies a given true statement. But we are often interested in questions of implication and nonimplication between true statements. We have all heard and said things like "Theorems A and B are equivalent." or "Theorem C does not just follow from

Theorem D." Computability theory and proof theory can both be used to analyze, and hence compare, the strength of theorems and constructions. For example, when we have a principle such as "Every infinite binary tree has an infinite path", we can ask how difficult it is to compute such a path from a given tree. We can also ask how much axiomatic power is necessary to prove that this principle holds. The first kind of question leads to the program of Computable Mathematics. One version of the second kind of question leads to the program of Reverse Mathematics. I will give an introduction to these research programs, focusing on combinatorial principles, and discuss how the close connection between computability and definability yields a fruitful interplay between them.

GEOMETRY OF SURFACES IN SPACE
Week 3. André Nevins

Lecture 1: I will review the basic notions regarding the geometry of surfaces in space, namely the principal curvatures, Gauss map, and Gauss curvature

Lecture 2: I will present the basic theorems regarding the geometry of surfaces in space, namely Gauss-Bonnet Theorem and the Gauss Egregium Theorem

Lecture 3: I will explain the geometric meaning of total curvature, Willmore energy, and the basic foundational results.

QUADRATIC FORMS
Weeks three through five. Akhil Mathew.

A quadratic form in n variables is a function of the form $q(x_1, \ldots, x_n) = \sum_{i,j} a_{ij} x_i x_j$, where the a_{ij} form a symmetric matrix; this makes sense over any field of characteristic $\neq 2$, such as $\mathbb{Q}, \mathbb{R}, \mathbb{C}, \mathbb{F}_q$ and others. There are a number of basic questions one can ask about quadratic forms, such as:
1) When are two quadratic forms isomorphic (up to a change of basis)?
2) When does a quadratic form represent an element, i.e., when is that element in the image? Example: Lagrange's famous theorem that every positive integer is the sum of four squares.

These questions have a long history and have led to much interesting mathematics. In the course, I'll try to survey some results in the theory, e.g., the Hasse-Minkowski theorem which answers these questions for quadratic forms over Q.

UNIQUE FACTORIZATION INTO LYNDON WORDS AND THEIR GENERALIZATIONS
Special talk July 8. Amanda Burcroff.

Lyndon words are combinatorial objects with applications throughout mathematics and computer science, including the study of Lie algebras, de Bruijn sequences, and data compression. We will investigate factorizations of arbitrary words into Lyndon words and a weakening of the Lyndon axioms which still leads to unique factorization.

UNDERSTANDING PDEs Week four. Will Feldman.

The course will introduce several fundamental PDE (partial differential equations), e.g. Laplace, Heat, Wave, and Transport. I will introduce some tools to gain intuitive understanding of the 'physical' meaning of the PDE. I will likely discuss variational principles and special solutions (scale invariant solutions, traveling waves, shocks).

PDEs AND EXPLOSIONS OF BACTERIA
Week five. Chris Henderson.

E coli (and several other organisms) secrete and sense chemicals which influence their movement by causing them to aggregate; that is, the e coli "tend" to move towards their neighbors. This behavior canbe modeled mathematically by a partial differential equation, and one famous recent result about this model is that if the initial population is large enough, then all bacteria aggregate to one infinite density point (the explosion!).

In this course, we will go over the derivation of the PDE and the proof of blow-up and non-blow-up. The focus will be less on the biology and more on the mathematical analysis. But all the work will be elementary and should be accessible to anyone who has taken calculus.

PROOF BY PICTURES
Special talks July 29, July 30. Emily Peters.

Proof by pictures I: diagram algebras and knots.

Take a piece of string, jumble it up, and fuse the ends together: now you have a mathematical knot. Do it again. Are the two knots you created the same? Can they be manipulated into the same shape? If not – how do you know? Knot invariants, that's how! Take your knot and replace it with a simpler thing: a number or a polynomial, say. Just make sure you do it in a way so that the thing doesn't change when the knot shifts around ...

Proof by pictures II: The n-color theorems.

Given a map, say of the forty-eight contiguous states of the US, how many colors do you need to color it in so that two states that share a border are always colored different colors? What properties of a map determine how many colors are required? And what does this have to do with yesterday's knots?

CATEGORY THEORY AND LIFE
Special talk August 14. Eugenia Cheng.

Category theory is often thought of as "very abstract algebra" with its main applications being in branches of pure mathematics, theoretical physics, and the foundations of computer science. However in this talk I will apply basic concepts of category theory to important questions of life such as prejudice, privilege, blame and responsibility. I will argue that abstraction has a purpose and that broad applicability is one of the powerful consequences. The talk will serve as an both an introduction to the ideas of category theory and a suggestion of how to convince skeptics of the relevance of abstract mathematics. I will introduce the category theory concepts from scratch so no prior knowledge is needed.