SIMPLICIAL SETS, NERVES OF CATEGORIES, KAN COMPLEXES, ETC

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These notes are taken from Peter May's classes in REU 2018. Some notations may be changed to the note taker's preference and some detailed definitions may be skipped and can be found in other good notes such as [2] or [3]. The note taker is responsible for any mistakes.

1. SIMPLICIAL APPROACH TO DEFINING HOMOLOGY

Defnition 1. *A simplical set/group/object K is a sequence of sets/groups/objects K_n for each $n \geq 0$ with face maps: $d_i : K_n \to K_{n-1}, 0 \leq i \leq n$ and degeneracy maps: $s_i : K_n \to K_{n+1}, 0 \leq i \leq n$ satisfying certain commutation equalities.*

Images of degeneracy maps are said to be degenerate.

We can define a functor:

$$\begin{array}{ccc} \text{ordered abstract simplicial complex} & \to & \text{sSet}, \\ K & \mapsto & K^s, \end{array}$$

where

$$K_n^s = \{v_0 \leq \cdots \leq v_n | \{v_0, \cdots, v_n\} \text{ (may have repetition) is a simplex in } K\}.$$

$$\begin{array}{rl} \text{Face maps:} & d_i : K_n^s \to K_{n-1}^s, 0 \leq i \leq n \text{ is by deleting } v_i; \\ \text{Degeneracy maps:} & s_i : K_n^s \to K_{n+1}^s, 0 \leq i \leq n \text{ is by repeating } v_i. \end{array}$$

In this way it is very straightforward to remember the equalities that face maps and degeneracy maps have to satisfy.

The simplical viewpoint is helpful in establishing invariants and comparing different categories. For example, we are going to define the integral homology of a simplicial set, which will agree with the simplicial homology on a simplical complex, but have the virtue of avoiding the barycentric subdivision in showing functoriality and homotopy invariance of homology. This is an observation made by Samuel Eilenberg.

To start, we construct functors:

$$\text{sSet} \xrightarrow{F} \text{sAb} \xrightarrow{C} \text{Ch}_{\mathbb{Z}}.$$

The functor F is the free abelian group functor applied levelwise to a simplical set.

The functor C is defined as such: given a simplical abelian group A, let $C_n(A) = A_n$, and $d : C_n(A) \to C_{n-1}(A)$ by $d = \sum_{i=0}^{n} (-1)^i d_i$.

The commutation equalities of simplicial sets will guarantee that $d^2 = 0$, so that d makes $C_*(A)$ into a chain complex of abelian groups. Taking homology of the chain complex gives the homology of the simplicial set.

Remark. The functor C is an equivalence of categories, called Dold-Thom correspondence. An explanation of this can be found in [1].

For general topological spaces, we can define

$$\begin{array}{ccc} Top & \to & \text{sSet}, \\ X & \mapsto & S(X), \end{array}$$

where $S_n(X) = Map(\Delta^n, X)$, the set of continuous maps from the standard topological n-simplex to X. It can be as bad as space filling curves.

For $k : \Delta^n \to X$, define the face and degeneracy maps as

$$\begin{array}{rccc} d_i k = k \circ \delta_i : & \Delta^{n-1} & \to & X, \\ s_i k = k \circ \sigma_i : & \Delta^{n+1} & \to & X. \end{array}$$

The coface map δ_i is inclusion of the face opposing the i-th vertex, while the codegeneracy map σ_i is the projection to the face along the edge connecting the i-th and $i+1$-th vertex. Explicitly, they are given by:

$$\begin{array}{rccc} \delta_i: & \Delta^{n-1} & \to & \Delta^n, \\ & (t_0,\dots,t_{n-1}) & \mapsto & (t_0,\dots,0,\dots,t_{n-1}); \\ \sigma_i: & \Delta^{n+1} & \to & \Delta^n, \\ & (t_0,\dots,t_{n+1}) & \mapsto & (t_0,\dots,t_i+t_{i+1},\dots,t_{n+1}). \end{array}$$

The maps δ_i and σ_i actually live in the simplex category Δ.

Defnition 2. *The simplex category Δ has objects isomorphism classes of finite ordered sets, or ordinals, $[n]=\{0,1,\cdots,n\}$, and morphisms non-strict order preserving maps.*

The category Δ is generated by morphisms $\delta_i:[n-1]\to[n]$ skipping i and $\sigma_i:[n+1]\to[n]$ repeating i in the sense that all morphisms are compositions of these.

A simplicial object is indeed a contravariant functor from Δ to the category of the object, which is a more concise way to say the same words as the commutation equalities.

Defnition 3. *The geometric realization/totalization of a simplicial set is the functor:*

$$\begin{array}{ccc} sSet & \to & Top, \\ K & \mapsto & |K| \text{ or } T(K), \end{array}$$

where

$$|K| = \coprod_{i\geq 0} K_i \times \Delta^i/(k,\alpha_* s)\sim(\alpha^* k, s),$$

for $k\in K_i, s\in\Delta^j, \alpha:[j]\to[i]$ composites of face and degeneracy maps.

Remark. In $|K|$, every point (k,s) is uniquely identified with a pair (l,t) such that l is nondegenerate and t is in the interior of a geometric simplex.

The topological standard n-simplexes Δ^n have their counterparts Δ^n_* as simplicial sets, which are exactly the contravariant functors on Δ corepresented by $[n]$. These simplicial standard n-simplexes geometrically realize to topological standard n-simplexes. To see this, it is convincing to think about the nondegenerate elements in these simplicial sets, which will realize to the faces of the desired topological simplex.

In summary, we have the functors as below and the coface and codegeneracy maps in Δ realized to those in Top.

$$\begin{array}{ccccc} \Delta & \to & \text{sSet} & \to & Top \\ [n] & \mapsto & \Delta^n_* = \Delta(-,[n]) & \mapsto & |\Delta^n_*| = \Delta^n. \end{array}$$

In this way, the total singular complex functor S is the corepresented functor on Top restricted to Δ.

Remark. In the next section, we make more use of this idea that an embedding of Δ into any other category gives the standard n-simplexes as image of $[n]$ in the target category.

In fact, the total singular complex and the geometric realization form an adjoint equivalence.

$$\text{sSet} \underset{S}{\overset{|-|}{\underset{\longleftarrow}{\overset{\longrightarrow}{\sim}}}} Top.$$

This is to say, for any $K\in\text{sSet}$, $X\in Top$, there are natural isomorphism of sets:

$$\begin{array}{rcl} sSet(K,S(X)) & \cong & Top(|K|,X), \\ \text{by } f(k)(s) & = & f(k,s) \text{ (check well definedness)}, \end{array}$$

and the unit and counit maps are equivalences:

$$\begin{array}{rcl} K & \xrightarrow{\sim} & S(|K|); \\ |S(X)| & \xrightarrow{\sim} & X. \end{array}$$

2. NERVE OF A CATEGORY

The idea of simplicial sets is so beautiful and opens up new visions.

Notation.

A-space	is the category of T_0 Alexandroff-spaces.
Posets	is the category of partially ordered sets.
$\mathcal{OSC}$	is the category of ordered simplicial complexes.
$\mathcal{ASC}$	is the category of abstract simplicial complexes.
sSet	is the category of simplicial sets.
Top	is the category of (compactly generated weak Hausdorff) spaces.
Cat	is the category of small categories.

Road map of what we have done and what we are doing today:

$$\begin{tikzcd}
 & Top \arrow[d, shift left, "S"] \arrow[d, phantom, "④"] & \\
 & sSets \arrow[u, shift left, "T"] \arrow[d, dotted, no head] & \\
Cat \arrow[ur, "N"] & & \mathcal{OSC} \arrow[ul, "③"'] \arrow[d] \\
 & & \mathcal{ASC} \arrow[dl, "(K)"] \\
A-spaces \arrow[r, shift left] \arrow[uur, hook, "⑤"] & Posets \arrow[l, shift left, "\sim"] \arrow[uur, "②"] \arrow[l, phantom, "①"] & \Delta \arrow[l, hook']
\end{tikzcd}$$

① We have seen that the categories of A-spaces and Posets are actually isomorphic.

To go from A-spaces to Posets, give the underlying set of an A-space the ordering $x \leq y$ if $U_x \subset U_y$, where U_x is the intersection of the open sets containing x. To go from Posets to A-spaces, give the underlying set of a poset the topology with minimal basis the sets $U_x = \{w \mid w \leq x\}$.

② To go from Posets to $\mathcal{OSC}$, take the n-simplexes to be $\{v_0 < \cdots < v_n\}$. There is also a functor going from $\mathcal{OSC}$ to Posets by first forgetting to $\mathcal{ASC}$ then taking the Poset of face simplexes as objects and inclusions as order. The composites of these two functors give subdivisions of the two categories.

③ $\mathcal{OSC}$ is not good for products, among other things, so we introduce simplicial sets.

To go from $\mathcal{OSC}$ to sSet, it is basically allowing repetition if the $\mathcal{OSC}$ had come from a Poset:

$$K \mapsto K^s,\ K^s_n = \{v_0 \leq \cdots \leq v_n | \{v_0, \cdots, v_n\} \text{ (may have repetition) is a simplex in } K\}.$$

④ Next we construct maps between sSet and Top.

To go from Top to sSet, use the geometric simplex $\Delta^n_t = \{(t_0, \ldots, t_n) | t_i \geq 0, \sum t_i = 1\}$ (The subscript "t" means "topological" here.) and define $S_n(X) = \{\Delta^n_t \to X\}$ with face and degeneracy maps as in section 1.

To go from sSet to Top,

$$T(K) = \coprod_{i \geq 0} K_i \times \Delta^i_t / (k, \sigma_i t) \sim (s_i k, t), (k, \delta_i s) \sim (d_i k, s),$$

$$\text{for } k \in K_n,\ t \in \Delta^{n+1}_t \text{ and } s \in \Delta^{n-1}_t.$$

The geometric realization is not a simplicial complex, but it is a CW complex.

⑤ There is an embedding of Poset in Cat: objects are elements, there is a morphism $x \to y$ if and only if $x \leq y$.

The nerve is a functor from Cat to sSet.

We first make two observations:

(1) The map we have constructed $\Delta \to sSet$ through ② and ③ indeed is the following, standard simplicial n-simplex: $\Delta^n_s : \Delta^{op} \to Set$ such that $\Delta^n_s[m] = \Delta([m], [n])$. (The subscript "s" means "simplicial" here.)

Then we can recover $K_n = sSet(\Delta^n_s, K)$. This is the Yoneda lemma of corepresented functors.

(2) The total singular complex S is defined as $S_n(X) = Map(\Delta^n_t, X)$.

Analogously, let $\Delta_c : \Delta \to Cat$ be the categorical n-simplex. (The subscript "c" means "categorical" here.) Explicitly, Δ^n_c looks like $\bullet \to \bullet \to \cdots \to \bullet$ with $n+1$ objects. The nerve is just

$$N_n(\mathcal{C}) = Fun(\Delta^n_c, \mathcal{C}).$$

Explicitly,

$$N_n(\mathcal{C}) = \begin{cases} Obj(\mathcal{C}); & n = 0; \\ X_0 \xrightarrow{f_1} X_1 \to \ldots \xrightarrow{f_n} X_n, & n \geq 1. \end{cases}$$

and face and degeneracy maps

$$d_i([f_1|\cdots|f_n]) = \begin{cases} [f_2|\cdots|f_n] & i = 0; \\ [f_1|\cdots|f_i f_{i-1}|\cdots|f_n] & 1 \leq i \leq n-1; \\ [f_1|\cdots|f_{n-1}] & i = n. \end{cases}$$

$$s_i([f_1|\cdots|f_n]) = \begin{cases} [\mathrm{id}|f_1|\cdots|f_n] & i = 0; \\ [f_1|\cdots|f_i|\mathrm{id}|\cdots|f_n] & 1 \leq i \leq n. \end{cases}$$

Here comes the miracle:

$$Fun(\mathscr{C}, \mathscr{D}) \xrightarrow{N} sSet(N\mathscr{C}, N\mathscr{D})$$

is a bijection. Therefore N is a fully faithful embedding of Cat into sSet.

What is more, the morphisms between functors between categories - the natural tranformations - are also taken to morphisms between maps between spaces - homotopies - after geometric realization.

To make this precise, let $\mathscr{I} = \Delta^1_c$. (objects: 0,1, one non-identity morphism $I : 0 \to 1$. It realizes to the unit interval.) A "homotopy" between $F, G : \mathscr{C} \to \mathscr{D}$ is a map

$$h : \mathscr{C} \times \mathscr{I} \to \mathscr{D}$$

such that $h(x, 0) = F(x), h(x, 1) = G(x),$ for $f : x \to y, h(f, \mathrm{id}_0) = Ff, h(f, \mathrm{id}_1) = Gf.$

$$\begin{tikzcd}
h(x,0) \arrow[rr, "h(f,\mathrm{id}_0)"] \arrow[dr, equal] \arrow[dd, "h(\mathrm{id}_x,I)"'] & & h(y,0) \arrow[dr, equal] \arrow[dd, "h(\mathrm{id}_Y,I)"] & \\
& Fx \arrow[rr, "Ff"] \arrow[dd] \arrow[dr, "h(f,I)"] & & Fy \arrow[dd] \\
& Gx \arrow[rr, "Gf"] \arrow[dr, equal] & & Gy \arrow[dr, equal] \\
h(x,1) \arrow[rr, "h(f,\mathrm{id}_1)"] & & h(y,1) &
\end{tikzcd}$$

If we stare at this commutative diagram, we see that by setting the unlabeled arrows as $\eta_x = h(\mathrm{id}_x, I)$ and $\eta_y = h(\mathrm{id}_y, I)$, a homotopy between two functors is exactly a natural transformation.

Last time we said the geometric realization commutes with cartesion products. The nerve does also, as can be proved either directly or knowing that nerve is a right adjoint. Thus

$$TN(\mathscr{C} \times \mathscr{I}) \cong T(N\mathscr{C} \times N\mathscr{I}) \cong TN(\mathscr{C}) \times TN(\mathscr{I}) \cong TN(\mathscr{C}) \times I,$$

so that geometric realization indeed take a "homotopy" between functors to a homotopy between their geometric realizations. In particular, adjoint equivalent pairs of functors realize to homotopy equivalence of spaces. An easy corollary is that any category with initial object or terminal object realizes to a contractible space.

Now we have established two routes from A-space to sSet, and they coincide.

3. PULLBACKS AND PUSHOUTS, EXAMPLES

We have seen Cartesian products and disjoint unions in different categories and characterization of them by universal properties.

The universal property for the product $X \prod Y$ is that it fits in the commutative square below and any Z that makes the diagram commute will admit a unique map to $X \prod Y$ to make the whole diagram commute. In other words, $X \prod Y$ is the terminal object to fit in the commutative square.

$$\begin{tikzcd}
Z \arrow[dr, dotted] \arrow[drr, bend left] \arrow[ddr, bend right] & & \\
& X \prod Y \arrow[r] \arrow[d] & Y \arrow[d] \\
& X \arrow[r] & *
\end{tikzcd}$$

Here, * is the terminal object (for example, 0 in Ab, pt in Top).

The products in Set, Ab, Top are the Cartesian products.

Defnition 4. *The pullback of a diagram*

$$\begin{tikzcd}
& Y \arrow[d, "g"] \\
X \arrow[r, "f"] & A
\end{tikzcd}$$

$X \times_A Y$ is the terminal object to fit in the following square:

$$\begin{tikzcd}
Z \arrow[dr, dotted] \arrow[drr, bend left] \arrow[ddr, bend right] & & \\
& X \times_A Y \arrow[r] \arrow[d] & Y \arrow[d, "g"] \\
& X \arrow[r, "f"] & A
\end{tikzcd}$$

In Set, Ab, Top the pullback is explicitly

$$Z = \{(x, y) \in X \times Y | f(x) = g(y)\}.$$

Dually, the universal property for the coproduct $X \coprod Y$ is that it is the initial object to fit in the following commutative square:

$$\begin{tikzcd}
\varnothing \arrow[r] \arrow[d] & Y \arrow[d] \arrow[ddr, bend left] & \\
X \arrow[r] \arrow[drr, bend right] & X \coprod Y \arrow[dr, dotted] & \\
& & Z
\end{tikzcd}$$

Here, $\varnothing$ is the initial object (for example, 0 in Ab, $\varnothing$ in Top).

The coproduct in Set, Top is the disjoint union, in Ab it is the direct sum, and in Grp it is the free group.

Defnition 5. *The pushout of a diagram*

$$\begin{tikzcd}
A \arrow[r, "g"] \arrow[d, "f"] & Y \\
X &
\end{tikzcd}$$

$X \cup_A Y$ is the initial object to fit in the following commutative square:

$$\begin{tikzcd}
A \arrow[r, "g"] \arrow[d, "f"] & Y \arrow[d] \arrow[ddr, bend left] & \\
X \arrow[r] \arrow[drr, bend right] & X \cup_A Y \arrow[dr, dotted] & \\
& & Z
\end{tikzcd}$$

In Set, Top the pushout is explicitly

$$Z = X \sqcup Y / \big(f(a) \sim g(a), \forall a \in A\big).$$

Example. A CW complex is constructed inductively by pushouts.

Take a discrete set as X^0. Inductively glue up $(n+1)$-disks to X^n, called the n-skeleton, via their boundaries S^n by a specified attaching map f_n to get X^{n+1}. Gluing here is technically taking the following pushout:

$$\begin{tikzcd}
\coprod S^n \arrow[r, "f_n"] \arrow[d, "i"] & X^n \arrow[d] \\
\coprod D^{n+1} \arrow[r] & X^{n+1}
\end{tikzcd}$$

The union of the skeletons $X = \cup X^n$ is called a CW complex. Here, CW = closure finite (any compact set inersects only finitely many cells) + weak topology (a subset is closed iff its intersection with any cell is closed). The topology is such that a map out of a CW complex is continuous iff it is so restricted to skeletons.

Any point of a CW complex is the interior of a unique cell. This resembles the case of a geometric realization $T(K)$ for a simplicial set K, and the latter is indeed a CW complex with cells given by nondegenerate simplices (noticing that $(\Delta^n, \partial\Delta^n) \cong (D^n, S^{n-1})$):

$$\begin{tikzcd}
\coprod\limits_{\substack{k \in K_n \\ \text{nondegenerate}}} S^{n-1} \arrow[r, "\text{face maps}", "\text{glued together}"'] \arrow[d, "i"] & T(K)^{n-1} \arrow[d] \\
\coprod\limits_{\substack{k \in K_n \\ \text{nondegenerate}}} D^n \arrow[r] & T(K)^n
\end{tikzcd}$$

The notion of a CW complex is of similar nature to but generalizes the notion of a simplicial complex. The difference is that in a simplicial complex, each attaching map is homeomorphic to its image, but this does not always hold in a CW complex. The following example gives S^n a CW structure with one 0-cell and one n-cell, and the attaching map maps everything to the same point.

$$\begin{tikzcd}
S^{n-1} \arrow[r, "f"] \arrow[d, "i"] & * \arrow[d] & \\
D^n \arrow[r] & S^n \arrow[r, equal] & D^n/S^{n-1}
\end{tikzcd}$$

We mentioned in section 1 that

$$TSX \to X$$

is a weak equivalence, i.e. an isomorphism on π_n. Here is a quick proof that it is an epimorphism. As any nonzero element of $\pi_n X$ gives a nondegenerate element k in $S_n(X)$ with all face maps constant to the base point, the cell k gives $S^n \to TSX^n$ that will map to the chosen element.

Next we will see how to compute the fundamental group of a pushout.

Let $X = U \cup V$, $x \in U \cap V$, and suppose $X, U, V, U \cap V$ are all connected and based at the point x.

Then there is a commutative diagram by the functoriality of π_1:

$$\begin{tikzcd}
\pi_1(U \cap V) \arrow[r] \arrow[d] & \pi_1(U) \arrow[d] \arrow[ddr, bend left] & \\
\pi_1(V) \arrow[r] \arrow[drr, bend right] & \pi_1(U) *_{\pi_1(U\cap V)} \pi_1(V) \arrow[dr, dotted, "\cong"] & \\
& & \pi_1(X)
\end{tikzcd}$$

Here, $\pi_1(U) *_{\pi_1(U\cap V)} \pi_1(V)$ is the amalgamated free product, the name of the pushout in Grp.

It turns out that the dotted arrow is an isomorphism. In other words, the functor $\pi_1 : \mathrm{Top} \to \mathrm{Grp}$ preserves pushouts. The proof is to show that π_1 satisfies the universal property so it is indeed the pushout. This is the Van Kampen theorem.

Now we are going to illustrate the pullback. For a (small) category $\mathcal{C}$, write $\mathcal{OC}$ as its set of objects and $\mathcal{MC}$ as its set of morphisms. It has maps between these two sets specifying the identity map and the source and target of a morphism.

$$\begin{tikzcd}
\mathcal{OC} \arrow[r, "I"] & \mathcal{MC} \arrow[l, bend right, "S"'] \arrow[l, bend left, "T"]
\end{tikzcd}$$

I, S, T are defined by

$$\begin{aligned}
I(x) &= \mathrm{id}_x, \\
S(f : x \to y) &= x, \\
T(f : x \to y) &= y.
\end{aligned}$$

They satisfy

$$\begin{aligned}
T \circ I &= \mathrm{id}, \\
S \circ I &= \mathrm{id}.
\end{aligned}$$

The pullback of the diagram

$$\begin{tikzcd}
\mathcal{MC} \times_{\mathcal{OC}} \mathcal{MC} \arrow[r, "\pi_2"] \arrow[d, "\pi_1"] & \mathcal{MC} \arrow[d, "S"] \\
\mathcal{MC} \arrow[r, "T"] & \mathcal{OC}
\end{tikzcd}$$

is explicitly $\{(f, g) \in \mathcal{MC} \times \mathcal{MC} | T(f) = S(g)\}$, i.e. composable maps. So composition of morphisms is given by a map:

$$C : \mathcal{MC} \times_{\mathcal{OC}} \mathcal{MC} \to \mathcal{MC}.$$

defined by $C(f, g) = g \circ f$.

It satisfies

$$\begin{aligned}
T(g \circ f) = T(g) &\Leftrightarrow T \circ C = T \circ \pi_2, \\
S(g \circ f) = S(f) &\Leftrightarrow S \circ C = S \circ \pi_1.
\end{aligned}$$

Likewise, the unit and associativity laws of composition can both be written as diagrams.

The description above actually gives an alternative abstract definition of a small category.

The abstract definition is easy to adapt to enriched categories. Briefly speaking, for $\mathcal{V}$ a symmetric monoidal category, $\mathcal{C}$ is enriched in $\mathcal{V}$ if $\mathcal{MC}$ is a collection of objects of $\mathcal{V}$.

Next we will show that the nerve functor has a left adjoint.

Recall that the nerve of a category is defined as:

$$\begin{aligned}
N &: Cat \to sSet \\
N_n(\mathcal{C}) &= Fun(\Delta_n^c, \mathcal{C})
\end{aligned}$$

First, we check which simplicial sets come from the nerve construction.
Let $K \in \mathrm{sSet}$. We are going to construct maps:

$$\delta : K_n \to \underbrace{K_1 \times_{K_0} K_1 \times_{K_0} \cdots \times_{K_0} K_1}_{n \text{ numbers}}.$$

When $n = 0$, it is the identity map on K_0.
When $n = 1$, it is the identity map on K_1.
When $n = 2$, the existence of δ comes from using the identity $d_1 d_0 = d_0 d_2$ in the following diagram:

$$\begin{tikzcd}
K_2 \arrow[dr, dotted, "\delta"] \arrow[drr, bend left, "d_0"] \arrow[ddr, bend right, "d_2"'] & & \\
& K_1 \times_{K_0} K_1 \arrow[r, "\pi_2"] \arrow[d, "\pi_1"] & K_1 \arrow[d, "d_1"] \\
& K_1 \arrow[r, "d_0"] & K_0
\end{tikzcd}$$

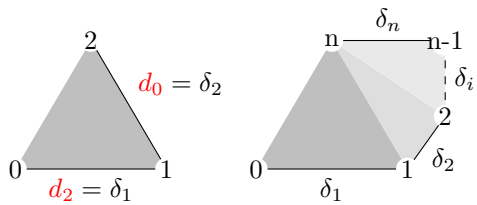

FIGURE 1. An illustration of δ_i's

For general n, define $\tilde{d}_n^{n-i} = d_{i+1} \cdots d_n, (taking \tilde{d}_n^0 = id)$ to be the iterated last face map. Take the i-coordinate map of δ as

$$\delta_i := d_0^{i-1} \tilde{d}_n^{n-i} (= \tilde{d}_{n-i+1}^{n-i} d_0^{i-1}) : K_n \to K_1, 1 \leq i \leq n.$$

The equivalent definition in the parenthesis is due to the identities $d_0 d_j = d_{j-1} d_0, 1 \leq j \leq n$.

Intuitively, it picks out the edge connecting the $(i-1)$-th and i-th vertices of the simplex. See Figure 1.

To rigorously check that these coordinates δ_i give a map δ, consider the following diagram:

$$\begin{tikzcd}
K_n \arrow[rrr, "\delta_n"] \arrow[dr, "\delta_3"] \arrow[ddr, "\delta_2"] \arrow[ddrr, "\delta_1"'] & & & K_1 \\
& & K_1 \arrow[ur, dotted] \arrow[d, "d_1"] & \\
& & K_1 \arrow[r, "d_0"] \arrow[d, "d_1"] & k_0 \\
& K_1 \arrow[r, "d_0"] & K_0 &
\end{tikzcd}$$

The iterated pullback, or the limit of the whole part involving K_0 and K_1, gives $K_1 \times_{K_0} \cdots \times_{K_0} K_1$, the target of δ. So it suffices to show the whole diagram commutes. This can be seen by making use of:

$$\begin{aligned}
\delta_i &= d_0^{i-1} \tilde{d}_n^{n-i} &&= d_0(d_0^{i-2} \tilde{d}_n^{n-i}), \\
\delta_{i-1} &= \tilde{d}_{n-i+2}^{n-i+1} d_0^{i-2} &&= d_2(\tilde{d}_{n-i+2}^{n-i} d_0^{i-2}),
\end{aligned}$$

and the commutative diagram:

$$\begin{tikzcd}
K_n \arrow[dr, "d_0^{i-2}\tilde{d}_n^{n-i}"] \arrow[ddr, bend right, "\tilde{d}_{n-i+2}^{n-i}d_0^{i-2}"'] \arrow[drr, bend left, "\delta_i"] & & \\
& K_2 \arrow[d, "\delta"] \arrow[dr, bend left, "d_0"] \arrow[dd, bend right, "d_2"'] & \\
& K_1 \times_{K_0} K_1 \arrow[r, "\pi_2"] \arrow[d, "\pi_1"] & K_1 \arrow[d, "d_1"] \\
& K_1 \arrow[r, "d_0"] \arrow[ll, bend left, "\delta_{i-1}"] & K_0
\end{tikzcd}$$

On the one hand, if $K = N(\mathcal{C})$, we would have that

$$\begin{aligned}
K_0 &= N_0(\mathcal{C}) &&= \mathcal{OC}, \\
K_1 &= N_1(\mathcal{C}) &&= \mathcal{MC}, \\
K_2 &= N_n(\mathcal{C}) &&= n \text{ composable morphisms.}
\end{aligned}$$

¿From the intuitive description Figure 1, the δ's are all isomorphisms. But for a general simplicial set, this may not be true.

On the other hand, if the δ's constructed above are all isomorphisms for K, we can define

$$\mathcal{OC} = K_0, \mathcal{MC} = K_1,$$
$$I = s_0, S = d_1, T = d_0,$$
$$C : K_1 \times_{K_0} K_1 \xrightarrow{\delta_2^{-1}} K_2 \xrightarrow{1} K_1.$$

This gives a category whose nerve is isomorphic to K.

In summary, we have:

Proposition 1. *A simplicial set K is isomorphic to $N(\mathcal{C})$ for a category $\mathcal{C}$ iff the δ's are all isomorphisms.*

Using this, we can prove that the nerve functor is a right adjoint.

Mimicking the previous construction, we define:

$$\Pi : \quad \text{sSet} \quad \to \text{Cat}$$

$$\mathcal{O}\Pi(K) = K_0,$$
$$\mathcal{M}\Pi(K) = \text{Free}(K_1)/\sim,$$
$$d_0 x d_2 x \sim d_1 x, \forall x \in K_2.$$

Here, $\text{Free}(K_1)$ is the free category generated by K_1, i.e.

$$\text{Free}(K_1) = \coprod_{n \geq 1} \underbrace{K_1 \times_{K_0} K_1 \times_{K_0} \cdots \times_{K_0} K_1}_{n \text{ numbers}}.$$

Two examples of Π are given in Figure 2, representing the effect of the free construction and the quotient.

Theorem 1.

$$\text{Cat}(\Pi K, \mathcal{C}) \cong \text{sSet}(K, N\mathcal{C}).$$

The correspondence of $F_0 : \mathcal{O}\Pi K \to \mathcal{OC}, F_1 : \mathcal{M}\Pi K \to \mathcal{MC}$ and $\tilde{F} : K \to N\mathcal{C}$ is that

$$\tilde{F}_0 = F_0, \tilde{F}_1 = F_1,$$
$$K_n \xrightarrow{\delta} K_1 \times_{K_0} \cdots \times_{K_0} K_1 \xrightarrow{F_1 \times \cdots \times F_1} N_n\mathcal{C}.$$

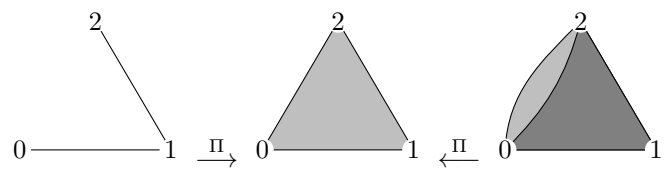

FIGURE 2. An illustration of Π

The subdivision of Poset can be extended to the subdivision of sSet. So we can define the subdivision of a category as:

$$\mathrm{Sd}(\mathcal{C}) = \Pi \mathrm{Sd} N(\mathcal{C}).$$

It is a mysterious theorem that for any category $\mathcal{C}$, $\mathrm{Sd}(\mathrm{Sd}(\mathcal{C}))$ is a Poset.

Remark. Sd is a left adjoint composed with a right adjoint, thus it is very hard to analyse in general. For example, it is not clear what is the double subdivision of BC_2 (the category with a single object and authomorphisms indexed by C_2).

4. SUBDIVISION

This section is written up in Chapter 12-13 of [4].

We have seen when a simplicial set comes from the nerve of a category. We can also consider the category of ordered simplicial complexes and ask when does a simplicial set come from a ordered simplicial complex. (The latter has a better behaved geometric realization because it's attaching maps are homeomorphic to its images.) Furthurmore, we can also ask what is the intersection of both, i.e. when does a simplicial set both come from the nerve of a category and an ordered simplicial set.

Defnition 6. *Let $K \in$ sSet. We define the following properties for K to be:*

(A) Property A, the non degenerate simplex property: If x is nondegenerate so is each $d_i x$.
(B) Property B, the distinct vertex property: The vertices of a nondegenerate simplex are distinct.
(C) Property C, the unique simplex property: Given a set of $n+1$ distinct vertices, there is at most one n-simplex whose vertices are those.

Example. Take $B\mathbb{Z}/2$, the category with one object and automorphisms indexed by $\mathbb{Z}/2$. Let $g \in \mathbb{Z}/2$ be the non-identity element. Then $\bullet \xrightarrow{g} \bullet \xrightarrow{g} \cdots \bullet$ is the unique nondegenerate n-simplex. But all of its inner faces are degenerate because $g^2 = 1$. Thus $B\mathbb{Z}/2$ does not satisfy Property A.

Lemma 1. *Property B $\Rightarrow$ Property A.*

Proof. If $d_i x$ is degenerate for $x \in K_n$, i.e. $d_i x = s_j y$ for some $y \in K_{n-2}$ and $0 \le j \le n-2$, using simplicial identities, we can show that some 1-face of x is also degenerate, i.e. equals to $s_0 y$ for some $y \in K_0$. But $d_0 s_0 y = d_1 s_0 y$, implying that x has two same vertices. □

We notice that classical ordered simplicial complexes, by definition, satisfy properties B and C and therefore also A. The converse is also true.

Proposition 2. *$K = L^s$ for some $L \in \mathcal{OSC} \Leftrightarrow K$ satisfies properties B and C.*

Proof. Recall from Section 2 that $K_n = \mathrm{sSet}(\Delta^n_s, K)$ where $\Delta^n_s([m]) = \Delta([m],[n])$.

One way to see this is that all simplexs in Δ^n_s are generated by the n-simplex, the identity map on $[n]$, and face and degeneracy maps. The point then is that every morphism of simplicial sets out of Δ^n_s, respecting face and degeneracy maps, is determined by its image of the identity map on $[n]$.

The other way is that observing the previous way is just a special case by taking $\mathcal{C} = \Delta, c = [n]$ in the following Yoneda lemma: For $\mathcal{C}$ any category, $c \in \mathcal{C}$ an object, the corepresented functor on c is:

$$Y(c) : \mathcal{C}^{op} \to \mathrm{Set}, b \mapsto \mathcal{C}(b,c).$$

Then for any contravariant functor $F : \mathcal{C}^{op} \to \mathrm{Set}$,

$$Nat(Y(c), F) \xrightarrow{\cong} F(c).$$

The correspondence is that for natural transformation $\eta : Y(c) \to F$, take $\eta_c(\mathrm{id}_c) \in F(c)$; for an element $x \in F(c)$, η_d maps $f : d \to c \in Y(c)(b)$ to $f^*(x) \in F(b)$.

In this perspective, any element of K_n as an image of $[n]$ inherits an ordering as $[n]$ is ordered. The problem is that as the simplex varies, the ordering is not consistent. Property B and C ensure that this is a total order.

□

We had a notion of subdivision of ordered simplicial complexes. If $|K| \xrightarrow{f} |L|$ is a map, it is very rare that f comes from a simplicial map, i.e. is homotopic to the geometric realization of a simplicial map. But there exists a large enough n such that the composite $|\mathrm{Sd}^n K| \cong |K| \xrightarrow{f} |L|$ comes from a simplicial map.

Subdivision is really important notion. The subdivision of $\mathcal{OSC}$ is very close to the subdivision of Poset, where otherwise there are only finitely many maps between any two objects. It can be generalized to sSet as well. And there is an amazing fact:

Theorem 2. *For a simplicial set K,*

(1) K has property A $\Leftrightarrow$ $\mathrm{Sd}K$ has property A;
(2) K has property A $\Leftrightarrow$ $\mathrm{Sd}K$ has property B;
(3) K has property B $\Leftrightarrow$ $\mathrm{Sd}K$ has property C.

Consequently, if K satisfy A, then $\mathrm{Sd}K$ satisfy B, so that Sd^2K satisfies C.

The definition of subdivision of sSet comes in two flavors: conceptual flavor and combinatorial flavor.

The **conceptual flavor** starts with the algebra construction of tensor product, and continues in similar way as geometric realizations.

Let R be a commutative ring. The tensor product $M \otimes_R N$ of R-modules M, N is the R-module with a bilinear map $M \times N \to M \otimes_R N$ such that any bilinear map out of $M \times N$ factors uniquely through a linear map out of $M \otimes_R N$.

It is constructed as $F(M) \times F(N)/\sim$, where $F(M), F(N)$ is the free R-module on elements of M, N respectively, and the equivalence relation is $(m_1 + m_2, n) \sim (m_1, n) + (m_2, n), (m, n_1 + n_2) \sim (m, n_1) + (m, n_2), (mr, n) \sim (m, rn)$.

We can also form the tensor product of a contravariant and a covariant functor just to blow your mind. Let us start by translating the tensor product of modules in this new context.

In any category, the coequalizer of two maps $f, g : X \to Y$, $Coeq(f, g)$ is the initial object in the following diagram receiving a map h such that $h \circ f = h \circ g$.

$$\begin{tikzcd} X \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & Y \arrow[r, "h"] \arrow[d] & Coeq(f,g) \arrow[dl, dotted] \\ & Z & \end{tikzcd}$$

If a category has finite coproducts, then it has all pushouts if and only if it has all coequalizers. The dual definition gives equalizers and the relation of pullbacks and equalizers.

An alternative definition of the tensor product of a right R-mod M and a left R-mod N is the equalizer:

$$M \otimes R \otimes N \underset{(R \otimes N \to N)}{\overset{(M \otimes R \to M)}{\rightrightarrows}} M \otimes N \longrightarrow M \otimes_R N$$

Exercise: compute $\mathbb{Z}/n \otimes \mathbb{Z}/m$. (When R is omitted, it is understood to be $\mathbb{Z}$.)

To generalize, we introduce more category concepts. Let $\mathcal{V}$ be a symmetric monoidal category. A category $\mathcal{C}$ is enriched in $\mathcal{V}$ means that for any $c, d \in \mathcal{OC}$, we have a morphism object $\mathcal{C}(c, d) \in \mathcal{OV}$ and composition is given by a morphism in $\mathcal{V}$: $\mathcal{C}(d, e) \otimes \mathcal{C}(c, d) \to \mathcal{C}(c, e)$. Assume more $\mathcal{V}$ is closed symmetric monoidal, meaning that it has internal hom objects $\mathcal{V}(y, z) \in \mathcal{OV}, \forall y, z \in \mathcal{OV}$ and the

following is true:

$$\mathcal{V}(x \otimes y, z) = \mathcal{V}(x, \mathcal{V}(y, z)), \forall x, y, z \in \mathcal{OV}.$$

$\mathcal{V}$ is closed symmetric monoidal is equivalent to that $\mathcal{V}$ is enriched in itself.

Example. Consider the monoidal category Ab of abelian groups with tensor product. It is closed symmetric monoidal as one can add/subtract morphisms so that the morphisms also form an abelian group. A category $\mathcal{R}$ of one object enriched in Ab, also known as a monoid in Ab, is a ring R, as we see by looking at the endomorphisms of the object.

Defnition 7. *Given enriched functors $F : \mathcal{C}^{op} \to \mathcal{V}$ and $G : \mathcal{C} \to \mathcal{V}$. Here, enriched means that $F : \mathcal{C}(c, d) \to \mathcal{V}(F(d), F(c))$ is a morphism in $\mathcal{V}$ and similar for G. Then the tensor product of the functors F and G is the equalizer of the following diagram:*

$$\coprod_{c,d \in \mathcal{OC}} F(d) \otimes \mathcal{C}(c, d) \otimes G(c) \underset{\text{covariance of } G}{\overset{\text{contravariance of } F}{\rightrightarrows}} \coprod_{e \in \mathcal{OC}} F(e) \otimes G(e) \longrightarrow F \otimes_{\mathcal{C}} G.$$

Example. Let $\mathcal{V} = \text{Ab}$, $\mathcal{C} = \mathcal{R}$ as described before. A covariant functor out of $\mathcal{R}$ to Ab is a left R-mod, while a contravariant functor out of $\mathcal{R}$ to Ab is a right R-mod. The tensor product of the two functors is exactly the tensor product of the two modules in the classical sense.

Example. Let $\mathcal{V} = \text{Spaces}$, $\mathcal{C} = \Delta$ and G be a topological group. Think of G as a topologically enriched category with one object and G as the space of morphisms from that one object to itself.

Take the contravariant functor to be the nerve $N_*(G) : \Delta^{op} \to \text{Spaces}$. Then $N_n(G)$ is the space G^n. Take the covariant functor to be $\Delta_t^\bullet : \Delta \to \text{Spaces}$.

The tensor product of these two functors, $N_*(G) \otimes_\Delta \Delta_t^*$, is the geometric realization of $N_*(G)$, and it gives a model for the classifying space BG.

For general $K \in \text{sSet}$, the tensor product gives the geometric realization:

$$\coprod_n K_n \times \Delta_t^n / \sim \quad =: K \otimes_\Delta \Delta_t^\bullet \quad = |K| = T(K).$$

Example. Let $\mathcal{V} = \text{sSet}$, $\mathcal{C} = \Delta$. We can embed Set to sSet by taking all face and degeneracy maps to be the identity. Take the covariant functor to be the simplicial standard simplexes: $\Delta_s^\bullet : \Delta \to \text{sSet}$. Take the contravariant functor a simplicial set viewed as a simplicial simplicial set $K : \Delta^{op} \to \text{Set} \to \text{sSet}$. Then

$$\coprod_n K_n \times \Delta_s^n / \sim \quad =: K \otimes_\Delta \Delta_s^\bullet \quad \cong K.$$

Now, we already know what is the barycentric subdivision $\text{Sd}K$ of $K \in \mathcal{OSC}$: it has n-simplices with vertices $\sigma_0 \subset \cdots \subset \sigma_n$ where σ_i is a i-simplex of K. Since Δ_s^n is a simplicial set that comes from a ordered simplicial complex, we can define the subdivision of Δ_s^n using that of $\mathcal{OSC}$. Furthermore, we can define:

Defnition 8. *For $K \in \text{sSet}$, the subdivision of K is:*

$$\text{Sd}(K) \quad := K \otimes_\Delta \text{Sd}(\Delta_s^\bullet) \quad = \coprod_n K_n \times \text{Sd}(\Delta_s^n) / \sim .$$

For $K \in \mathcal{OSC}$, we can define a map of simplicial sets $\text{Sd}K \xrightarrow{\xi} K$, for example, by mapping each σ_i to its largest vertex. But this map does not realize to a homeomorphism. However, there does exist a homeomorphism $|\text{Sd}(K)| \to |K|$.

Next, we will give the **combinatorial flavor** of the subdivision of simplicial sets, which is good for proving theorems. (Every proof in [4] in this part is using the combinatorial description and proof by contradiction.)

Defnition 9. *(Definition 12.2.1 in* [4]*.) For $K \in \text{sSet}$, the q-level of the subdivision of* $\text{Sd}K$ *is*

$$(\text{Sd}K)_q = \{(x; S_0, \cdots, S_q) | n \geq 0, x \in K_n, S_i \text{ subsets of } [n] \text{ and } S_0 \subseteq \cdots \subseteq S_q\} / \sim,$$

where for $\mu : [m] \to [n]$, $(\mu^*; S_0 \subseteq \cdots \subseteq S_q \subseteq [m]) \sim (x, \mu(S_0) \subseteq \cdots \subseteq \mu(S_q) \subseteq [n])$.

Just like the case of geometric realization, every element in $\mathrm{Sd}K$ is uniquely identified with a pair of a nondegenerate simplex and an "interior point" of a $\mathrm{Sd}\Delta^s_n$. Explicitly, it can written in minimal form $(x; S_0, \cdots, S_q)$ where $S_q = [n]$ and $x \in K_n$ nondegenerate. In the minimal form, an element is denegerate iff $S_i = S_{i+1]}$ for some i.

Using this explicit definition, one can prove Theorem 2, see [4]. This theorem is strange. There is nothing like this in Poset. However, we are not giving the proofs here. Instead, we see how to define subdivision and characterize Properties A,B and C in Cat.

Similarly, we can define subdivision using the nerve.

Defnition 10. *For $\mathcal{C} \in \mathrm{Cat}$,*

$$\mathrm{Sd}^c(\mathcal{C}) = \Pi \mathrm{Sd}^s N\mathcal{C}.$$

(Here, the decoration "c" means "categorical" and "s" means "simplicial". They will be omitted and understood as in the appropriate category.)

Explicitly,

Defnition 11.

$$\mathcal{O}(\mathrm{Sd}(\mathcal{C})) = \{(a_0 \xrightarrow{f_1} \cdots \xrightarrow{f_n} a_n)\}/\sim,$$

where $\sim$ is similar as before and omitted here. We write (f, n) for short.

In this context the minimal form is (f, n) with no identity in the f_i's.

Proposition 3.

$N\mathcal{C}$ has Property A $\Leftrightarrow$ *$\mathcal{C}$ has no retracts $a \xrightarrow{i} b \xrightarrow{r} a$, i.e.$\{r \circ i = \mathrm{id} \Rightarrow a = b, i = r = id\}$.*

$N\mathcal{C}$ has Property B $\Leftrightarrow$ *$\mathcal{C}$ has no loops, i.e.$\{a \xrightarrow{f} b, b \xrightarrow{g} a \Rightarrow a = b, f = g = \mathrm{id}\}$.*

$N\mathcal{C}$ has Property C $\Leftrightarrow$ *$\mathcal{C}$ has one way property, i.e. for any ordered set of objects $C_0, \cdots, C_n$, there is at most one sequence of non-identity morphisms connecting them.*

$\Leftrightarrow$ *For any distinct pairs of objects C_0, C_1, there is at most one morphism $C_0 \to C_1$.*

Defnition 12. *We say $\mathcal{C} \in \mathrm{Cat}$ has Property A,B or C if $N\mathcal{C}$ has.*

The counterpart of Theorem 2 in Cat is:

Theorem 3. *For $\mathcal{C} \in \mathrm{Cat}$, the following are always true:*

(1) $\mathrm{Sd}\mathcal{C}$ has Property B;
(2) $\mathcal{C}$ has Property B $\Leftrightarrow$ $\mathrm{Sd}\mathcal{C}$ is a poset.

The above theorem implies immediately that $\mathrm{Sd}^2\mathcal{C}$ is always a poset.

The notion of subdivision is used in a beautiful theorem: there is a homotopy theory of categories which is equivalent to the homotopy theory of spaces, also equivalent to the homotopy theory of posets. The first part is proved by Thomason [**?**] and it uses the second subdivision of sSet in a fundamental way while, strangely, subdivision of Cat plays no role in it. The second part is implicit in Thomason and made explicit by Raptis [**?**].

This theorem also has an equivariant version for finite group. This first part is proved by six authors Bohmann-Mazur-Osorno-Ozornova-Ponto-Yarnall [**?**]; the second part is proved by May-Stephan-Zakharevich [**?**].

5. ON AND BEYOND

We have seen what is a simplicial set. In sSet there are some special ones called horns $\Lambda^q_i \subset \Delta^q_s$, for $0 \le i \le q$. It is Δ^q_s with its non-degenerate q-simplex and its i-th face removed and looks just as its name suggests. If $0 < i < q$, Λ^q_i is an inner horn; otherwise it is an outer horn.

The horns are used to formulate the following problem: Given all the faces of a q-simplex other than the i-th one, is there a q-simplex that realizes these faces? It is the same as asking whether there exists the dotted arrow in the diagram:

$$\begin{tikzcd} \Lambda_i^q \arrow[r] \arrow[d] & K \\ \Delta_s^q \arrow[ur, dotted] & \end{tikzcd}$$

Let K come from the nerve of a category. Given d_0, d_2, there is always a filler $d_1 = d_0 d_2$ and it is unique. Given d_1, d_2, there may not be a filling. However, if the cateogory is a groupoid, meaning every morphism is invertible, then there is a unieque filler $d_0 = d_1 d_2^{-1}$.

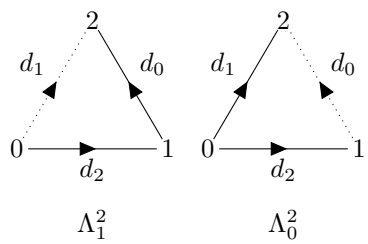

FIGURE 3. Λ_1^2 and Λ_0^2

Defnition 13. *$K \in$ sSet is a Kan complex if every horn has a filler.*

Example. The total singular complex SX is always a Kan complex. This is because the whole q-simplex deformation retracts to anyhorn. Using the precomposition we construct the horn filler.

Example. We have also the notion of a simplicial group (i.e. a functor $\Delta^{op} \to$ Grp with all face and degeneracy maps group homomorphisms). Every simplicial group is a Kan complex. For a proof see " simplicial objects in algebraic topology" by May or " simplicial homotopy theory" by Goerss and Jardine.

Our goal is to use sSet to do topology. First, we define homotopy groups of simplicial sets. We may simply define them as the homotopy groups of the geometric realizations, but for Kan complexes, we may also define homotopy groups combinatorially, which we now explain.

For $x, y \in K_n$ with $d_i x = d_i y, \forall i$, a homotopy $h : x \to y$ is an element $h \in K_{n+1}$ such that $d_n h = x, d_{n+1} h = y, d_i h = s_{n-1} d_i x = s_{n-1} d_i y, i < n$.

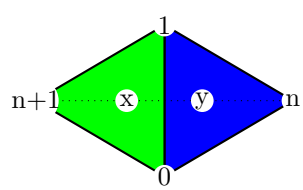

FIGURE 4. homotopy

Homotopy is an equivalence relation in Kan complexes, but not necessarily in general simplicial sets. Idea of proof:

- Reflexive: $d_n s_n x = x = d_{n+1} s_n x$.
- Symmetric: symmetry in figure 4.
- Transitive (special to Kan complexes): Proof by picture, see figure 5.

Now we suppose K is a pointed Kan complex with base point $\varnothing \in K_0$ and all images of its degeneracies.

Defnition 14. *The n-th homotopy group of a Kan complex K is:*

$$\Pi_n(K, \varnothing) = \{x \in K_n | d_i x = \varnothing, \forall i\} / \sim .$$

Here, $\sim$ is homotopy.

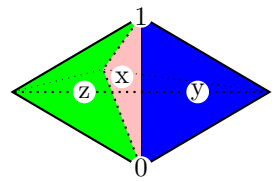

FIGURE 5. transitivity

It can be shown that Π_n is a group and $\Pi_n(K, \varnothing) \cong \Pi_n(|K, \varnothing|)$. Take $K = SX$ as an example and the identification is clear.

Second, we define mapping spaces.

Defnition 15. *For $K, L \in$ sSet, the mapping simplicial set is defined as:*

$$Map(K, L)_q = \mathrm{sSet}(K \times \Delta_s^q, L).$$

Remember as q varies Δ_s^q gives a covariant functor $\Delta \to$ sSet. So $Map(K, L)$ is a contravariant functor $\Delta^{op} \to$ Set, i.e. a simplicial set. This shows that sSet is self enriched.

If L is a Kan complex, $Map(K, L)$ is also a Kan complex. This is similar to Milnor's theorem that if X is compact, Y is a CW complex, then $Map(X, Y)$ is a CW complex.

The 0-simplices of $Map(K, L)$ is $\mathrm{sSet}(K, L)$. We have an equivalence relation of homotopy here because $Map(K, L)$ is a Kan complex. We can therefore define $\Pi_0 Map(K, L) =: [K, L]$.

Let $f, g : K \to L$. We can define homotopy $f \sim g$ more directly. A homotopy from f to g is given by maps $h_i : K_q \to L_{q+1}$ such that $d_0 h = f, d_{q+1} h = g$ with some more commutation relations. The tedious relations can be recovered by deciphering the conceptual idea that indeed h is a map $K \times I \to L$ for $I = \Delta_s^1$. Here, homotopy realizes to chain homotopy. Thus, homotopic maps of simplicial sets induce the same map on homology. This is a proof that the singular homology is invariant under homotopy equivalence.

In section 3 we saw that a simplicial set K is isomorphic to the nerve of a category $\mathcal{C}$ iff the Segal maps $\delta : K_q \to K_1 \times_{K_0} \cdots \times_{K_0} K_1$ are all isomorphisms.

We are going to give some other characterization criteria.

Theorem 4. $K \in$ sSet. *TFAE:*

(1) $K \cong N\mathcal{C}$.
(2) Every inner *horn has a* unique *filler.*
(3) For $n \geq 2$ and any n-tuple $x_i \in K_1, 1 \leq i \leq n$ such that $d_0 x_{i-1} = d_1 x_i, 2 \leq i \leq n$, there exists a unique $y \in K_n$ such that $\delta_i^* y = x_i$. (δ_i is defined in section 3, and this is a restatement of Segal maps being isomorphisms.)

$\mathcal{C}$ is a groupoid in (1) iff every *horn has a* unique *filler in (2).*

A quasicategory is a simplicial set such that every inner horn has a filler. To visualize this, we can view 0-simplices as objects, 1-simplices as morphisms of a "category", but composition is not unique here as there may be different horn fillers. These different compositions are "homotopic" by a (non-unique) higher simplex, and higher and higher. Quasicategories are also called "$(\infty, 1)$ categories". There are thousands of pages on this by Lurie.

REFERENCES

[1] https://mathoverflow.net/questions/47271/from-chain-complex-to-simplicial-abelian-group
[2] https://math.mit.edu/~ebelmont/ssets.pdf
[3] https://ocw.mit.edu/courses/mathematics/18-904-seminar-in-topology-spring-2011/final-paper/MIT18_904S11_finlSmplicial.pdf
[4] http://math.uchicago.edu/~may/FINITE/FINITEBOOK/FINITEBOOKCollatedDraft.pdf