TENSOR PRODUCT OF FUNCTORS

SAAD SLAOUI

We set out to develop a framework in which to naturally express the notion of the barycentric division of a simplicial set, based on our knowledge of that of the standard simplicial n-simplex. The right concept turns out to be a categorical generalization of the tensor product of R-modules over a given ring R. After building up the categorical language necessary to fit this motivating example into a more general context, we come back to earth and provide a concise conceptual formulation of the geometric realization and barycentric subdivision of a simplicial set.

Let $\mathscr{A}b$ denote the category of abelian groups and group homomorphisms. Given two abelian groups A and B, recall that one may define the *tensor product* of A and B, denoted by $A \otimes B$, via the following universal property:

$$\begin{tikzcd} A \times B \arrow[r, "\forall \varphi", "\mathbb{Z}\text{-bilinear}"'] \arrow[d] & C \\ A \otimes B. \arrow[ur, dashed, "\exists ! \overline{\varphi}\ \mathbb{Z}-\text{linear}"'] & \end{tikzcd}$$

That is, the tensor product of A and B is a pair $(A \otimes B, A \times B \to A \otimes B)$ where $A \otimes B$ is an abelian group and $A \times B \to A \otimes B$ is a $\mathbb{Z}$-bilinear map such that every $\mathbb{Z}$-bilinear map out of $A \times B$ factors uniquely as a $\mathbb{Z}$-linear map through $A \otimes B$ under the provided map. This defines $A \otimes B$ uniquely up to isomorphism. Concretely, $A \otimes B$ may be constructed by taking the free abelian group $F(A \times B)$ with the cartesian product of A and B as generating set, and taking its quotient by the subgroup consisting of all elements of the form $(a + a', b) - (a, b) - (a', b)$, $(a, b + b') - (a, b) - (a, b')$ for $a, a' \in A$ and $b, b' \in B$. Then, the universal map $A \times B \to A \otimes B$ is simply the composite of the canonical inclusion $A \times B \to F(A \times B)$ with the quotient map $F(A \times B) \to A \otimes B$. Denoting the image of $(a, b) \in A \times B$ in $A \otimes B$ by $a \otimes b$, it follows that elements of $A \otimes B$ are finite linear combinations of the form $\Sigma_i a_i \otimes b_i$ (after absorbing the integral scalar multiples into either factor).

More generally, given a (not necessarily commutative) ring R, a right R-module M and a left R-module N, one may mirror the above construction and define the *tensor product* of M and N to be a pair $(M \otimes_R N, f : M \times N \to M \otimes_R N)$ where $M \otimes_R N$ is an abelian group and $f : M \times N \to M \otimes_R N$ is an *R-bilinear map* (in the sense that $f(mr, n) - f(m, rn) = 0$ whenever $r \in R, m \in M, n \in N$) with the property that every R-bilinear map out of $M \times N$ factors uniquely through it as an R-linear map. Concretely, $M \otimes_R N$ can be obtained as a *quotient* of the tensor product $M \otimes N$ in $\mathscr{A}b$ by the subgroup generated by elements of the form $(mr, n) - (m, rn)$ with $r \in R, m \in M, n \in N$.

We describe a categorical construction of which the above connection between $M \otimes N$ and $M \otimes_R N$ is an instance. Let $\mathscr{C}$ be any category, and let $A \underset{g}{\overset{f}{\rightrightarrows}} B$ be a diagram consisting of two objects and two morphisms between them in $\mathscr{C}$. Define the *coequalizer* of this diagram to be the pair $(\mathrm{coeq}(f,g), h : B \to \mathrm{coeq}(f,g))$ in $\mathscr{C}$ making the resulting diagram

$$A \underset{g}{\overset{f}{\rightrightarrows}} B \xrightarrow{h} \mathrm{coeq}(f,g)$$

commute, and which is *initial* with respect to this property, meaning that for any other pair $(Z, k : B \to Z)$ making the same diagram commute there exists a unique morphism $\eta : \mathrm{coeq}(f,g) \to Z$ such that $\eta h = k$. When working in $\mathscr{A}b$, we may use the fact that hom sets have an abelian group structure to show that $\mathrm{coeq}(f,g)$ $= \mathrm{coker}(f-g)$ is just the quotient of B by the image of the difference of the two morphisms between A and B.

Now, observe that the right R-module structure (resp. left R-module structure) of M (resp. N) is captured by the existence of an R-linear map $\alpha : M \otimes R \to M$ (resp. $\beta : R \otimes N \to N$). We may then form the coequalizer:

$$M \otimes R \otimes N \underset{id_M \otimes \beta}{\overset{\alpha \otimes id_N}{\rightrightarrows}} M \otimes N \longrightarrow \mathrm{coeq}(\alpha \otimes id_N, id_M \otimes \beta). \tag{1}$$

By the above discussion, we see that this coequalizer pair corresponds precisely to the pair $(M \otimes_R N, M \otimes N \to M \otimes_R N)$ introduced earlier.

Next, we set out to reinterpret situation (1) in the language of functors. Recall that the data of a group is equivalent to that of a category with one object in which every morphism is an isomorphism. We seek an analogous interpretation of the notion of a ring R, as some category $\mathscr{R}$ with one object whose hom set has a ring structure, so that we may speak of R-modules as functors from the category $\mathscr{R}$ to $\mathscr{A}b$ (by exploiting the alternative definition of an R-module as an abelian group M equipped with a ring homomorphism $R \to \mathrm{End}_{\mathscr{A}b}(M) = \mathscr{A}b(M,M)$).

To do so, we introduce in broad strokes the notion of a *symmetric monoidal category* as a natural generalization of the internal "product" structure enjoyed by $\mathscr{A}b$ under the tensor product. Say a category $\mathscr{D}$ is a *symmetric monoidal category* if it comes equipped with a *multiplication* bifunctor $\otimes : \mathscr{D} \times \mathscr{D} \to \mathscr{D}$ and a designated *unit object* $1 \in ob\mathscr{D}$, such that the resulting product structure on $ob\mathscr{D}$ is associative, commutative and unital up to natural isomorphism. In particular, for every object x of $\mathscr{D}$, we are provided natural isomorphisms $x \otimes 1 \xrightarrow{\sim} x \xleftarrow{\sim} 1 \otimes x$. We further require the multiplication bifunctor to satisfy some coherence conditions in the form of commutative diagrams which we omit from our discussion. For more details, see for instance section E.2 of Riehl [1].

In general, we say that a category $\mathscr{A}$ is *enriched* over a symmetric monoidal category $\mathscr{D}$ if the hom sets of $\mathscr{A}$ are objects of $\mathscr{D}$, the identity morphisms in $\mathscr{A}$ are indicated by $\mathscr{D}$-morphisms $1 \to \mathscr{A}(x,x)$ for each object x in $\mathscr{A}$, and the maps definining the composition law in $\mathscr{A}$ are morphisms in $\mathscr{D}$ of the form:

$$\mathscr{A}(y,z) \otimes \mathscr{A}(x,y) \to \mathscr{A}(x,z).$$

Further, we say that a functor $F : \mathscr{A} \to \mathscr{C}$ between two categories enriched over $\mathscr{D}$ is *$\mathscr{D}$-enriched* if the induced maps on hom sets under F

$$\mathscr{A}(x, y) \to \mathscr{C}(F(x), F(y))$$

are morphisms in $\mathscr{D}$ for all objects x, y of $\mathscr{A}$. In our context, we shall always require $\mathscr{D}$ to be *closed*, in the sense that $\mathscr{D}$ is enriched over itself in such a way that the hom-tensor adjunction

$$\mathscr{D}(x \otimes y, z) \cong \mathscr{D}(x, \mathscr{D}(y, z))$$

holds in $\mathscr{D}$.

In the case that a closed symmetric monoidal category $\mathscr{D}$ also happens to have all small products, small coproducts, equalizers and coequalizers (implying that it is what is referred to as a *bicomplete* category), we call $\mathscr{D}$ a *cosmos*. By equalizer, we mean the dual notion of coequalizer obtained by formally reversing all arrows. As our construction of the tensor product of functors requires the underlying symmetric monoidal category to be closed and to have coproducts and coequalizers, we shall assume in our construction that we are dealing with a cosmos.

Finally, we may define a *monoid object* in any symmetric monoidal category $\mathscr{D}$ to be an object M of $\mathscr{D}$ equipped with designated "multiplication" and "identity" morphisms $\mu : M \otimes M \to M$ and $\eta : 1 \to M$ fitting into two commutative diagrams expressing that these morphisms give M an associative, unital multiplicative structure:

(2)

$$\begin{tikzcd}
(M \otimes M) \otimes M \arrow[r, "\sim"] \arrow[d, "\mu\otimes id"'] & M \otimes (M \otimes M) \arrow[r, "id\otimes\mu"] & M \otimes M, \arrow[d, "\mu"] \\
M \otimes M \arrow[rr, "\mu"'] & & M
\end{tikzcd}
\qquad
\begin{tikzcd}
M \otimes 1 \arrow[r, "id\otimes\eta"] \arrow[dr, "\sim"'] & M \otimes M \arrow[d, "\mu"] & 1 \otimes M. \arrow[l, "\eta\otimes id"'] \arrow[dl, "\sim"] \\
& M &
\end{tikzcd}$$

In the above, the indicated isomorphisms are governed by the underlying symmetric monoidal structure of $\mathscr{D}$.

Coming back to our motivating example, observe that we may view $\mathscr{A}b$ as a cosmos with monoidal structure given by the tensor product and unit object given by $\mathbb{Z}$. The associative, commutative and unital properties of the tensor product are immediate consequences of its definition by a universal property; $\mathscr{A}b$ is enriched over itself by giving hom sets an abelian group structure under pointwise addition in such a way that the hom-tensor adjunction is satisfied, and it can be verified to be bicomplete. Then, a monoid object in $\mathscr{A}b$ consists of an abelian group R together with two group homomorphisms $\mu : R \otimes R \to R$, $\eta : \mathbb{Z} \to R$ satisfying the commutative diagrams in (2). Notice that the data of a group homomorphism out of $\mathbb{Z}$ consists precisely in its behavior on $1 \in \mathbb{Z}$, so that $\eta : \mathbb{Z} \to R$ can be understood as picking out an element $\eta(1)$ in R. Finally, since μ is a group homomorphism, the additive and multiplicative structure of R combine into the structure of a (not necessarily commutative) ring. Conversely, any ring can be viewed as an abelian group equipped with such a multiplication and identity morphism in $\mathscr{A}$. Hence rings are precisely monoid objects in the cosmos $\mathscr{A}b$. There is in fact yet another way to speak of rings:

Construction 0.3. Let R be a monoid object in $\mathscr{A}b$. Define a category $\mathscr{R}$ to have only one object $*$ and to have as unique morphism set $\mathscr{R}(*,*) = R$ the underlying set of R, with composition of morphisms given by $\mu : R \otimes R \to R$ and identity morphism given by $\eta(1) \in R$. Commutativity of the diagrams making R a monoid object precisely express the fact that the composition law in $\mathscr{R}$ is associative and unital, hence that $\mathscr{R}$ is a valid category. Further observe that the abelian group structure of $R \in \mathscr{A}b$ allows us to naturally view $\mathscr{R}$ as a category enriched over $\mathscr{A}b$. From this viewpoint, $\mathscr{R}(*,*)$ appears as an abelian group equipped with a second associative, unital binary operation under composition. To say that the category $\mathscr{R}$ is enriched over $\mathscr{A}b$ is to say that composition in $\mathscr{R}$ fits into a group homomorphism

$$\mathscr{R}(*,*) \otimes \mathscr{R}(*,*) \to \mathscr{R}(*,*),$$

which precisely enforces distributivity laws between the two operations. Hence we retrieve the original ring R in the structure of the unique morphism set of $\mathscr{R}$.

Then, a covariant $\mathscr{A}b$-enriched functor $F : \mathscr{R} \to \mathscr{A}b$ consists precisely in a choice of abelian group N together with a functorial assignment

$$\mathscr{R}(*,*) \to \mathscr{A}b(N,N)$$

which is also a group homomorphism. Together, these properties mean that F encodes precisely the data of a left R-module structure on N. Also observe that the hom-tensor adjunction provides us with an adjoint group homomorphism to the above:

$$\mathscr{R}(*,*) \otimes N \xrightarrow{\beta} N,$$

reminiscent of the alternative viewpoint on R-modules.

Dually, a contravariant $\mathscr{A}b$-enriched functor $G : \mathscr{R}^{op} \to \mathscr{A}b$ consists in a choice of abelian group M together with a *right* R-module structure encoded by the ring homomorphism

$$\mathscr{R}^{op}(*,*) \to \mathscr{A}b(M,M),$$

corresponding by adjunction to a map

$$M \otimes \mathscr{R}(*,*) \xrightarrow{\alpha} M$$

by symmetry of the tensor product.

Taking the tensor product $M \otimes_R N$ then coincides precisely with taking the coequalizer of the diagram

$$M \otimes \mathscr{R}(*,*) \otimes N \underset{id_M \otimes \beta}{\overset{\alpha \otimes id_N}{\rightrightarrows}} M \otimes N,$$

where $M \otimes N$ denotes the tensor product of M and N taken in $\mathscr{A}b$.

This leads us to formulating the general pattern in which this construction naturally fits, the main difference being that the domain category under consideration may have more than one hom set.

Construction 0.4. Let $\mathscr{D}$ be a cosmos, let $\mathscr{C}$ be any small category enriched over $\mathscr{D}$, and suppose we are given a contravariant $\mathscr{D}$-enriched functor $F : \mathscr{C}^{op} \to \mathscr{D}$ and a covariant $\mathscr{D}$-enriched functor $G : \mathscr{C} \to \mathscr{D}$. Then for each pair of objects $(x, y) \in ob\mathscr{C}^2$, F and G induce $\mathscr{D}$-morphisms at the level of hom objects:

$$\mathscr{C}(x,y) \longrightarrow \mathscr{D}(F(y), F(x)), \qquad \mathscr{C}(x,y) \longrightarrow \mathscr{D}(G(x), G(y))$$

Since $\mathscr{D}$ is closed and by symmetry of $\otimes$, these maps correspond to adjoint morphisms:

$$F(y) \otimes \mathscr{C}(x,y) \xrightarrow{\tilde{\alpha}} F(x), \qquad \mathscr{C}(x,y) \otimes G(x) \xrightarrow{\tilde{\beta}} G(y)$$

Now, recall that the coproduct $\coprod_{z \in ob\mathscr{C}} F(z) \otimes G(z)$ comes equipped with canonical inclusions $l_x : F(x) \otimes G(x) \to \coprod_{z \in ob\mathscr{C}} F(z) \otimes G(z)$ for each $x \in ob\mathscr{C}$. Hence for each pair $(x, y) \in ob\mathscr{C}^2$, we obtain a diagram in $\mathscr{D}$:

(5)
$$\begin{tikzcd}
 & F(x) \otimes G(x) \arrow[dr, "l_x"] & \\
F(y) \otimes \mathscr{C}(x,y) \otimes G(x) \arrow[ur, "\tilde{\alpha}\otimes id"] \arrow[dr, "id\otimes\tilde{\beta}"'] \arrow[rr, shift left] \arrow[rr, shift right] & & \coprod_{z \in ob\mathscr{C}} F(z) \otimes G(z) \\
 & F(y) \otimes G(y) \arrow[ur, "l_y"'] &
\end{tikzcd}$$

We may further consider the coproduct $\coprod_{(x,y) \in ob\mathscr{C}^2} F(y) \otimes \mathscr{C}(x,y) \otimes G(x)$. By its defining universal property, maps

$$\coprod_{(x,y) \in ob\mathscr{C}^2} F(y) \otimes \mathscr{C}(x,y) \otimes G(x) \to z$$

are in one-to-one correspondence with families of maps

$$F(y) \otimes \mathscr{C}(x,y) \otimes G(x) \to z,$$

one for each pair $(x, y) \in ob\mathscr{C}^2$.

Thus the two families of maps $F(y) \otimes \mathscr{C}(x,y) \otimes G(x) \rightrightarrows \coprod_{z \in ob\mathscr{C}} F(z) \otimes G(z)$ obtained by the respective horizontal composites in diagram (5) above when ranging over all pairs of objects in $\mathscr{C}$ allow us to form the following diagram in $\mathscr{D}$:

$$\coprod_{(x,y) \in ob\mathscr{C}^2} F(y) \otimes \mathscr{C}(x,y) \otimes G(x) \underset{id\otimes\beta}{\overset{\alpha\otimes id}{\rightrightarrows}} \coprod_{z \in ob\mathscr{C}} F(z) \otimes G(z).$$

Define the *tensor product* of the functors $F : \mathscr{C}^{op} \to \mathscr{D}$ and $G : \mathscr{C} \to \mathscr{D}$ over $\mathscr{C}$ to be the coequalizer of this diagram.

We are now in a position to give conceptual formulations of the notions of geometric realization and barycentric subdivision of a simplicial set. Let K be a given simplicial set, interpreted as a contravariant functor $K : \Delta^{op} \to \mathscr{S}et$ from the category of finite ordered sets and order preserving maps to sets. Notice that we may view K as a functor $K^t : \Delta^{op} \to \mathscr{U}$ into the category of spaces by giving each K_n the discrete topology. Next, let $\Delta^t_* : \Delta \to \mathscr{U}$ denote the covariant functor sending $[n]$ to the standard topological n-simplex, and whose action on face maps, resp. degeneracy maps is given by the usual face inclusion $(t_0, ..., t_n) \mapsto (t_0, ..., t_{i-1}, 0, t_{i+1}, ..., t_n)$, resp. topological degeneracy map $(t_0, ..., t_n) \mapsto (t_0, ..., t_{i-2}, t_i + t_{i+1}, t_{i+2}, ..., t_n)$, with action on other maps in Δ induced from these two cases by functoriality. We may then obtain the geometric realization of K precisely as the space resulting from taking the tensor product of K^t and Δ^t_* over Δ:

$$|K| = K^t \otimes_\Delta \Delta^t_*.$$

Similarly, given a simplicial set K, we may view K as a functor $K^s : \Delta^{op} \to s\mathscr{S}et$ taking values in the category of simplicial sets. This can be done by giving each K_n a trivial simplicial set structure whose p-simplices are given by the set K_n in all degrees, and whose face maps and degeneracy maps are all taken to be the identity map on K_n. Notice that K^s must then take Δ-morphisms to simplicial maps, namely natural transformations of functors, but naturality of the maps induced from the set maps given by the original functor K is immediate from the construction. Next, recall that we may build from each object $[n]$ of the category Δ a standard simplicial n-simplex Δ^s_n given by the represented functor $\Delta^s_n := \Delta(-, [n])$. We may associate to each standard simplicial n-simplex a canonical barycentric subdivision, namely the simplicial set $\Delta'_n : \Delta^{op} \to \mathscr{S}et$ acting on objects and morphisms via

$$[p] \mapsto \{(S_0, ..., S_p) \mid S_i \subseteq S_{i+1}, 0 \leq i \leq p-1\},$$
$$(\mu : [p] \to [q]) \mapsto (\mu^* : (S_0, ..., S_q) \mapsto (S_{\mu(0)}, ..., S_{\mu(p)}))$$

In words, the simplicial set Δ'_n has for p-simplices non-decreasing chains of subsets of $[n]$ of length $(p+1)$, and it takes Δ-morphisms to maps at the level of simplices induced by selecting a length $(p+1)$-chain from a given length $(q+1)$-chain, a process which is well-defined precisely as a result of the monotonic increasing property of Δ-morphisms. Now, observe that the assignment $[n] \mapsto \Delta'_n$ gives rise to a covariant functor $\Delta'_* : \Delta \to s\mathscr{S}et$ which sends a monotonic map $\gamma : [n] \to [m]$ to the simplicial map given at the level of p-simplices by $\gamma_* : \Delta'_n[p] \to \Delta'_m[p]$, $(S_0, ..., S_p) \mapsto (\gamma(S_0), .., \gamma(S_p))$. The barycentric subdivision of K can then be defined to be the simplicial set obtained as the tensor product of the functors K^s and Δ'_* over Δ:

$$SdK := K^s \otimes_\Delta \Delta^{'}_*.$$

References

[1] Riehl, Emily. Category Theory in Context. 219-220. Available for download at math.jhu.edu/˜eriehl/context.pdf