A BRIEF INTRODUCTION TO KNOT THEORY

YU XIAO

Abstract. We begin with basic definitions of knot theory that lead up to the Jones polynomial. We then prove its invariance, and use it to detect amphichirality. While the Jones polynomial is a powerful tool, we discuss briefly its shortcomings in ascertaining equivalence. We shall finish by touching lightly on topological quantum field theory, more specifically, on Chern-Simons theory and its relation to the Jones polynomial.

Contents

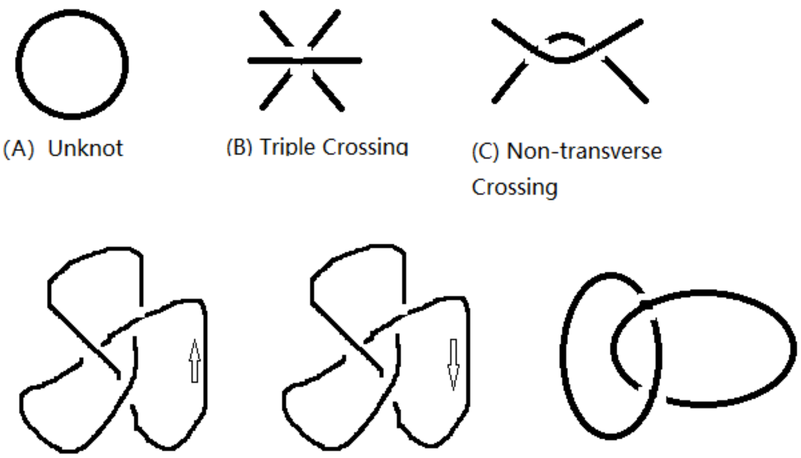
1. Definitions

Generally, it suffices to think of knots as results of tying two ends of strings together. Since strings, and by extension knots exist in three dimensions, we use *knot diagrams* to present them on paper.

One should note that different diagrams can represent the same knot, as one can move the string around without cutting it to achieve different-looking knots with the same makeup. Many a hair scrunchie was abused though not irrevocably harmed in the writing of this paper.

Date: AUGUST 14, 2017.

Figure 1



(A) Unknot (B) Triple Crossing (C) Non-transverse Crossing

(D) Trefoil of One Orientation (E) Trefoil of Other Orientation (F) Hopf Link

As always, some restrictions exist: (1) each crossing should involve two and only two segments of strings; (2) these segments must cross transversely. (See Figure 1.B and 1.C for examples of triple crossing and non-transverse crossing.)

An *oriented knot* is a knot with a specified orientation, corresponding to one of the two ways we can travel along the string. On a knot diagram, we can indicate the orientation via an arrow. (See Figure 1.D and 1.E for the two orientations of the trefoil knot.)

Sometime we use multiple pieces of strings. So we define *link* to be a generalization of a knot, which can be made up of multiple pieces of strings. Number of components in a link is the number of strings used. Likewise, a *link diagram* is a generalization of a knot diagram. An *oriented link* is a link with all its components' orientations specified. (See Figure 1.F for an example of the Hopf link.)

All of the above sounds quite intuitive. We start with a slightly more rigorous but still quite simple definition.

Definition 1.1. A *knot* $K \subset \mathbb{R}^3$ is a subspace of points homeomorphic to a circle.

Like actual knots, our mathematical knots need to be flexible and and deformable as well to model physical changes and manipulations. To that end, we introduce the idea of *isotopy*.

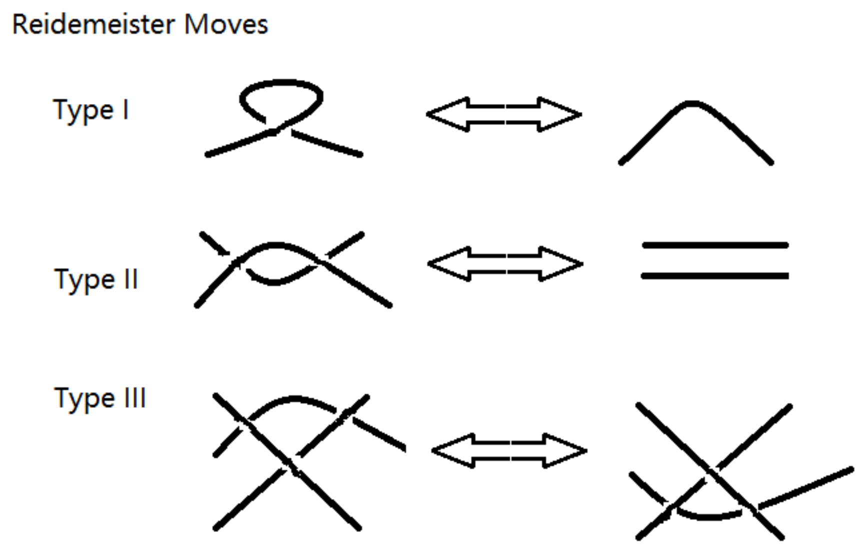
Definition 1.2. (**Isotopy**) Let X be a topological space. An *isotopy* of X is a continuous map $h : X \times [0,1] \to X$ such that $h(x,0) = x$ and $h(\cdot, t)$ is an embedding for each $t \in [0,1]$.

This is not yet enough as all knots are isotopic to the trivial knot. To see why it is so, one need only consider the Bachelor's Unknotting —by pulling on the strings, we can make the knot smaller and smaller. Since mathematical strings have no thickness, instead of getting a small clump like we would in real life, we obtain a point that disappears and leaves us with a trivial knot.Therefore, the space containing the knot must move continuously along with the knot. We therefore arrive at a more complicated and rigorous knot definition.

Definition 1.3. (**Knot**) A *knot* is a smooth embedding $f : S^1 \to \mathbb{R}^3$. Two knots are considered equivalent if related by a smooth isotopy of $\mathbb{R}^3$.

Now that we have defined what knots are, we now turn to something to help us to study them, manipulate them, and study their behavior under these manipulations.

Definition 1.4. (**Reidemeister Moves**) The Reidemeister moves refer to three types of moves. Type I is adding/removing a twist, Type II crossing/uncrossing two strands, and Type III sliding a strand past a crossing.



It is clear that applying these moves does not change the knot and preserves the isotopy. In fact, two diagrams of the same knot can always be made to arrive at each other through a finite sequence of Reidemeister moves.

Theorem 1.5. ***(Reidemeister Theorem)*** *Two diagrams of links are isotopic if and only if one can be transformed into the other by a finite sequence of Reidemeister moves.*

This theorem, the proof of which shall not be given here but could be found in Reidemeister's book, will be incredibly useful in proving knot invariants defined in terms of knot diagrams. Clearly, if we can show that some quantity remains unchanged by the Reidemeister moves, it is then a knot invariant, which doesn't change for equivalent knot diagrams.

Definition 1.6. (**Knot Invariant**) A *knot invariant* is something (for example a number, matrix, or polynomial) that is associated with a knot. A *link invariant* is defined similarly for links.

The knot polynomials form one important subset of knot invariants. They have the advantage of being easy to compute and being easy to compare the obtained computations.

The Jones polynomial is one such knot polynomial.

2. The Jones Polynomial

Though it was Vaughan Jones who discovered and named the Jones polynomial in 1984, his construction is somewhat complicated. The properties of the Jones polynomial were then used by Louis Kauffman to obtained an easier process. In this paper, we shall give the definition by Jones' method, and provide Kauffman's method as well.

2.1. **The Jones Method.**

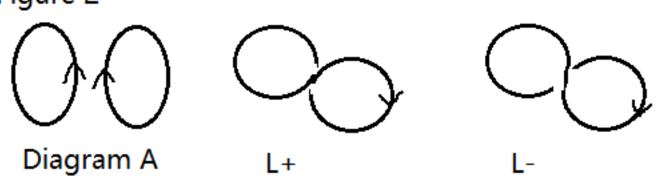
Definition 2.1. (**Laurent Polynomial**) A *Laurent polynomial* with coefficients in a field $\mathbb{F}$ is an expression of the form $p = \sum_k p_k X^k, p_k \in \mathbb{F}$, with X as a formal variable, k an integer, and a finite number of non-zero p_k with k negative.

Definition 2.2. (**Jones Polynomial**) The Jones polynomial is an assignment of a Lanrent polynomial $V_L(t)$ in the variable $\sqrt{t}$ to oriented links L following these three axioms.

- (JP1) Two equal links have the same polynomial
- (JP2) For the unknot ○, $V_K(t) = 1$
- (JP3) If three links L_+, L_-, and L_0 have diagrams that are identical apart from within a certain region where L_+ corresponds to ⤱, L_- to ⤲, and L_0 to ⥮, then $\frac{1}{t}V_{L_+} - tV_{L_-} = (\sqrt{t} - \frac{1}{\sqrt{t}}V_{L_0})$

To be fair, this looks less like a definition and more like a calculational method. There is no guarantee yet that such a polynomial is knot invariant. But in fact, (JP3) is sufficient for calculating $V_L(t)$ inductively on all links. Consider diagram A in the following Figure 2.

Figure 2



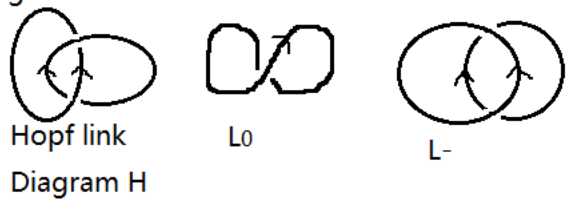
Clearly, here both L_+ and L_- are unknots, L_0 is A. (JP3) gives

$$\frac{1}{t}V_{L_+} - tV_{L_-} = (\sqrt{t} - \frac{1}{\sqrt{t}}V_A)$$

We can then obtain $V_A = -\sqrt{t} - \frac{1}{\sqrt{t}}$.

Now consider the Hopf link diagram H in the following Figure 3.

Figure 3

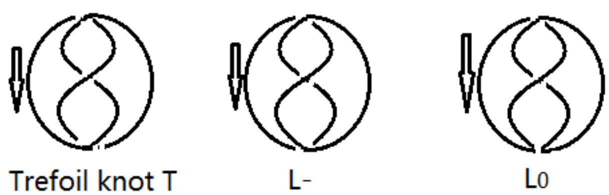


By (JP3), we have $\frac{1}{t}V_H - tV_{L_-} = (\sqrt{t} - \frac{1}{\sqrt{t}}V_{L_0})$.

Previously we have obtained $V_{L_0} = V_A = -\sqrt{t} - \frac{1}{\sqrt{t}}$. Meanwhile, L_0 is an unknot. So $V_{L_0} = 1$. We obtain $V_H = -\sqrt{t} - t^2\sqrt{t}$.

Taking this process a step further, we can successfully calculate the Jones polynomial of the trefoil.

Figure 4



Again, by (JP3) we have $\frac{1}{t}V_T - tV_{L_-} = (\sqrt{t} - \frac{1}{\sqrt{t}}V_{L_0})$. Since here L_- is the unknot and L_0 is now H, we can easily calculate $V_T = -t^4 + t^3 + t$.

The process should be clear. The crux of the matter is that any knot or link can be untied by changing a sufficient number of crossings. So there must be a crossing that can simplify the knot if changed. Should we designate the diagram with this particular crossing as the L_+, then the L_- would be simpler and L_0 has one less crossing. With some combination and induction, voila! We have the desired Jones polynomial.

One might ask, what is so special about these coefficients in JP3? They are actually not all that special. The Jones polynomial is in fact a special case of the HOMFLY polynomial.

Definition 2.3. (**the HOMFLY Polynomial**) For L_+ denoting ⤫, L_- denoting ⤪, and L_0 denoting ≍, the HOMFLY polynomial is a polynomial that satisfies $AP(L_+) + BP(L_-) + CP(L_0) = 0$.

Additionally, the HOMFLY polynomial also generalized the Alexander polynomial, a knot invariant which assigns a polynomial with integer coefficients to each knot type.

2.2. **The Kauffman Method.** Kauffman introduced bracket polynomials to ease the process. The bracket polynomials are not knot invariants themselves as we will see later in this paper. Their use lies in specifying a relationship between more complex link diagrams and simpler ones.

Definition 2.4. (**Crossing Resolutions**) Given a crossing of the form $\times$, the 0-resolution of this crossing is $)($, the 1-resolution is $\asymp$.

Figure 5

(A) Trefoil (B) 0-resolution (C) 1-resolution

Definition 2.5. (**Bracket Polynomial**) The *bracket polynomial* of a link diagram D is a Laurent polynomial in one variable A and denoted as $\langle D \rangle$. It is determined by the following three rules:

- (BP1) $\langle \bigcirc \rangle = 1$
- (BP2) $\langle D \sqcup \bigcirc \rangle = (-A^2 - A^{-2})\langle D \rangle$
- (BP3) $\langle \times \rangle = A\langle)(\rangle + A^{-1}\langle \asymp \rangle$

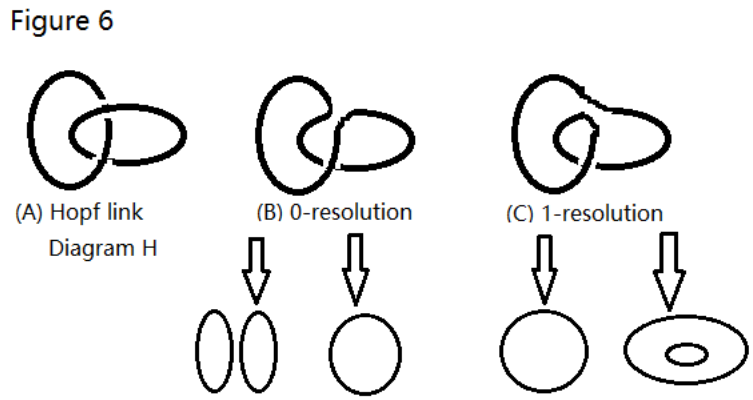
Remarks 2.6. (1) Note that (BP1) only refers to the trivial knot. It is not applicable to all knot diagrams of the unknot.

(2) In (BP2), $D \sqcup \bigcirc$ is to add an extra circle onto diagram D. The original diagram D and the circle should not cross. For example, $\langle \bigcirc\bigcirc \rangle = (-A^2 - A^{-2})\langle \bigcirc \rangle = -A^2 - A^{-2}$.

(3) We can write (BP3) more comfortably as the following. Suppose that we have a diagram D with a fixed crossing, and denote D_0 and D_1 as the corresponding 0-resolution and 1-resolution of the crossing. (BP3) can then be written as $\langle D \rangle = A\langle D_0 \rangle + A^{-1}\langle D_1 \rangle$.

The heart of bracket polynomials lies in (BP3). By recursively using (BP3), we can compute bracket polynomials of diagrams using diagrams with fewer crossings until we arrive at diagrams with no crossings. The advantage of (BP3) over (JP3) is that the number of crossings is strictly reduced when (BP3) is applied. We can then obtain an explicit formula for the Kauffman bracket polynomial.

Take the Hopf link for example. Repeated application of (BP3) gives us the following relations.



Therefore, $\langle H\rangle = A(A\langle \bigcirc\bigcirc\rangle + A^{-1}\langle\bigcirc\rangle) + A^{-1}(A\langle\bigcirc\rangle + A^{-1}\langle\bigcirc\bigcirc\rangle)$. With $\langle\bigcirc\bigcirc\rangle = -A^2 - A^{-2}$, we obtain $\langle H\rangle = -A^4 - A^{-4}$.

Definition 2.7. (**Smoothing**) A *smoothing* of a given link diagram D is a diagram in which every crossing of D has been resolved.

For a link diagram D with n crossings, it has 2^n smoothings that allow us to calculate the bracket polynomial of D as shown in the example above.

We can number the crossings $1, ..., n$. For $\epsilon_1, \epsilon_2, ..., \epsilon_n \in 0, 1$, we denote $D_{\epsilon_1\epsilon_2...\epsilon_n}$ as the smoothing of D where crossing i is resolved by an ϵ_i-resolution. Taking the example of Hopf link as shown above and labeling the top crossing as 1 and bottom crossing as 2, the smoothings shown in Figure 6 from left to right would correspondingly be $D_{00}, D_{01}, D_{10}, D_{11}$.

For a particular smoothing D_ϵ where $\epsilon = \epsilon_1\epsilon_2...\epsilon_n$, let $s_0(\epsilon)$ = number of 0-resolutions in D_ϵ, and $s_1(\epsilon)$ = number of 1-resolutions in D_ϵ.

Under this notation, a smoothing D_ϵ contributes a $A^{s_0(\epsilon)-s_1(\epsilon)}\langle D_\epsilon\rangle$ to the bracket polynomial $\langle D\rangle$.

Theorem 2.8. *There exists a unique polynomial that satisfies the axioms of the Kauffman bracket polynomial.*

Proof. For a link diagram D with n crossings, the analysis carried out above already gifts us with

$$\langle D\rangle = \sum_{\epsilon\in\{0,1\}^n} A^{s_0(\epsilon)-s_1(\epsilon)}\langle D_\epsilon\rangle$$

As D_ϵ consists of a certain number k of non-crossing loops, repeated application of (BP2) gives us $\langle D_\epsilon\rangle = (-A^2 - A^{-2})^{k-1}$.

Therefore, $\langle D\rangle = \sum_{\epsilon\in\{0,1\}^n} A^{s_0(\epsilon)-s_1(\epsilon)}(-A^2 - A^{-2})^{k-1}$.

We must now check that the obtained polynomial satisfies the original three rules.

(BP1): $\langle\bigcirc\rangle = (-A^2 - A^{-2})^0 = 1$. (BP1) is satisfied.

(BP2): By adding a disjoint unknot to diagram D, each smoothing then has an additional disjoint unknot. $\langle D \sqcup \bigcirc \rangle = \sum_{\epsilon \in \{0,1\}^n} A^{s_0(\epsilon)-s_1(\epsilon)}(-A^2 - A^{-2})^k$. Distributing gives us, $\langle D \sqcup \bigcirc \rangle = (-A^2 - A^{-2})\langle D \rangle$. (BP2) is satisfied.

(BP3): Let D_0 and D_1 denote the diagrams resolved by different resolutions at a particular crossing. We have:

$$\langle D_0 \rangle = \sum_{\epsilon_0 \in \{0,1\}^n} A^{s_0(\epsilon_0)-s_1(\epsilon_0)-1}(-A^2 - A^{-2})^{k(\epsilon_0)-1};$$

$$\langle D_1 \rangle = \sum_{\epsilon_1 \in \{0,1\}^n} A^{s_0(\epsilon_1)-s_1(\epsilon_1)+1}(-A^2 - A^{-2})^{k(\epsilon_1)-1}.$$

$$\begin{aligned} A\langle D_0 \rangle + A^{-1}\langle D_1 \rangle = &\sum_{\epsilon_0 \in \{0,1\}^n} A^{s_0(\epsilon_0)-s_1(\epsilon_0)}(-A^2 - A^{-2})^{k(\epsilon_0)-1} \\ &+ \sum_{\epsilon_1 \in \{0,1\}^n} A^{s_0(\epsilon_1)-s_1(\epsilon_1)}(-A^2 - A^{-2})^{k(\epsilon_1)-1}. \end{aligned}$$

Using $\epsilon_0 + \epsilon_1 = \epsilon$, we may obtain:

$$A\langle D_0 \rangle + A^{-1}\langle D_1 \rangle = \sum_{\epsilon \in \{0,1\}^n} A^{s_0(\epsilon)-s_1(\epsilon)}(-A^2 - A^{-2})^{k(\epsilon)-1} = \langle D \rangle.$$

Thus, (BP3) is satisfied.

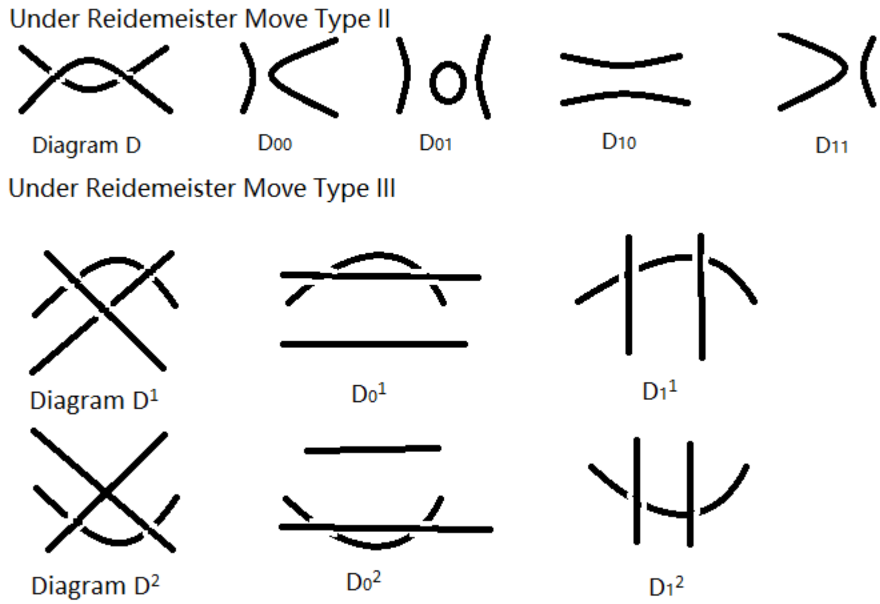
$$\langle D \rangle = \sum_{\epsilon \in \{0,1\}^n} A^{s_0(\epsilon)-s_1(\epsilon)}(-A^2 - A^{-2})^{k-1}.$$

This polynomial comes directly from the three rules with no element of choice and satisfies them all. It is then unique and the explicit expression of the Kauffman bracket polynomial. □

As the bracket polynomial is intended to help us arrive at the Jones polynomial, it is then natural for us to explore the behavior of bracket polynomials under the Reidemeister moves. Or just for plain curiosity's sake.

Lemma 2.9. *The bracket polynomial is invariant under Type II and Type III Reidemeister moves.*

Figure 7



Proof. (1) Reidemeister Move Type II

As shown in Figure 7 for Reidemeister Move Type II, we start with Diagram D, label the left and right crossing 1 and 2 respectively, and obtain the four smoothings as shown by resolving these crossings.

By the proof of Theorem 2.8, we obtain $\langle D\rangle = A^2\langle D_{00}\rangle + \langle D_{01}\rangle + \langle D_{10}\rangle + A^{-2}\langle D_{11}\rangle$. By (BP2), $\langle D_{01}\rangle = (-A^2 - A^{-2})\langle D_{00}\rangle$. Therefore the three terms of $\langle D_{00}\rangle, \langle D_{01}\rangle$, and $\langle D_{11}\rangle$ all cancel out and leave us with $\langle D\rangle = \langle D_{10}\rangle$.

Therefore, the bracket polynomial is knot invariant under Reidemeister Move Type II.

(2)Reidemeister Move Type III

We could of course apply the same steps to Type III Moves. However, that would require a total of eight smoothings. A cleaner approach is to only consider the crossing of the two upper strands. Application of (BP3) shows that

$$\langle D^1\rangle = A\langle D_0^1\rangle + A^{-1}\langle D_1^1\rangle$$

$$\langle D^2\rangle = A\langle D_0^2\rangle + A^{-1}\langle D_1^2\rangle$$

As it is clear to see that $\langle D_1^1\rangle = \langle D_0^2\rangle$ and $\langle D_1^1\rangle = \langle D_1^2\rangle$, we obtain that $\langle D^1\rangle = \langle D^2\rangle$.

Therefore, the bracket polynomial is knot invariant under Reidemeister Move Type III.

□

Remarks 2.10. By this proof, we can see that the coefficients of (BP3) were specifically chosen so that invariance under Type II and Type III are satisfied.

Yet the bracket polynomial is not invariant under Type I Reidemeister moves.

Figure 8

Diagram D^3 D_0^3 D_1^3

$$\langle D^3\rangle = A\langle D_0^3\rangle + A^{-1}\langle D_1^3\rangle = A\langle D_0^3\rangle + A^{-1}(-A^2 - A^{-2})\langle D_0^3\rangle = -A^{-3}\langle D_0^3\rangle$$

The attempt to compensate for this deficiency leads us to the Jones polynomial.

To this end, we introduce *writhe*. Recall that an *oriented* link diagram is one in which all components have a specified direction. Given an orientation, we can define positive and negative crossings, with positive crossing as and negative as . Since a knot only has one component, we can define positive and negative crossings for knots without specifying the orientation involved.

Definition 2.11. (**Writhe**) Let $n_+(D)$ be the number of positive crossings in diagram D, and $n_-(D)$ be the number of negative crossings in D. Then the *writhe* of D is $w(D) = n_+(d) - n_-(D)$.

Writhe is also not a link invariant, but we do know how it varies. Type I Reidemeister move going from D^3 to D_0^3 changes the writhe by one. $w(D^3) = w(D_0^3) - 1$. Therefore, some combination of the bracket polynomial and the writhe can give us a link invariant.

Untwisting D^3 changes the bracket polynomial by multiplying $-A^{-3}$. We shall try to offset this multiplication by using writhe.

Lemma 2.12. *For link diagram D, the polynomial $-A^{-3w(D)}\langle D\rangle$ is a link invariant.*

Proof. Since both the bracket polynomial and the writhe are link invariant under Type II and Type III Reidemeister moves, we only need to check for Type I.

$$-A^{-3w(D^3)}\langle D^3\rangle = -A^{-3w(D_0^3)+3}\cdot(-A^{-3})\langle D_0^3\rangle = -A^{-3w(D_0^3)}\langle D_0^3\rangle$$

□

This winding road has finally led us to a link invariant! It is precisely the Jones polynomial. Here we have a differently worded definition for the Jones polynomial.

Definition 2.13. (**Jones Polynomial**) Given an oriented link L, *the Jones polynomial* of L denoted $V_L(t)$ is obtained by setting $A = t^{-1/4}$ in $-A^{-3w(D)}\langle D\rangle$

Since computation of the Jones polynomial generally follows the method of breaking the knot up into simpler knots with simpler Jones polynomials, it is then natural to consider constructing more complicated knots.

Definition 2.14. (**Knot Sum**) The connected sum of two knots K and K', denoted $K\#K'$, is formed by attaching the knots with respect to the orientation of each knot. It is done by removing a small arc on each knot, and then gluing the knots together by their boundary respecting orientation.

Theorem 2.15. *Given knots K_1 and K_2, $V(K_1\#K_2) = V(K_1)V(K_2)$*

Proof. We go through the same procedure as we do when computing the Jones polynomial of K_1 alone. Except that here, instead of being reduced to the unknot in the computation of $V(K_1)$, K_1 is instead reduced to K_2. Therefore, the Jones polynomial of $K_1\#K_2$ is then merely the Jones polynomial of K_1 multiplied with $V(K_2)$ for every term. The desired result of $V(K_1\#K_2) = V(K_1)V(K_2)$ follows by factoring. □

The Jones polynomial has many uses. We shall demonstrate one of them in the following section: detecting chiral knots.

3. Amphichirality

Definition 3.1. (**Amphichirality and Chirality**) A knot is *amphichiral* if it is equivalent to its mirror image in $\mathbb{R}^3$. Otherwise, it is *chiral*.

Figure 9 gives the mirror images of a trefoil.

Figure 9

(A) Left-Handed Trefoil (B) Right-Handed Trefoil

For a knot K with diagram D, let K^{flip} and D^{flip} denote the corresponding mirror knot and diagram.

Every smoothing ϵ of D corresponds to the dual smoothing $\hat{\epsilon}$ of D^{flip}, with $\hat{\epsilon}$ obtained by interchanging every 0-resolution and 1-resolution in ϵ.

That is, $(D^{flip})_{\hat{\epsilon}} = D_\epsilon$, with $s_0(\epsilon) = s_1(\hat{\epsilon})$ and $s_1(\epsilon) = s_0(\hat{\epsilon})$.

$$\langle D^{flip}\rangle(A) = \sum_{\epsilon\in\{0,1\}^n} A^{s_0(\epsilon)-s_1(\epsilon)}\langle(D^{flip}_\epsilon)\rangle(A) = \sum_{\epsilon\in\{0,1\}^n} A^{-(s_0(\hat{\epsilon})-s_1(\hat{\epsilon}))}\langle D_{\hat{\epsilon}}\rangle(A).$$

Since $\langle D_\epsilon\rangle = (-A^2 - A^{-2})^{k-1}$, $\langle D_\epsilon\rangle(A) = \langle D_\epsilon\rangle(A^{-1})$.

Therefore,

$$\langle D^{flip}\rangle(A) = \sum_{\epsilon\in\{0,1\}^n} (A^{-1})^{s_0(\hat{\epsilon})-s_1(\hat{\epsilon})}\langle D_{\hat{\epsilon}}\rangle(A^{-1}) = \langle D\rangle(A^{-1})$$

Meanwhile, $w(D) = -w(D^{flip})$, since positive and negative crossings are interchanged in D and D^{flip}. We can then obtain the following lemma.

Lemma 3.2. *Given knot K, $V_K^{flip}(t) = V_K(t^{-1})$.*

We return to the example of left-handed and right-handed trefoils. For the left-handed trefoil, $V(t) = -t^{-4} + t^{-3} + t^{-1}$. For the right-handed trefoil. $V(t) = -t^4 + t^3 + t$. As the two trefoils have different Jones polynomials, they are then distinct from each other. We summarize this into a theorem, whose proof is apparent.

Theorem 3.3. *Given knot K, if $V_K(t) \neq V_K(t^{-1})$, then K is chiral.*

4. Limitation of the Jones Polynomial

One limitation of the Jones polynomial must be pointed out. While it is true that when $V_K(t) \neq V_K(t^{-1})$, K is chiral. The reverse is not always true. $V_K(t) = V_K(t^{-1})$ does not guarantee amphichirality. There exist chiral knots with symmetric Jones polynomials.

Figure 10

A chiral knot with symmetric Jones polynomial

The Jones polynomials of the knot above and of its mirror image are both $V(t) = t^3 + t^{-3} - t^2 - t^{-2} + t + t^{-1} - 1$. However, this is a chiral knot.

5. A Bit on Chern-Simons Theory

Up until now, it is somewhat obvious that we have been treating knots, which are naturally three-dimensional, as two-dimensional objects. These mathematical definitions of the Jones polynomial always involve looking at the two-dimensional projection of the knot, giving a two-dimensional method of computation, and proving the obtained polynomial to be independent of the projection.

But the Jones polynomial's importance is not only restricted to mathematics. Indeed, it has a wide range of connections and application in physics. Construction of knot polynomials (though not always Jones) appear quite often: in Temperley-Lieb algebras and their generalizations, in solutions of the Yang-Baxter equation, in conformal field theory, etc.

Therefore physicsts are also invested in this problem: can we then find a intrinsically three-dimensional definition of the Jones polynomial?

We look towards Chern-Simons Theory. Chern-Simons Theory is a three-dimensional topological quantum field theory of the Schwartz type with correlation functions of the system computed by the path integral of metric independent action functional. In condensed-matter physics, it is used to describe the topological order in fractional quantum Hall effect states. In mathematics, it can be used to calculate knot invariants such as the Jones polynomial.

The Chern-Simons action is as follows with A as the G-gauge connection on principal bundle on base space M:

$$S = \frac{k}{4\pi} \int_M Tr(A \wedge dA + \frac{2}{3} A \wedge A \wedge A) \tag{5.1}$$

We shall not go into complicated details here. It suffices to know that the Yang-Mills theory with a Chern-Simons action is soluble and provides a framework for understanding the Jones polynomial in three-dimensional terms. This was proven by Edward Witten in 1989. The Yang-Mills theory is a gauge theory based on compact reductive Lie algebra and seeks to describe the behavior of elementary particles using non-Abelian Lie groups. It unifies electromagnetc force, weak forces, and quantum chromodynamics.

Acknowledgments. Many thanks to Professor Peter May for giving me this opportunity to participate in REU 2018. It is a pleasure to thank my mentor Drew Moore for his help and patience. I would also like to thank Mao-Tian Tan for his suggestions on physics-related readings.

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