SIMPLICIAL HOMOLOGY AND THE CLASSIFICATION OF COMPACT SURFACES

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Abstract. This paper provides an introduction to simplicial homology groups of topological spaces as well as a proof of the classification theorem of compact surfaces. The classification theorem shows that every surface is homeomorphic to one of the standard surfaces, while the introduction of simplicial homology groups provides a basis for the proof that each of the standard surfaces is indeed topologically distinct from the rest.

Contents

1. Introduction

This paper has two main objectives: the introduction of simplicial homology, and its application in the classification of compact surfaces. The classification theorem states that there are only four types of compact, closed surfaces: the sphere, the projective plane, the n-fold torus, and the m-fold projective plane. With our knowledge of simplicial homology groups, we can deduce that each of these types of surfaces is not homeomorphic to any of the others. For example, there are many surfaces with "different" triangulations that are homeomorphic to the 3-fold torus, and they are all topologically "distinct" from the 5-fold torus and the 17-fold projective plane.

We will begin our proof of the classification theorem by first showing that all surfaces can be "represented" by a polygon whose edges are identified in pairs. Using operations that do not change the identification space, i.e that produce homeomorphic surfaces, we show that every surface is homeomorphic to one of the standard ones.

All together, we achieve an understanding of what types of surfaces there are, while proving that every standard type of surface is not homeomorphic to any other type.

2. Some Basic Definitions

We begin with the definitions of topological spaces. In this paper, we are concerned specifically with Hausdorff topological spaces. In the proof of the classification theorem of surfaces, we further narrow our scope to two-dimensional manifolds.

Definition 2.1. A **topological space** is a pair (X, τ) where X is a set and τ is a collection of subsets of X, defined to be the open sets, such that:

(1) The empty set and X are open sets in τ.
(2) Every finite union of open sets in τ is also an open set in τ.
(3) Every finite intersection of open sets in τ is also an open set in τ.

We say that τ is the topology on X.

Definition 2.2. Let (X, τ) be a topological space, and let S be a subset of X, $S \subseteq X$. The **subspace topology** σ on S is a subcollection of the open sets of τ:

$$\sigma = \{S \cap U | U \in \tau\}$$

Note that (S, σ) is itself a topological space.

Definition 2.3. A topological space X is **Hausdorff** if for every for every $x_1, x_2 \in X, x_1 \neq x_2$, there exist disjoint neighbourhoods U_1, U_2 such that $x_1 \in U_1, x_2 \in U_2$.

Definition 2.4. A **manifold** is a Hausdorff topological space such that some neighbourhood of every point is topologically homeomorphic to the unit ball in Euclidean space.

We say that a manifold locally resembles Euclidean space. In addition, we say that a topological space is an n-dimensional manifold if it locally resembles n-dimensional Euclidean space near each point.

Definition 2.5. Let (X, τ_X) be a topological space, and let $\sim$ be an equivalence relation on X. The **identification space** $X/\sim$ is the codomain of the identification map $q : X \to X/\sim$. The open sets of the quotient space $X/\sim$ are defined to be those whose preimage are open sets in X.

In section 4, we will "glue" edges of polygons together. Saying that two edges are glued together is the same as saying they are identified with each other.

3. Simplicial Homology

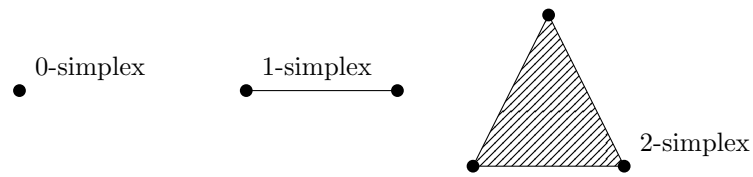
3.1. **Simplicial Complexes.** We now define simplices and simplicial complexes. These are crucial to the construction of n-chains, which in turn allow us to define simplicial homology groups.

Definition 3.1. An **n-simplex** σ is the smallest convex hull of $n+1$ points in general position.

A set of $n+1$ points in Euclidean space are said to be in **general position** if they do not occupy an $n-1$-dimensional hyperplane. For example, three points are in general position if they are not collinear, i.e. on the same line.

From this definition, we can see that each simplex is uniquely defined by its vertices. By choosing an ordering of the vertices, we give the simplex an **orientation**. We require oriented simplices for the purposes of defining simplicial homology groups.

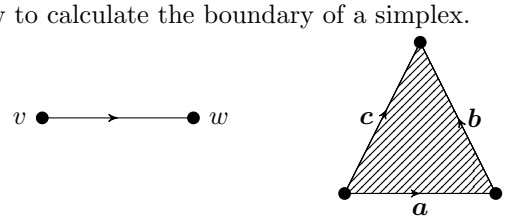
Examples of a 0-simplex, a 1-simplex, and a 2-simplex are shown below.



0-simplexes are vertices, 1-simplices are lines, 2-simplices are triangles, 3-simplices are tetrahedrons, and so on. Each simplex contains all its lower dimensional simplices. For example, a 2-simplex contains three 1-simplices (its edges) and three 0-simplices (its vertices).

Definition 3.2. The **face** of an n-simplex σ is an $n-1$-simplex contained in σ. The "formal" sum of all faces of σ is called the **boundary** of σ, denoted $\partial\sigma$.

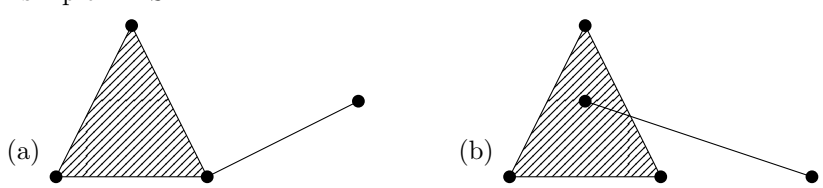
Below we have an a 1-simplex and a 2-simplex. We will use these as examples to demonstrate how to calculate the boundary of a simplex.



On the left, we have a 1-simplex with vertices v, w oriented v to w. Its faces are the vertices v, w, and its boundary is $w - v$. On the right, we have a 2-simplex with oriented edges a, b, c. These edges are the faces of this 2-simplex. If the 2-simplex is oriented counterclockwise, its boundary $a + b - c$. Otherwise, if it is oriented clockwise, its boundary is $-a - b + c$.

Definition 3.3. A **simplicial complex** $K = (V, S)$ is a collection of vertices V and simplices S in Euclidean space satisfying the following properties:

(1) Every vertex $v \in V$ is the vertex of at least one and at most finitely many simplices in S.
(2) Every face of a simplex in S is itself an element of S.
(3) The intersection of two simplices is a common face of each, i.e. is itself a simplex in S.



In the figure above, (a) is a simplicial complex. However, (b) is not, as it does not satisfy property (3) of the definition. The intersection of the triangle and the line is a vertex which is not a face of the triangle and part of a line which is not itself a full simplex.

Definition 3.4. The **geometric realization** of a simplicial complex $K = (V, S)$ is a subset $|K|$ of Euclidean space obtained from the embedding of K in Euclidean space, together with the subspace topology. $|K|$ is therefore a topological space.

Definition 3.5. A topological space M is **triangulable** if it is homeomorphic to the geometric realization of a simplicial complex K. We say that $|K|$ is a **triangulation** of M.

Triangulations of a space are not unique. A given triangulable topological space is homeomorphic to the geometric realizations of many different simplicial complexes.

Definition 3.6. Let M be a topological space with a triangulation $|K|$. Let $C_n(M)$ be the free abelian group generated by the ordered n-simplices e^n_α contained in K. Elements of $C_n(M)$, called **n-chains**, can be written as $\sum_\alpha n_\alpha e^n_\alpha$ with coefficients $n_\alpha \in \mathbb{Z}$.

In other words, we know that every simplicial complex K contains various simplices, perhaps of different dimensions. We can label the n-dimensional simplices e^n_α. An n-chain is a linear combination of these n-dimensional simplices. We can now define the **boundary function** $\partial_n : C_n(M) \to C_{n-1}(M)$ that takes each oriented n-simplex σ with vertices $v_1, ..., v_n$ to its boundary. The notation $\hat{v_i}$ denotes the elimination of that v_i from the sum.

$$\partial_n(\sigma) = \sum_i (-1)^i [v_0, ...\hat{v_i}, ...v_n]$$

The boundary of an n-simplex is a $n-1$-chain, a sum of oriented $n-1$-dimensional simplices. Consequently, the boundary functions are homomorphisms from each free abelian group $C_n(M)$ to the following $C_{n-1}(M)$, and so we can create the chain complex below.

$$0 \longrightarrow C_n(M) \xrightarrow{\partial_n} C_{n-1}(M) \xrightarrow{\partial_{n-1}} \cdots \xrightarrow{\partial_1} C_0(M) \xrightarrow{\partial_0} 0$$

The following lemma shows that the boundary of a boundary is always zero.

Lemma 3.7. *The composition of two boundary functions is the zero homomorphism, $\partial_n \circ \partial_{n+1} = 0$.*

Proof.

$$\begin{aligned}
\partial_n \circ \partial_{n+1}(\sigma) &= \partial_n \sum_i (-1)^i [v_0, ...\hat{v_i}, ...v_{n+1}] \\
&= \sum_{j=0}^{n+1} (-1)^j \sum_{i=j+1}^{n+1} (-1)^{i-1} [v_0, ...\hat{v_j}, ...\hat{v_i}, ...v_{n+1}] \\
&+ \sum_{j=0}^{n+1} (-1)^j \sum_{i=0}^{j-1} (-1)^{i-1} [v_0, ...\hat{v_i}, ...\hat{v_j}, ...v_{n+1}] \\
&= 0
\end{aligned}$$

The elements of the two remaining sums cancel, and we are left with zero. □

3.2. **Simplicial Homology Groups.**

Definition 3.8. The **rank** of a group G is the smallest cardinality of a generating set of G. The rank of an abelian group is the cardinality of the maximal linearly independent subset.

Note that the rank of an abelian group determines the largest free abelian group in A.
We will define the function $r : G \to \mathbb{N}$ that returns the rank of a group G.

Proposition 3.9. *Let G be an abelian group of finite rank and H a subgroup of G. Then $r(G/H) = r(G) - r(H)$.*

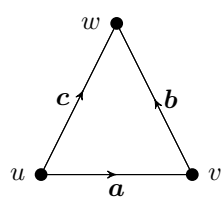
We can now define the simplicial homology of a space M. In Lemma 3.7 we saw that the image of ∂_{n+1} is in the kernel of ∂_n. We will denote $im(\partial_{n+1})$ as $Z_n(M)$ and $ker(\partial_n)$ as $B_n(M)$.

Definition 3.10. Let $n \geq 0$, and M be a triangulable manifold. The nth **simplicial homology group** $H_n(M)$ of a manifold M is the abelian group obtained by the quotient group of the kernel of ∂_n by the image of ∂_{n+1}.

$$H_n(M) = Z_n(M)/B_n(M)$$

Because each boundary function takes n-chains of oriented n-simplices to their boundaries, the kernel of ∂_n, $Z_n(M)$, is exactly those n-chains in which the elements of the boundary cancel each other out. This happens when each face of an n-simplex in the chain is the face of exactly one other n-simplex which is oriented in the opposite direction, in other words the n-chain is a sort of n-dimensional ball. The image of ∂_{n+1}, $B_n(M)$, is the subset of these balls that in the space are the boundaries of $n+1$-dimensional simplices. The quotient group $H_n(M)$ consists of the balls that are not the boundaries of higher dimensional simplices, in other words, the n-dimensional holes in the space.

For example, let us consider the 1-chain below, $a + b - c$. It is composed of three 1-simplices that form a loop, a one-dimensional ball. There is a one-dimensional hole inside of this loop, because there is no inside to the triangle.



The boundary of a is $v - u$, the boundary of b is $w - v$, and the boundary of $-c$ is $u - w$. Together, the boundary of $a + b - c$ is:

$$\begin{aligned}\partial(a + b - c) &= \partial(a) + \partial(b) + \partial(-c) \\ &= (v - u) + (w - v) + (u - w) \\ &= 0\end{aligned}$$

Note that this is also an example that illustrates why the boundary of a boundary is the zero homomorphism, Lemma 3.7.

Definition 3.11. The **direct sum** of abelian groups A and B, denoted $A \oplus B$, is the Cartesian product of the groups. Multiplication $\cdot$ of the direct sum is defined component-wise, i.e. for $a_1, a_2 \in A, b_1, b_2 \in B$ we have:

$$(a_1, b_1)(a_2, b_2) = (a_1 \cdot a_2, b_1 \cdot b_2)$$

Remark 3.12. One way to compute $H_1(M)$, apart from using the definition, is through the abelianization of the fundamental group of M, the direct sum of the elements of the fundamental group. This is possible when the fundamental group is known.

Definition 3.13. We define the **Euler-Poincaré characteristic** of a triangulable topological space M of dimension n to be the alternating sum of the ranks of the simplicial homology groups of any triangulation K.

$$\chi(M) = \sum_{i=0}^{n} (-1)^i r(H_i(K))$$

In this paper we will call this the Euler characteristic.

Theorem 3.14. *Let M be a topological space, and K a triangulation of M of dimension n. Let p_i be the number of i-dimensional simplices in K. Then*

$$\chi(M) = \sum_{i=0}^{n} (-1)^i p_i$$

Proof. By proposition 3.9,

$$\begin{aligned} r(H_i(M)) &= r(Z_i(M)/B_i(M)) \\ &= r(Z_i(M)) - r(B_i(M)) \end{aligned}$$

Indeed,

$$\begin{aligned} r(C_i(M)) &= r(ker(\partial_i)) + r(im(\partial_i)) \\ &= r(Z_i(M)) + r(B_{i-1}(M)) \end{aligned}$$

p_i is the number of i-dimensional simplices in K, and so

$$r(C_i(M)) = p_i = r(Z_i(M) + B_{i-1}(M))$$

Note that $B_{n+1}(M) = 0$, and $B_{-1}(M) = 0$ for a topological space whose triangulation only contains simplices of degree $\leq n$.

$$\begin{aligned}
\chi(M) &= \sum_{i=0}^{n}(-1)^i r(H_i(M)) \\
&= \sum_{i=0}^{n}(-1)^i r(Z_i(M)) - r(B_i(K)) \\
&= \sum_{i=0}^{n}(-1)^i r(Z_i(M)) - \sum_{i=0}^{n}(-1)^i r(B_{i-1}(M)) \\
&= \sum_{i=0}^{n}(-1)^i r(Z_i(M)) + \sum_{i=0}^{n}(-1)^{i-1} r(B_{i-1}(M)) \\
&= \sum_{i=0}^{n}(-1)^i r(Z_i(M)) + \sum_{i=0}^{n-1}(-1)^{i-1} r(B_i(M)) \\
&= \sum_{i=0}^{n}(-1)^i r(Z_i(M)) + \sum_{i=0}^{n}(-1)^{i-1} r(B_i(M)) \\
&= \sum_{i=0}^{n}(-1)^i r(Z_i(M)) - r(B_i(M)) \\
&= \sum_{i=0}^{n}(-1)^i p_i
\end{aligned}$$

□

Theorem 3.15. *If X and Y are homeomorphic triangulable topological spaces, then $H_n(X)$ and $H_n(Y)$ are isomorphic for any $n \in \mathbb{Z}$.*

A proof for this theorem can be found in Chapter Two of *Algebraic Topology* by Allen Hatcher [1].

A given topological space M is homeomorphic to the geometric realizations of many different simplicial complexes. Theorem 3.15 shows that the simplicial homology groups, and in turn the Euler characteristic, are topological invariants. In other words, the nth simplicial homology group is the same regardless of the triangulation chosen.

Remark 3.16. The case $n = 2$, two dimensional surfaces, is the most well-known use of the Euler characteristic. Let v be the number of vertices, e the number of edges, and t the number of triangles of a given triangulation of a surface M. Then $\chi(M) = v - e + t$.

Definition 3.17. Let L and M be n-dimensional manifolds. The **connected sum** L and M is obtained by removing an n-dimensional unit ball from each and identifying the boundaries of the holes together. See Definition 2.5

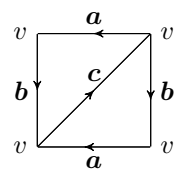
Lemma 3.18. *Let L and M be n-dimensional manifolds. Let l and m be the Euler characteristics for L and M, respectively. The Euler characteristic of the connected sum of L and M is $l + m$.*

A proof for this lemma can be found in *Algebraic Topology* by Allen Hatcher [1].

Definition 3.19. A surface S is **orientable** if $H_2(S) \cong \mathbb{Z}$.

Intuitively, a surface is orientable if it has an "inside" and an "outside", like the sphere. The Klein bottle is an example of a non-orientable surface. The homology group H_2 is a topological invariant, and so orientability is as well. See Hatcher [1].

We will now compute the simplicial homology groups of the torus. A triangulation of the torus T is represented in the following diagram:



We will denote the upper left triangle U, oriented counterclockwise, and the lower right triangle as L, oriented clockwise. The orientations are arbitrarily chosen.

$C_0(T)$ is composed of the linear combinations of the vertices, $C_1(T)$ is composed of the linear combinations of the lines, $C_2(T)$ is composed of the linear combinations of the triangles, and $C_3(T)$ and higher are empty, as there are no tetrahedrons or higher dimensional simplices in the triangulation.
Subsequently, we can compute:

$$C_0(T) = \mathbb{Z}, \text{ the free abelian group generated by } v.$$
$$C_1(T) = \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}, \text{ generated by } a, b, c.$$
$$C_2(T) = \mathbb{Z} \oplus \mathbb{Z}, \text{ generated by } U, L.$$
$$C_3(T) = 0$$

$$0 \xrightarrow{\partial_3} C_2(T) \xrightarrow{\partial_2} C_1(T) \xrightarrow{\partial_1} C_0(T) \xrightarrow{\partial_0} 0$$

The nth simplicial homology group is determined by the the kernel of ∂_n and the image of ∂_{n+1}. We will begin with the computation of $H_0(T)$:
∂_0 is the zero homomorphism, and so all of $C_0(T) = \mathbb{Z}$ is in the kernel.
$\partial_1(a) = v - v = 0 = \partial_1(b) = \partial_1(c)$, so the image of ∂_1 in this case is always 0.
$H_0(T) = ker(\partial_0)/im(\partial_1) = \mathbb{Z}/0 = \mathbb{Z}$.

Now we will compute $H_1(T)$:
As shown above, ∂_1 is the zero homomorphism in this case, and so all of $C_1(T) = \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}$ is in the kernel. It is generated by a, b, c, but with a change of basis this is equivalent to the free abelian group generated by $a, b, (a + b + c)$.
$\partial_2(U) = a + b + c = \partial_1(L)$. Let $p, q \in \mathbb{Z}$.
By definition, $\partial_2(C_2(T)) = \partial_2(p\partial_2(U) + q\partial_2(L)) = (p + q)(a + b + c)$.
The image of ∂_2 is $\mathbb{Z}$, generated by $(a + b + c)$.
$H_1(T) = ker(\partial_1)/im(\partial_2) = \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}/\mathbb{Z} = a, b, (a + b + c)/(a + b + c) = \mathbb{Z} \oplus \mathbb{Z}$.

We will now compute $H_2(T)$:
To find the kernel of ∂_2, we set $(p + q)(a + b + c) = 0$. This occurs when $p = -q$.
So, $ker(\partial_2) = \mathbb{Z}$, generated by $a + b + c$.

$C_3(T) = 0$, so the image of ∂_3 is zero.
$H_2(T) = ker(\partial_2)/im(\partial_3) = \mathbb{Z}/0 = \mathbb{Z}$.

The rank of $H_0(T)$, $r(H_0(T))$, is 1, which indicates that the torus has one connected component. $r(H_1(T)) = 2$, indicating that there are two one-dimensional holes. $r(H_2(T)) = 1$, indicating that there is one cavity.

4. Preliminaries to the Classification of Compact Surfaces

The main theorem that we will prove is the classification of compact surfaces, which states that every closed, compact, triangulable surface is homeomorphic to one of the standard ones. We will begin with definitions that will allow us to make this statement and the subsequent proof rigorous.

A **surface** is a two-dimensional manifold. A surface is **compact** if it is compact as a topological space, that is, for every open covering of the surface there exists a finite sub-cover. A surface is **closed** if it does not have a boundary. The sphere, the torus, the projective plane, and the Klein bottle are all examples of closed surfaces. A disk is not closed because it has a boundary, S^1.

Remark 4.1. Note that in other contexts, the word "closed" is used to mean a set that contains all of its limit points. In this sense, a closed disk is closed as a topological set, although it is not closed as a manifold.

As discussed in Definition 3.5, a surface is **triangulable** if it is homeomorphic to the geometric realization of a simplicial complex. Because we are working with two-dimensional manifolds, the corresponding simplicial complexes will be composed of only vertices, edges, and triangles.

Theorem 4.2. *Every surface can be triangulated.*

A proof for this theorem can be found in Appendix E of A Guide to the Classification Theorem for Compact Surfaces [2].

Recall from Definition 3.17 that if X and Y are disjoint surfaces, the connected sum of X and Y is obtained by removing a disk from each and identifying the boundaries of the holes together.

Another way of conceptualizing a connected sum of surfaces is by gluing one end of a cylinder to each hole. This is called adding a handle. The standard orientable surface of genus n is a sphere with n handles sewn on. It is homeomorphic to the n-fold torus, the connected sum of n tori.

A non-orientable surface can be obtained, for example, by cutting a disk out of a surface and sewing the boundary of a Möbius strip to the boundary of the hole. In three-dimensional space, the Möbius strip must intersect itself to make this possible. Any surface with a Möbius strip sewn in is non-orientable, and does not have an inside or an outside. All other surfaces are orientable. See Definition 3.19.

The following theorem and corollary are the basis for our approach to the classification theorem of surfaces.

Theorem 4.3. *If S is a closed, compact surface, then S is homeomorphic to the quotient space obtained from gluing a collection of disjoint triangles together along their edges in pairs.*

Proof. By Theorem 4.2, S is triangulable.

This proof relies on the fact that S is a manifold, and so around every point it is locally homeomorphic to a disk.

First, we will show that every edge e is contained in exactly two triangles. This proves that the edges are glued in pairs. For contradiction, let there be an edge e_1 that is contained in only one triangle, and an edge e_2 that is contained in more than two triangles. Let a be a point on e_1. Because e_1 is the edge of only one triangle, it is only ever possible to find a half disk around a, contradicting the fact that there must exist a disk around every point. Now, let b be a point on e_2. If e_2 is the intersection of more that two triangles, then the neighbourhood of b will not be homeomorphic to a disk, a contradiction. Therefore every edge is contained in exactly two triangles.

Now, we will show that for every vertex v each triangle that contains v shares edges with exactly two other triangles that also contain v. For contradiction, let there be two sets of triangles, A and B, each containing triangles that have v as a vertex. Define the sets such that none of the triangles in A share edges with triangles in B. Removing v from S would cause the neighbourhood of v to be disconnected, and so this neighbourhood without v would not be homeomorphic to a punctured disk. □

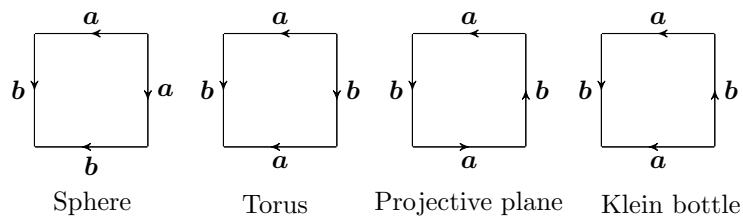
Corollary 4.4. *If S is a closed, compact surface, then S is homeomorphic to the space obtained from identifying a polygon's edges in pairs.*

Proof. Theorem 4.3 showed that there exists a triangulation of S that consists solely of triangles with edges glued together in pairs. Beginning with one of these triangles, we can glue its neighbouring triangles to its edges to form a planar region. Regardless how many of the triangles we paste, we will still have a polygon. Eventually we will have glued all the triangles, and still have a polygon.
In Theorem 4.3, we showed that every edge is the intersection of exactly two triangles. Therefore, an edge of the polygon must be identified with exactly one other edge, which consequently must be an edge of the polygon as well.
Therefore, S is homeomorphic to the space obtained from identifying a polygon's edges in pairs. □

Definition 4.5. Let P be a planar polygon. We can label each edge of P with a letter. A **labeling scheme** of P is a word composed of the letters that represent the edges of P in the order of our chosen orientation of the polygon.

An edge e will appear as e in the **labeling scheme** if its orientation is in the same direction as the orientation of the polygon. Otherwise, it will appear as e^-, the inverse of e. We will use $\sim$ to denote equivalence between labeling schemes. Labeling schemes are equivalent if they represent homeomorphic surfaces.

Below we have polygons that represent the sphere, the torus, the projective plane, and the Klein bottle.



The corresponding labeling schemes are:

(1) $abb^-a^- \sim aa^-$, the sphere.
(2) aba^-b^-, the torus.
(3) $abab \sim aa$, the projective plane.
(4) aba^-b, the Klein bottle.

Definition 4.6. Let e be an edge oriented from the vertex v to the vertex w. Then v is the **initial point** of e, and w is the **end point**.

Definition 4.7. A labeling scheme is a **proper scheme** if every edge e appears twice, either both times as e, or as e and e^-.

Corollary 4.4 shows that the polygonal representation of a surface will always have a proper scheme. Furthermore, the edges must be pasted in pairs, and so there must be an even number of edges. A loop is a path along a surface whose initial point is the same as its end point.

If a surface is orientable, then for every edge e in its polygonal representation, both e and e^- appear. Otherwise, if the surface is non-orientable, e appears twice. This is because the initial points (and the end points) of e must be glued together when reconstructing the surface. If e appears twice, then the polygon must be twisted in the gluing process, which corresponds with the intuition that a non-orientable surface does not have distinct "inside" from its "outside".

There are five operations that can be performed on a polygon and its labeling scheme to produce homeomorphic surfaces, i.e. equivalent labeling schemes.

- Rotate: A rotation is a cyclic permutation of the polygon's labeling scheme which has no effect on the space represented.
- Reflect: The reflection of $ab...cd$ is $d^-c^-...b^-a^-$. Because the relative orientations and orderings of the edges don't change, reflection is a homeomorphism.
- Relabel: Relabeling e_1 to e_2 has no effect on the surface represented.
- Cutting and gluing: Gluing matches up a pair of edges that are identified together, matching the initial points together and the end points together. Cutting a polygon is the same as cutting a surface. Because we label the cut, we are remembering how to glue it back together, so the identification space stays the same.
- Canceling and uncanceling: Canceling is deleting adjacent pairs $...ee^-...$ or $...e^-e...$ from a labeling scheme. Canceling does not change the surface, as these edges were to be pasted together anyway. The reverse, uncanceling, is adding a pair of adjacent edges ee^- or e^-e anywhere in the scheme, which also does not affect the surface.

Creating holes by cutting does not change the Euler characteristic of the surface because the triangulation of the hole in the surface has n edges and n vertices. As a

result, the triangulation of the polygon of a surface has the same Euler characteristic as the surface, because we must make cuts in the surface to form the polygon. The Euler characteristic of a surface is introduced in Definition 3.13, and is discussed further in Remark 3.16.

We will now prove some lemmas that will allow us to prove the classification theorem of compact surfaces.

Lemma 4.8. *Let w be a proper scheme of the form $w = ay_1by_2a^-y_3b^-y_4$, where the y_i are either labeling schemes or empty. Then $w \sim aba^-b^-y_4y_3y_2y_1$.*

Proof. Note that the y_i are not necessarily proper labeling schemes, for example there could be an edge in y_1 that is identified with an edge in y_2.
We will perform the operations mentioned above to prove the equivalence of the labeling schemes from the statement of the theorem.

$$w = ay_1by_2a^-y_3b^-y_4 \tag{4.9}$$

$$\sim y_4ay_1c, c^-by_2a^-y_3b^- \text{ (cut)} \tag{4.10}$$

$$\sim y_1cy_4a, a^-y_3b^-c^-by_2 \text{ (rotate)} \tag{4.11}$$

$$\sim c^-by_2y_1cy_4y_3b^- \text{ (glue and rotate)} \tag{4.12}$$

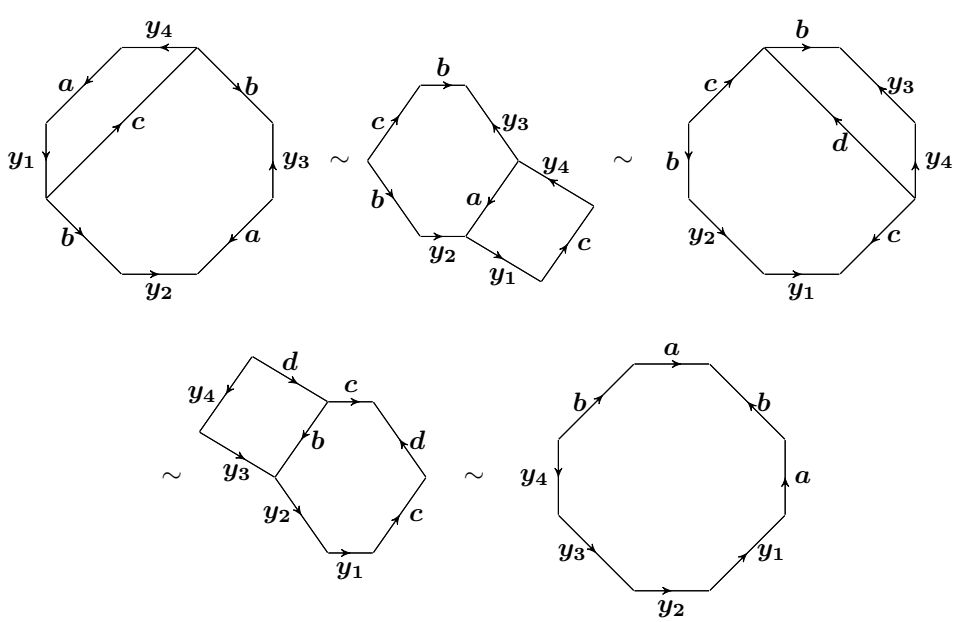
$$\sim c^-by_2y_1cd, d^-y_4y_3b^- \text{ (cut)} \tag{4.13}$$

$$\sim y_2y_1cdc^-b, b^-d^-y_4y_3 \text{ (rotate)} \tag{4.14}$$

$$\sim cdc^-d^-y_4y_3y_2y_1 \text{ (glue and rotate)} \tag{4.15}$$

$$\sim aba^-b^-y_4y_3y_2y_1 \text{ (relabel } c \text{ to } a,\ d \text{ to } b) \tag{4.16}$$

Below is the visual representation of the equivalent polygons. In order left to right, starting on the first row, they represent the schemes in (4.10), (4.12), (4.13), (4.15), and (4.16).



□

By repeating the process outlined in this lemma, we obtain the following corollary.

Corollary 4.17. *If w is a proper scheme that contains oriented pairs then $w \sim (a_1b_1a_1^-b_1^-)(a_2b_2a_2^-b_2^-)...(a_nb_na_n^-b_n^-)y$ where y is either a proper non-orientable labeling scheme or empty.*

Lemma 4.18. *Let $w = y_0ay_1a$ be a proper labeling scheme of a non-orientable surface, where the y_i are labeling schemes or empty. Then $w \sim aay_0y_1^-$.*

Proof. As in the previous lemma, the y_i are not necessarily proper labeling schemes.

$$w = y_0ay_1a \tag{4.19}$$

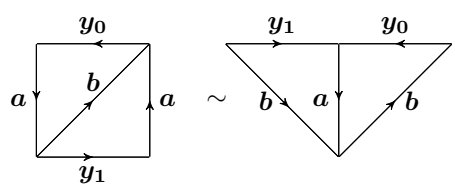
$$\sim y_0ab, b^-y_1a \text{ (cut)} \tag{4.20}$$

$$\sim by_0a, a^-y_1^-b \text{ (rotate first word, reflect second word)} \tag{4.21}$$

$$\sim by_0y_1^-b \text{ (glue)} \tag{4.22}$$

$$\sim aay_0y_1^- \text{ (rotate, relabel } b \text{ to } a) \tag{4.23}$$

Below is the visual representation of the equivalent polygons. On the left is a representation of the labeling scheme in (4.20), on the right a representation of (4.22).



□

By repeating the process outlined in this lemma, we obtain the following corollary.

Corollary 4.24. *If w is a proper scheme of a non-orientable surface then $w \sim (a_1a_1)(a_2a_2)...(a_na_n)y$ where y is either a proper orientable scheme or empty.*

Lemma 4.25. *If w is a proper scheme of the form $w = y_0cca^-b^-aby_1$ then $w \sim y_0aabbccy_1$.*

Proof. This lemma is showing that a surface that contains the connected sum of a torus and a projective plane is homeomorphic to a surface that contains the connected sum of three projective planes.

$$w = y_0cca^-b^-aby_1 \tag{4.26}$$

$$\sim aby_1y_0cca^-b^- \text{ (rotate)} \tag{4.27}$$

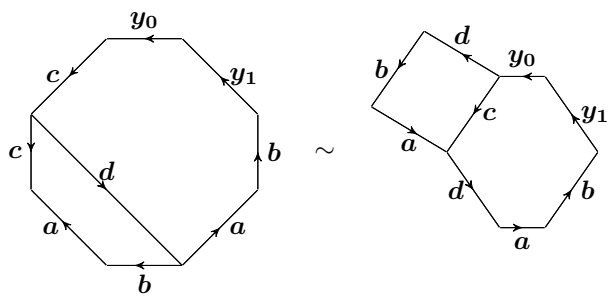
$$\sim aby_1y_0cd, d^-ca^-b^- \text{ (cut)} \tag{4.28}$$

$$\sim daby_1y_0c, c^-dba \text{ (rotate both words, reflect the second word)} \tag{4.29}$$

$$\sim daby_1y_0dba \text{ (glue)} \tag{4.30}$$

$$\sim caby_1y_0cba \text{ (relabel } d \text{ to } c) \tag{4.31}$$

Below is the visual representation of the equivalent polygons. On the left is a representation of the labeling scheme in (4.28), on the right a representation of (4.30).



We can now apply Lemma 4.24 first to c, then b, then a. After rotating y_0 to the beginning of the labeling scheme, we obtain our result, that $w \sim y_0aabbccy_1$. $\square$

Using what we know from Section 3.2 about simplicial homology, we can differentiate between the sphere, the torus, the projective plane, and the Klein bottle. A discussion of the Euler characteristic $\chi(M)$ can be found at the end of Section 3.2.

(1) The sphere: $\chi(S) = 2$.
- $H_0(S) = \mathbb{Z}$
- $H_1(S) = 0$
- $H_2(S) = \mathbb{Z}$

(2) The torus: $\chi(S) = 0$.
- $H_0(T) = \mathbb{Z}$
- $H_1(T) = \mathbb{Z} \oplus \mathbb{Z}$
- $H_2(T) = \mathbb{Z}$

(3) The projective plane: $\chi(\mathbb{R}P^2) = 1$.
- $H_0(\mathbb{R}P^2) = \mathbb{Z}$
- $H_1(\mathbb{R}P^2) = \mathbb{Z}_2$
- $H_2(\mathbb{R}P^2) = 0$

(4) The Klein bottle: $\chi(K) = 0$.
- $H_0(K) = \mathbb{Z}$
- $H_1(K) = \mathbb{Z} \oplus \mathbb{Z}_2$
- $H_2(K) = 0$

All four of these surfaces are connected, so for all four $H_0(M) = \mathbb{Z}$. The first homology groups of each of these surfaces are different, and so by Theorem 3.15, they are not homeomorphic. For both the sphere and the torus, $H_2(M) = \mathbb{Z}$, and so are both orientable. The projective plane and the Klein bottle are both non-orientable because their second homology groups are zero. While the torus and the Klein bottle have the same Euler characteristic, their homology groups differ, and so they are not homeomorphic. Similarly, the n-fold torus and the m-fold projective plane have each have unique homology groups, and so they are not homeomorphic.

5. Classification of Compact Surfaces

We will now prove the classification theorem of compact surfaces with the aid of the lemmas and theorems from the previous section. This proof relies first and foremost on the fact that if S is a surface, then S is triangulable, by Theorem 4.2. Furthermore, if S is a closed and compact surface, then S is homeomorphic to the space obtained from identifying a polygon's edges in pairs, by Corollary 4.4. Now that we have proven Corollaries 4.17, 4.24, and Lemma 4.25 of the previous section,

we are free in this section to focus on the conceptual aspects of the proof, while maintaining the rigor.

Theorem 5.1. *Let X be a closed, compact, triangulable surface. Then X is homeomorphic to one of the standard types of surfaces:*

(1) aa^-, the sphere
(2) aa, the projective plane
(3) $(a_1b_1a_1^-b_1^-)...(a_nb_na_n^-b_n^-)$, the n-fold torus
(4) $(a_1a_1)...(a_ma_m)$, the m-fold projective plane

Proof. X is a closed, compact surface, so by Corollary 4.4, X is homeomorphic to a polygon with a proper labeling scheme w. Let n denote the length of w. The length of w will always be even, as w is a proper scheme.
Consider the case in which $n = 2$. If w is orientable, then $w \sim aa^-$, and X is homeomorphic to a sphere, type (1). Otherwise, if w is non-orientable, then $w \sim aa$ after relabeling, and X is homeomorphic to a projective plane, type (2).
For $n \geq 4$, we can apply Lemma 4.17 that shows that $w \sim (a_1b_1a_1^-b_1^-)...(a_nb_na_n^-b_n^-)y$ where y is either a proper non-orientable labeling scheme or empty. If w is orientable, then y must be empty, and so X an n-fold torus, type (3).
If w is non-orientable, apply Lemma 4.24 which shows that
$w \sim (a_1a_1)...(a_ia_i)(a_1b_1a_1^-b_1^-)...(a_nb_na_n^-b_n^-)$. Then, by repeatedly applying Lemma 4.25, we see that $w \sim (a_1a_1)...(a_ma_m)$, the m-fold projective plane, and X is type (4). □

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References

[1] Hatcher, Allen. Algebraic Topology. Cambridge University Press, 2017.
[2] Gallier, Jean, and Dianna Xu. A Guide to the Classification Theorem for Compact Surfaces. Springer-Verlag Berlin An, 2015.
[3] Armstrong, M. A. Basic Topology. Springer, 2011.
[4] Breen, Casey. The Classification of Surfaces. http://math.uchicago.edu/ may/REU2017/REUPapers/Breen-Edelstein.pdf