CUBICAL STRUCTURES OF MULTI-FOLD INTERNAL CATEGORIES

INDRANEEL TAMBE

Abstract. Internal categories and internal functors are objects designed to abstract the notion of categories and functors within an ambient category $\mathcal{C}$. This leads to a category $\text{Cat}^\mathcal{C}$ of internal categories in $\mathcal{C}$. The $\text{Cat}$ procedure can then be iterated further, yielding structures called "$n$-fold internal categories" that can be interpreted as slight weakenings of standard strict higher categories, "cubically"-shaped in contrast to the "globular" shape of a standard strict higher category. In this paper we look at the concept of a "cubical internal $n$-category", which captures the structure of $n$-fold internal categories but makes the "cubical" behavior of the latter more apparent. These also generalize standard strict cubical higher categories in that a cubical internal $n$-category in $\text{Set}$ is equivalent to a "standard" cubical $n$-category.

Contents

1. Introduction and definitions 1
2. Double categories: internal categories in $\text{Cat}$ 4
3. Composition of 2-arrows in double categories 7
4. Cubical internal categories 9
5. Interpretation of cubical internal categories 14
Acknowledgments 15
References 16

1. Introduction and definitions

The idea of an internal category is to abstract the definition of a category $C$ by collecting the objects of $C$ into a single object $C_0$, the morphisms of $C$ into a single object $C_1$, and to capture all the categorical structure of $C$ with morphisms involving $C_0$ and $C_1$, and to do this all in the context of an ambient category $\mathcal{C}$.

We begin with the definition of an internal category. This definition follows that found in [1], though it is standard.

Definition 1.1. Let $\mathcal{C}$ be a category, assumed to admit all limits used in the definition. Then an internal category $C$ in $\mathcal{C}$ consists of the following data.

- Two objects $C_1, C_0$ in $\mathcal{C}$.
- Four $\mathcal{C}$-morphisms $s, t : C_1 \to C_0$ and $e : C_0 \to C_1$ and $m : C_1 \times_{C_0} C_1 \to C_1$.

Date: August 26, 2018.
Here \( C_1 \times C_0 \) denotes the pullback of the diagram

\[
\begin{array}{ccc}
C_1 & \xrightarrow{s} & C_0 & \xrightarrow{t} & C_1;
\end{array}
\]

this notation will be used throughout this section.

The above data is required to satisfy the following axioms (and all limits used in this definition are required to exist).

1. (Unit and composition coherence) The following diagrams commute:

\[
\begin{array}{ccc}
C_0 & \xrightarrow{s} & C_1 & \xrightarrow{t} & C_0; \\
\downarrow{id} & & \downarrow{c} & & \downarrow{id} \\
C_0 & \xrightarrow{s} & C_1 & \xrightarrow{t} & C_0.
\end{array}
\]

\[
\begin{array}{ccc}
C_0 \times C_1 & \xrightarrow{p_2} & C_1 \times C_0 & \xrightarrow{p_1} & C_1 \\
\downarrow{m \times C_0 id} & & \downarrow{id} & & \downarrow{id} \\
C_1 \times C_0 & \xrightarrow{m} & C_1 & \xrightarrow{s, id} & C_0.
\end{array}
\]

2. (Associativity and identity laws) The following diagrams commute:

\[
\begin{array}{ccc}
C_1 \times C_0 C_1 & \xrightarrow{id \times C_0 m} & C_1 \times C_0 C_1 & \xrightarrow{m \times C_0 id} & C_1 \times C_0 \times C_1 \\
\downarrow{id \times C_0 e} & & \downarrow{id} & & \downarrow{id} \\
C_1 \times C_0 C_0 & \xrightarrow{m} & C_1 & \xrightarrow{s, id} & C_0.
\end{array}
\]

Here \( C_0 \times id, C_1 \) and \( C_1 \times s, id \) respectively denote the pullbacks of the two diagrams

\[
\begin{array}{ccc}
C_0 & \xrightarrow{id} & C_0 & \xrightarrow{id} & C_1 \\
C_1 & \xrightarrow{s} & C_0 & \xrightarrow{id} & C_0,
\end{array}
\]

and \( e \times C_0, s \) and \( id \times t, C_0 e \) are defined as the morphisms induced via the two diagrams

\[
\begin{array}{ccc}
C_0 \times id, C_1 & \xrightarrow{p_2} & C_1 \times C_0 C_1 & \xrightarrow{p_1} & C_1 \times C_0 \times C_1 \\
\downarrow{p_2} & & \downarrow{t} & & \downarrow{s, id} \\
C_0 & \xrightarrow{e} & C_1 & \xrightarrow{s} & C_0 \\
C_1 & \xrightarrow{t} & C_0 & \xrightarrow{e} & C_0.
\end{array}
\]

which both commute by Axiom 1 and by the definitions of \( C_0 \times id, C_1 \) and \( C_1 \times s, id \) of \( C_0 \).

The above definition is meant to capture the essential structure of a category \( C \), while using an “ambient category” \( C \) in which to encapsulate the structural maps of \( C \)—hence, an “internal category.” In the case \( C = \text{Set} \), internal categories in \( C \) coincide exactly with small categories.

In what follows, we will often say a category \( C \) has enough limits if \( C \) admits all the limits that are used in any further constructs involving \( C \).

We now give the intuitive interpretations of the data and axioms defining internal categories in the case when the ambient category \( C \) is concrete. The objects \( C_1 \) and \( C_0 \) are to be thought of as the “set of arrows of \( C \)” and the “set of objects of \( C \)” respectively. Then, the morphisms \( s \) and \( t \) can be thought of as “source” and “target” maps, sending an arrow of \( C \) to the object in \( C_0 \) it originates from and to the object in \( C_0 \) it points toward, respectively. The morphism \( e \) can be thought
of as the map sending an object in \( C_0 \) to the identity arrow on that object. The object \( C_1 \times C_0 C_1 \) then is the set of pairs of “composable” arrows—that is, pairs of arrows such that the source of the first is the target of the second, and \( m \) gives the multiplication or composition map. (Note that composition is performed “right-to-left,” as is the usual convention.)

The unit coherence requirement enforces that the identity arrow on any object \( c \) of \( C \) does indeed have \( c \) as its source and target. The composition coherence requirement enforces that the composition of two arrows has the correct source and target: specifically, the source of \( m(g, f) \) should be the source of \( f \), and its target should be the target of \( g \). Finally, the unit and associativity laws simply enforce the usual associativity and identity laws for categories.

We can now define internal functors, which are exactly analogous to the usual functors of categories.

**Definition 1.2.** Let \( \mathcal{C} \) be a category with enough limits, and let \( C \) and \( D \) be internal categories in \( \mathcal{C} \). (Denote the structural morphisms of \( C \) as \( s^C, t^C, e^C \), and similarly for \( D \).) Then an **internal functor** \( F : C \rightarrow D \) in \( \mathcal{C} \) consists of two \( \mathcal{C} \)-morphisms \( F_1 : C_1 \rightarrow C_1 \) and \( F_0 : C_0 \rightarrow C_0 \), satisfying the following axioms (and with all limits used in this definition assumed to exist).

1. (Functorial coherence) The following diagram commutes:

\[
\begin{array}{ccc}
C_0 & \xrightarrow{s^C} & C_1 \\
\downarrow{F_0} & & \downarrow{F_1} \\
D_0 & \xrightarrow{s^D} & D_1 \\
\end{array}
\]

2. (Preservation of identity and composition) The following two diagrams commute:

\[
\begin{array}{ccc}
C_0 & \xrightarrow{e^C} & C_1 \\
\downarrow{F_0} & & \downarrow{F_1} \\
D_0 & \xrightarrow{e^D} & D_1 \\
\end{array} \quad \quad \begin{array}{ccc}
C_1 \times C_0 C_1 & \xrightarrow{m^C} & C_1 \\
\downarrow{F_1 \times C_0 F_1} & & \downarrow{F_1} \\
D_1 \times D_0 D_1 & \xrightarrow{m^D} & D_1 \\
\end{array}
\]

Here \( F_1 \times C_0 F_1 \) is defined as the morphism induced via the following diagram:

\[
\begin{array}{ccc}
C_1 & \xrightarrow{F_1} & D_1 \\
\downarrow{p_1} & & \downarrow{s^D} \\
C_1 \times C_0 C_1 & \xrightarrow{F_1 \times C_0 F_1} & D_1 \times D_0 D_1 \\
\downarrow{p_2} & & \downarrow{t^D} \\
C_1 & \xrightarrow{F_1} & D_1 \\
\end{array}
\]

which commutes by Axiom 1 and the definitions of \( C_1 \times C_0 C_1 \) and \( D_1 \times D_0 D_1 \).

In the case when \( \mathcal{C} \) is a concrete category, the definition of an internal functor \( F : C \rightarrow D \) in \( \mathcal{C} \) as given above can be interpreted as follows. The morphisms \( F_1 \) and \( F_0 \) give the “arrow function” and “object function” of the functor \( F \). The
functorial coherence axiom enforces that the arrow function $F_1$ takes an arrow $f : x \to y$ in the category $C$ to an arrow $F_1(f) : F_0(x) \to F_0(y)$, and the second axiom simply enforces that the functor $F$ preserves identity arrows and preserves composition of arrows.

Now let $\mathcal{C}$ be a category with enough limits. Let $X, Y, Z$ be internal categories in $\mathcal{C}$, and let $F : X \to Y$ and $G : Y \to Z$ be internal functors. We can define the composition $G \circ F$ as the internal functor $X \to Z$ whose object function is the composition $G_0 \circ F_0$ and whose arrow function is the composition $G_1 \circ F_1$; it can be verified that the resulting pair of morphisms is an internal functor as defined above. It is also true that the composition of internal functors defined in this way is associative; this follows directly from associativity of composition of morphisms in $\mathcal{C}$. And finally, we can define the internal identity functor on $X$, whose object function is the identity morphism on $C_0$ and whose arrow function is the identity morphism on $C_1$.

Armed with the above definitions and claims, we arrive at the following definition:

**Definition 1.3.** Let $\mathcal{C}$ be a category. Then we define the category $\text{Cat}\mathcal{C}$ whose objects are the internal categories of $\mathcal{C}$ and whose morphisms are the internal functors of $\mathcal{C}$. The composition of morphisms in $\text{Cat}\mathcal{C}$, and the identity morphisms of objects in $\text{Cat}\mathcal{C}$, are given by the composition of internal functors and the identity internal functors, defined above.

For example, $\text{CatSet}$ is exactly the same as $\text{Cat}$, the category of small categories (whose morphisms are functors between small categories).

Having defined $\text{Cat}\mathcal{C}$ for any category $\mathcal{C}$ with enough limits, there is nothing stopping us from iterating this procedure further and taking $\text{Cat}^n\mathcal{C}$ for any positive integer $n$. The objects of $\text{Cat}^n\mathcal{C}$ will be called $n$-fold internal categories in $\mathcal{C}$, or multi-fold internal categories in $\mathcal{C}$ for general $n$.

Though their definition is easy to write down, multi-fold internal categories contain much internal structure interacting in complicated ways; however, as will be explored in what follows, they can be visually represented with multidimensional cubes of arrows. This motivates the definition of cubical internal categories explored later in this paper, which is essentially a redefinition of multi-fold internal categories that reorganizes the data and axioms in a way that makes the “cubical” behavior apparent.

2. **Double categories: internal categories in $\text{Cat}$**

In order to motivate the definition of a cubical category given later, we first investigate 2-fold internal categories in $\text{Set}$. Note 2-fold internal categories in $\text{Set}$ are just internal categories in (the category of internal categories in $\text{Set}$), which are canonically equivalent to just internal categories in $\text{Cat}$, where $\text{Cat}$ is the category of (small) categories.

Internal categories in $\text{Cat}$ are also referred to as “double categories”; see for example [2], section 1 for definitions.

Let $C$ be a 2-fold internal category in $\text{Set}$, then $C$’s “arrow object” $C_1$ and “object object” $C_0$ are themselves (small) categories, so denote the arrow set and
A diagram similar to the one above commutes for all the target functions in $C$ mixing the source and target functions:

$$
\begin{array}{ccc}
C_{11} & \xrightarrow{S_1} & C_{10} \\
\downarrow{s_1} & & \downarrow{s_0} \\
C_{01} & \xrightarrow{S_0} & C_{00}
\end{array}
$$

A diagram similar to the one above commutes for all the target functions in $C$; we will denote the target functions as $T_1, T_0, t_1, t_0$, in a manner analogous to the source functions.

Also, functoriality of $S$ and $T$ together give some more commutative diagrams mixing the source and target functions:

$$
\begin{array}{ccc}
C_{11} & \xrightarrow{T_1} & C_{10} \\
\downarrow{s_1} & & \downarrow{s_0} \\
C_{01} & \xrightarrow{T_0} & C_{00}
\end{array}
\quad \quad
\begin{array}{ccc}
C_{11} & \xrightarrow{S_1} & C_{10} \\
\downarrow{t_1} & & \downarrow{t_0} \\
C_{01} & \xrightarrow{S_0} & C_{00}
\end{array}
$$

Next, let us look at the composition functions in $C$. As part of the data we are given a Cat$^1$-Set-morphism, in other words a (small) functor, $S : C_1 \to C_0$, which gives an arrow function $S_1 : C_{11} \to C_{01}$ and an object function $S_0 : C_{10} \to C_{00}$. At the same time, $C_1$ has its own source function $s_1 : C_{11} \to C_{10}$, and $C_0$ has its own function $s_0 : C_{01} \to C_{00}$. The functoriality of $S$ then implies the following diagram is commutative:

$$
\begin{array}{ccc}
C_{11} & \xrightarrow{S_1} & C_{10} \\
\downarrow{s_1} & & \downarrow{s_0} \\
C_{01} & \xrightarrow{S_0} & C_{00}
\end{array}
$$

A diagram similar to the one above commutes for all the target functions in $C$; we will denote the target functions as $T_1, T_0, t_1, t_0$, in a manner analogous to the source functions.

Also, functoriality of $S$ and $T$ together give some more commutative diagrams mixing the source and target functions:

$$
\begin{array}{ccc}
C_{11} & \xrightarrow{T_1} & C_{10} \\
\downarrow{s_1} & & \downarrow{s_0} \\
C_{01} & \xrightarrow{T_0} & C_{00}
\end{array}
\quad \quad
\begin{array}{ccc}
C_{11} & \xrightarrow{S_1} & C_{10} \\
\downarrow{t_1} & & \downarrow{t_0} \\
C_{01} & \xrightarrow{S_0} & C_{00}
\end{array}
$$

At this point it would be natural to try to view $C$ as some kind of (small) 2-category. Clearly $C$ is not a weak 2-category, because all the composition maps are required to be strictly associative. The set $C_{00}$ can be the set of objects, since it has no composition map, and $C_{11}$ can be the set of 2-morphisms, since it has two different types of composition $M_1$ and $m_1$.

However, it is yet unclear what should be considered the set of 1-morphisms, as there are two potential candidates, $C_{10}$ and $C_{01}$. Both of these sets have their “source” and “target” lying in $C_{00}$, and each has exactly one composition map, whose domain is a pullback over the object set $C_{00}$. Furthermore, each has identity arrows for $C_{00}$ (there is a unit function $e_0 : C_{00} \to C_{01}$, and also a unit function $E_0 : C_{00} \to C_{10}$ obtained as the object function from the small functor / Cat-morphism $E : C_0 \to C_1$ arising from the data of the structure on $C_1, C_0$ of an internal category in Cat) satisfying all the usual requirements. In fact these two sets of 1-morphisms have essentially identical structure and behavior, but they are also completely separate—there is no composition.

To examine the behavior of double categories / 2-fold internal categories further, let us combine the four commutative diagrams explained above (the one involving only target functions was not shown, and is exactly analogous to that involving
only source functions) into the following commutative diagram:

\[
\begin{array}{c}
\text{C}_{00} \xleftarrow{s_0} \text{C}_{10} \xrightarrow{t_1} \text{C}_{00} \\
\downarrow{S_0} \quad \downarrow{S_1} \quad \downarrow{S_0} \\
\text{C}_{01} \xleftarrow{s_1} \text{C}_{11} \xrightarrow{t_0} \text{C}_{01} \\
\downarrow{T_0} \quad \downarrow{T_1} \quad \downarrow{T_0} \\
\text{C}_{00} \xleftarrow{s_0} \text{C}_{10} \xrightarrow{t_1} \text{C}_{00}.
\end{array}
\]

We will refer to this combined commutative diagram involving all the possible compositions of all the source and target functions in \( C \) as the “cubical coherence” diagram. It suggests that we can view the source-target relations among the different types of objects in \( C \) with the following picture:

\[
\begin{array}{c}
| & | \\
\bullet & \bullet \\
\downarrow & \downarrow \\
\bullet & \bullet \\
\uparrow & \uparrow \\
| & |
\end{array}
\]

In the above picture, an element \( u \) of \( C_{11} \) is represented as a double-lined arrow. Given the one \( u \), by the cubical coherence diagram there are up to 4 possibly distinct ways of taking pairs of source and/or target functions of \( u \), leading to the four vertices, which are elements of \( C_{00} \). However, since \( C_{11} \) has two possible source functions and two possible target functions, this gives four possible 1-morphisms, but the horizontal ones are elements of \( C_{10} \) while the vertical ones are elements of \( C_{01} \). These can be treated as two separate types of 1-morphisms in our theory: each object in \( C_{00} \) has an identity arrow in \( C_{10} \) and also one in \( C_{01} \), and the arrows of \( C_{10} \) and \( C_{01} \) can both be composed in identical but separate ways along objects of \( C_{00} \). The definitions do not give any method of composing elements of \( C_{10} \) and \( C_{01} \) together, and so with our above visualization, we need only compose 1-arrows only horizontally or only vertically separately. We can then call the horizontal arrows \( S_1 u \) and \( T_1 u \) the “horizontal source and target” of the 2-morphism \( u \), and we can call the vertical arrows \( s_1 u \) and \( t_1 u \) the “vertical source and target” of \( u \).

Of course, all these choices of directions are arbitrary conventions. Furthermore, giving the 2-morphism \( u \) a particular direction is somewhat misleading—or rather, giving it only one direction is, and instead \( u \) could be represented e.g. as a square surface with a choice of direction and orientation, analogously to how the 1-arrows are represented as oriented 1-surfaces. This hints at the higher-dimensional visualization scheme we will explore later.

We have seen that the compositions of the different types of 1-arrows, in \( C_{10} \) and \( C_{01} \), behave in essentially the same way as composition of arrows in a standard 1-category. However, we still have not explored the two different possible compositions \( M_1 \) and \( n_1 \) on \( C_{11} \) and their interactions with sources, targets, and other compositions; we turn to this next.
3. Composition of 2-arrows in double categories

Let us now investigate the two different compositions on $C_{11}$ (the set of 2-arrows of $C$ a double category, viewed as a 2-fold internal category in Set) and determine how they can be incorporated into our visualization scheme and what kind of associativity laws they obey.

Further commutative diagrams can be derived involving $M_1$ and $m_1$, which are the two possible composition functions for 2-morphisms in $C_{11}$, and the source and target functions. We will give the more general versions of these diagrams later in the definition of cubical internal categories; essentially they follow from functoriality of $S$ and $T$. Here we will first explain how we can visualize their content.

The composition $M_1$ becomes composition of 2-arrows along horizontal 1-arrows, and according to the commutative diagrams for $M_1$, the vertical source of the $M_1$-composition of two 2-morphisms will be the composition of their individual vertical sources, and similarly for the vertical targets. This bears similarity to the “horizontal composition” of two 2-morphisms in a (“standard”, globularly-shaped strict) 2-category. However, $m_1$ behaves in exactly the same way, but flipped: $m_1$ is composition of 2-arrows along vertical 1-arrows, and the horizontal source and target of the $m_1$-composition of two 2-morphisms will be the composition of their individual horizontal sources.

So unlike in a “standard” globularly-shaped strict 2-category, the two types of composition of 2-arrows in $C$ are exactly analogous to each other. Visually, the two compositions $M_1$ and $m_1$ can be represented thus, respectively:

In the above, the appropriate compatibility conditions for 2-arrows to be composed are represented by “gluing” the squares for the arrows along the corresponding 1-faces.

We see the two different ways of composing 2-arrows in a 2-fold internal category are on very equal footing and arise in an easily visualized geometric way: they simply correspond to composition along the two different types (“vertical” and “horizontal”) of 1-arrows.

Having examined the basic behavior of the two compositions on $C_{11}$, we now turn to questions of their associativity. Each composition $M_1$ and $m_1$ is associative on its own (which is required already from the definition of an internal category). However, there is also a sort of associativity between the two compositions as well; algebraically this can be shown through commutative diagrams arising from the functoriality of $M$, and a more general version will be given later.
Here we will first explain how to visualize this type of associativity, which we call multi-associativity.

Given four elements of $C_{11}$, such that two pairs are horizontally compatible and the crossed pairs are vertically compatible, we can first vertically compose the vertically compatible pairs then horizontally compose the resulting 2-arrows, or we can first horizontally compose the the horizontally compatible pairs and then vertically compose the resulting 2-arrows. The multi-associativity of the two compositions on $C_{11}$ states that the result of these two possible ways of composing all four 2-arrows must be the same.

We can view the four multi-compatible arrows as fitting together in the following diagram:

```
• /•
  • /•
  • /•
• /•
```

The multi-associativity of 2-arrows in $C$ can then be described as follows: the possible compositions of any four 2-morphisms fitting together compatibly as in the above diagram can be performed in any order (either a pair of vertical compositions then a horizontal one, or a pair of horizontal compositions then a vertical one) and should both give the same result.

Thus we have a full visualization scheme for a 2-fold internal category $C$ in Set, which we now summarize. The elements of $C_{00}$ are represented as vertices, and the elements of $C_{10}$ are represented as horizontal arrows and the elements of $C_{01}$ are represented as vertical arrows, between vertices. The horizontal and vertical arrows can be composed associatively in strings, but horizontal and vertical arrows cannot be composed with each other.

The elements of $C_{11}$ are represented as oriented squares bound by a square of 1-arrows, with the vertical pairs and horizontal pairs aligned in the same direction. Then the different compositions of elements of $C_{11}$ are represented by “stacking” together squares in the two different possible directions. Furthermore, the “multi-associativity” of 2-arrow composition means that we can build any rectangular shape of 2-arrow squares, and the full sequence of compositions can be done in any order and must give the same result.

We see now that the algebraic structure of a double category is somewhat like a 2-category, but not quite. It is not a weak category, because all the compositions are required to be associative. However, it is not the same is a strict 2-category; instead of higher morphisms having a globular shape, as strict higher categories do, the higher morphisms of a multi-fold internal category have a cubical shape. Thus a double category presents a particular loosening of the notion of standard “globularly shaped” strict higher category, but in a way that is still “strict” in that associativity and identity for composition are still strict.

Having understood the behavior of 2-fold internal categories (in Set), we may wish to proceed to higher multi-fold internal categories. However, we must organize
all the data these contain, and the notation given by the multi-fold internal category structure not only is cumbersome but also obscures the “cubical” behavior and visualization of multi-fold internal categories.

Therefore, in the following section we will present the concept of a cubical internal n-category, which is essentially equivalent to an n-fold internal category but organizes the data in a way that clearly shows the different dimensions and types of morphisms of a multi-fold internal category and shows its cubical behavior.

4. Cubical internal categories

Here we describe a concept we call a cubical internal n-category. This structure is essentially equivalent to an n-fold internal category; however, the definition described below is more amenable to the visualization and understanding of multi-fold internal categories.

We note in the literature there is also the notion of a “(strict) cubical n-category”, which (as in [3], section 1) can be defined as an ordinary category enriched over the category of cubical sets (and “truncated at dimension n”), similarly to how strict globular n-categories are defined as ordinary categories enriched over the category of globular sets (and “truncated at dimension n”). A cubical n-category in this sense is equivalent to what is defined below as a cubical internal n-category in Set.

Notation 4.1. For a nonnegative integer N, we will denote \( \overline{N} := \{0, 1, 2, \ldots, N\} \).

Definition 4.2. Let \( N \) be a nonnegative integer, and let \( \mathcal{C} \) be a category. Then a cubical internal \( N \)-category \( \Theta \) in \( \mathcal{C} \) consists of the following data.

- For each \( A \subseteq \overline{N} \), an object \( \Theta(A) \) of \( \mathcal{C} \).
- For each \( A \subseteq \overline{N} \) with \( A \neq \emptyset \), for each \( a \in A \), two \( \mathcal{C} \)-morphisms \( s_a^{(\Theta, A)} : \Theta(A) \longrightarrow \Theta(A \setminus \{a\}) \).
- For each \( A \subseteq \overline{N} \) with \( A \neq \overline{N} \), for each \( x \in \overline{N} \setminus A \), a \( \mathcal{C} \)-morphism \( e_x^{(\Theta, A)} : \Theta(A) \longrightarrow \Theta(A \cup \{x\}) \).
- For each \( A \subseteq \overline{N} \) with \( A \neq \emptyset \), for each \( a \in A \), a \( \mathcal{C} \)-morphism \( m_a^{(\Theta, A)} : \Theta(A) \times_a \Theta(A) \longrightarrow \Theta(A) \), where \( \Theta(A) \times_a \Theta(A) \) denotes the pullback of the diagram

\[
\begin{array}{c}
\Theta(A) \quad \Theta(A) \setminus \{a\} \\
\downarrow s_a \quad \downarrow t_a \\
\Theta(A) \setminus \{a\} \quad \Theta(A) \quad \Theta(A) \setminus \{a\}
\end{array}
\]

(which is a notation that will be used throughout this section).

The above data is required to satisfy the following axioms, and all limits used in this definition are required to exist. Note: to ease notation, we will often leave off the superscripts on the structural morphisms defined above.

(1) (Lateral unit and composition coherence) For all \( A \subseteq \overline{N} \setminus \{\emptyset\} \), for all \( a \in A \), the following diagrams commute:

\[
\begin{array}{c}
\Theta(A \setminus \{a\}) \\
\Theta(A) \quad \Theta(A \setminus \{a\}) \\
\downarrow \downarrow \\
\Theta(A \setminus \{a\}) \quad \Theta(A)
\end{array}
\]

\[
\begin{array}{c}
\Theta(A \setminus \{a\}) \\
\Theta(A) \quad \Theta(A \setminus \{a\}) \\
\downarrow \downarrow \\
\Theta(A \setminus \{a\}) \quad \Theta(A)
\end{array}
\]

\[
\begin{array}{c}
\Theta(A \setminus \{a\}) \\
\Theta(A) \quad \Theta(A \setminus \{a\}) \\
\downarrow \downarrow \\
\Theta(A \setminus \{a\}) \quad \Theta(A)
\end{array}
\]

\[
\begin{array}{c}
\Theta(A \setminus \{a\}) \\
\Theta(A) \quad \Theta(A \setminus \{a\}) \\
\downarrow \downarrow \\
\Theta(A \setminus \{a\}) \quad \Theta(A)
\end{array}
\]

\[
\begin{array}{c}
\Theta(A \setminus \{a\}) \\
\Theta(A) \quad \Theta(A \setminus \{a\}) \\
\downarrow \downarrow \\
\Theta(A \setminus \{a\}) \quad \Theta(A)
\end{array}
\]

\[
\begin{array}{c}
\Theta(A \setminus \{a\}) \\
\Theta(A) \quad \Theta(A \setminus \{a\}) \\
\downarrow \downarrow \\
\Theta(A \setminus \{a\}) \quad \Theta(A)
\end{array}
\]

\[
\begin{array}{c}
\Theta(A \setminus \{a\}) \\
\Theta(A) \quad \Theta(A \setminus \{a\}) \\
\downarrow \downarrow \\
\Theta(A \setminus \{a\}) \quad \Theta(A)
\end{array}
\]

\[
\begin{array}{c}
\Theta(A \setminus \{a\}) \\
\Theta(A) \quad \Theta(A \setminus \{a\}) \\
\downarrow \downarrow \\
\Theta(A \setminus \{a\}) \quad \Theta(A)
\end{array}
\]

\[
\begin{array}{c}
\Theta(A \setminus \{a\}) \\
\Theta(A) \quad \Theta(A \setminus \{a\}) \\
\downarrow \downarrow \\
\Theta(A \setminus \{a\}) \quad \Theta(A)
\end{array}
\]

\[
\begin{array}{c}
\Theta(A \setminus \{a\}) \\
\Theta(A) \quad \Theta(A \setminus \{a\}) \\
\downarrow \downarrow \\
\Theta(A \setminus \{a\}) \quad \Theta(A)
\end{array}
\]
and

\[
\begin{array}{c}
\Theta(A) \xleftarrow{p_2} \Theta(A) \times_a \Theta(A) \xrightarrow{p_1} \Theta(A) \\
\downarrow{s_a} \quad \downarrow{m_a} \\
\Theta(A\setminus\{a\}) \xleftarrow{s_a} \Theta(A) \xrightarrow{t_a} \Theta(A\setminus\{a\}).
\end{array}
\]

(2) (Lateral associative and unit laws) For all \( A \subseteq N \setminus \{\emptyset\} \), for all \( a \in A \), the following diagram commutes:

\[
\begin{array}{c}
\Theta(A) \times_a \Theta(A) \times_a \Theta(A) \xrightarrow{id \times_a m_a} \Theta(A) \times_a \Theta(A) \\
\downarrow{m_a \times_a id} \quad \downarrow{m_a} \\
\Theta(A) \times_a \Theta(A) \xrightarrow{m_a} \Theta(A).
\end{array}
\]

and

\[
\begin{array}{c}
\Theta(A\setminus\{a\}) \times_{id,t} \Theta(A) \xrightarrow{e_a \times_{a,s} id} \Theta(A) \times_{a,s} \Theta(A) \xleftarrow{id \times_{a,t} e_a} \Theta(A) \times_{s,id} \Theta(A\setminus\{a\}) \\
\downarrow{p_1} \quad \downarrow{p_2} \\
\Theta(A\setminus\{a\}) \xleftarrow{p_2} \Theta(A).
\end{array}
\]

Here \( \Theta(A\setminus\{a\}) \times_{id,t} \Theta(A) \) and \( \Theta(A) \times_{s,id} \Theta(A\setminus\{a\}) \) respectively denote the pullbacks of the two diagrams

\[
\begin{array}{c}
\Theta(A\setminus\{a\}) \xrightarrow{id} \Theta(A\setminus\{a\}) \xleftarrow{t_a} \Theta(A) \\
\downarrow{s_a} \quad \downarrow{id} \\
\Theta(A) \xrightarrow{s_a} \Theta(A\setminus\{a\}) \xleftarrow{id} \Theta(A\setminus\{a\}).
\end{array}
\]

and \( e_a \times_{a,s} id \) and \( id \times_{a,t} e_a \) are defined as the morphisms induced via the two diagrams

\[
\begin{array}{c}
\Theta(A\setminus\{a\}) \times_{id,t} \Theta(A) \xrightarrow{e_a \times_{a,s} id} \Theta(A) \times_{a,s} \Theta(A) \xleftarrow{id \times_{a,t} e_a} \Theta(A) \times_{s,id} \Theta(A\setminus\{a\}) \\
\downarrow{p_1} \quad \downarrow{p_2} \\
\Theta(A\setminus\{a\}) \xleftarrow{p_2} \Theta(A) \xrightarrow{s_a} \Theta(A\setminus\{a\})
\end{array}
\]

and

\[
\begin{array}{c}
\Theta(A) \xleftarrow{p_1} \Theta(A) \times_a \Theta(A) \xrightarrow{id \times_{a,t} e_a} \Theta(A) \times_{s,id} \Theta(A\setminus\{a\}) \\
\downarrow{s_a} \quad \downarrow{p_2} \\
\Theta(A\setminus\{a\}) \xleftarrow{t_a} \Theta(A) \xrightarrow{e_a} \Theta(A\setminus\{a\}).
\end{array}
\]

which both commute by Axiom 1 and by the definitions of \( \Theta(A\setminus\{a\}) \times_{id,t} \Theta(A) \) and \( \Theta(A) \times_{s,id} \Theta(A\setminus\{a\}) \).
(3) (Cubical coherence) For all $A \subseteq \overline{N}$ such that $|A| \geq 2$, and for all $a, b \in A$ such that $a \neq b$, the following diagram commutes:

$$
\begin{array}{c}
\Theta(A \setminus \{a, b\}) \xrightarrow{s_b} \Theta(A \setminus \{a\}) \xrightarrow{t_b} \Theta(A \setminus \{a, b\}) \\
\downarrow s_a \quad \downarrow s_a \quad \downarrow s_a \\
\Theta(A \setminus \{b\}) \xrightarrow{s_b} \Theta(A) \xrightarrow{t_b} \Theta(A \setminus \{b\}) \\
\downarrow t_a \quad \downarrow t_a \quad \downarrow t_a \\
\Theta(A \setminus \{a, b\}) \xrightarrow{s_b} \Theta(A \setminus \{a\}) \xrightarrow{t_b} \Theta(A \setminus \{a, b\}).
\end{array}
$$

(4) (Multi-coherence of unit and composition) For all $A \subseteq \overline{N}$ such that $|A| \geq 2$, and for all $a, b \in A$ such that $a \neq b$, the following three diagrams commute:

$$
\begin{array}{c}
\Theta(A \setminus \{b\}) \times_a \Theta(A \setminus \{b\}) \xrightarrow{S_b} \Theta(A) \times_a \Theta(A) \xrightarrow{T_b} \Theta(A \setminus \{b\}) \times_a \Theta(A \setminus \{b\}) \\
\downarrow m_a \quad \downarrow m_a \quad \downarrow m_a \\
\Theta(A \setminus \{b\}) \xrightarrow{s_b} \Theta(A) \xrightarrow{t_b} \Theta(A \setminus \{b\}),
\end{array}
$$

$$
\begin{array}{c}
\Theta(A \setminus \{a, b\}) \xrightarrow{s_a} \Theta(A \setminus \{a\}) \xrightarrow{t_a} \Theta(A \setminus \{a, b\}) \\
\downarrow e_b \\
\Theta(A \setminus \{b\}) \xrightarrow{s_b} \Theta(A) \xrightarrow{t_b} \Theta(A \setminus \{b\}),
\end{array}
$$

$$
\begin{array}{c}
\Theta(A \setminus \{b\}) \times_a \Theta(A \setminus \{b\}) \xrightarrow{E_b} \Theta(A) \times_a \Theta(A) \\
\downarrow m_a \quad \downarrow m_a \\
\Theta(A \setminus \{b\}) \xrightarrow{e_b} \Theta(A).
\end{array}
$$

Here $S_b, T_b, E_b$ are defined as the morphisms induced via the following three diagrams:
and

\[
\begin{align*}
\Theta(A \setminus \{b\}) & \xrightarrow{e_b} \Theta(A) \\
\Theta(A \setminus \{b\}) \times_A \Theta(A \setminus \{b\}) & \xrightarrow{E_b} \Theta(A \setminus \{b\}) \times_A \Theta(A \setminus \{a\}) \\
\Theta(A \setminus \{b\}) & \xrightarrow{e_b} \Theta(A)
\end{align*}
\]

which all commute by Axiom 3 and the definition of $\Theta(A) \times_A \Theta(A)$.

(5) (Multi-associativity of $m$) For all $A \subseteq N$ such that $|A| \geq 2$, and for all $a, b \in A$ such that $a \neq b$, the following diagrams commute:

\[
\begin{align*}
\Xi & \xrightarrow{P_R} \Theta(A) \times_A \Theta(A) \\
\Xi & \xrightarrow{P_D} \Theta(A) \times_A \Theta(A)
\end{align*}
\]

Here $\Xi$ is defined as the limit of the diagram given below, which does not generally commute:

\[
\begin{align*}
\Theta(A) & \xrightarrow{t_b} \Theta(A \setminus \{b\}) \xleftarrow{s_b} \Theta(A) \\
\Theta(A \setminus \{a\}) & \xrightarrow{t_b} \Theta(A \setminus \{a, b\}) \xleftarrow{s_b} \Theta(A \setminus \{a\}) \\
\Theta(A) & \xrightarrow{t_b} \Theta(A \setminus \{b\}) \xleftarrow{s_b} \Theta(A)
\end{align*}
\]

and $P_L, P_R, P_U, P_D$ are defined as the morphisms induced via the two diagrams.
and

\[ \Theta(A) \xrightarrow{t_b} \Theta(A \setminus \{b\}) \xleftarrow{s_b} \Theta(A) \]

\[ \Theta(A) \times_b \Theta(A) \]

\[ \Theta(A) \xrightarrow{t_b} \Theta(A \setminus \{b\}) \xleftarrow{s_b} \Theta(A) \]

which both commute by Axioms 3 and 4 and the definition of \( \Theta(A) \times_n \Theta(A) \).

(Note that the four “corner” objects \( \Theta(A) \) in both of the above two diagrams defining \( P_L, P_R, P_U, P_D \) correspond exactly to the corner objects in the diagram defining \( \Xi \) above.)

In the above definition, Axiom 4 can be interpreted as asserting functoriality of \( s, t, e \) at all dimensions, and similarly Axiom 5 can be interpreted as asserting functoriality of \( m \) similarly. But the names given here for the Axioms match more closely with the discussion of the previous section.

Let us use the following terms:

**Notation 4.3.** Let \( N \) be a nonnegative integer, and let \( \Theta \) be an cubical internal \( N \)-category in a category \( C \) having enough limits. For a subset \( A \subseteq N \) of cardinality \( k \), we say that \( \Theta(A) \) is an object of arrows of dimension \( k \) in \( \Theta \) of type \( A \).

If \( C \) is a concrete category, then for \( u \in \Theta(A) \), we can say \( u \) is an arrow of \( \Theta \) of dimension \( k \) and of type \( A \), or sometimes just a \( k \)-arrow or an \( A \)-arrow. (We may also use “morphism” in place of “arrow.”)

Note that in a cubical internal \( n \)-category \( \Theta \), for a given integer \( k \) such that \( 0 \leq k \leq n \), there are \( \binom{n}{k} \) different types of arrows of dimension \( k \) in \( \Theta \) (corresponding to the \( \binom{n}{k} \) different subsets of \( \pi \) of cardinality \( k \)).

The equivalence between cubical internal \( n \)-categories and \( n \)-fold internal categories is realized as follows. Suppose we are given an \( n \)-fold internal category \( C \). Then \( C \) consists of two \((n-1)\)-fold categories \( C_1 \) and \( C_0 \), and these each consist of a pair of \((n-2)\)-fold internal categories, and so on. Proceeding inductively, we can write the lowest-level objects of arrows of \( C \) in the form \( C_{b_1b_2\cdots b_n} \), where \( b_1b_2\cdots b_n \) is a string of \( n \) binary digits. The strings of binary digits can be interpreted as picking out a particular subset \( A \) of \( \pi \), so that in the language used above, \( C_{b_1b_2\cdots b_n} \) is the object of arrows of type \( A \) of \( C \).

Next, there are multiple different source functions: the possible source functions acting on \( C_{b_1b_2\cdots b_n} \) are indexed by the digits of \( b_1b_2\cdots b_n \) having value 1. Given such a particular digit \( b_k = 1 \), the codomain of the source function \( s_k \) on \( C_{b_1b_2\cdots b_n} \) is the object \( C_{b_1\cdots b_{k-1}0b_{k+1}\cdots b_n} \), whose binary string has a 0 in place of 1 at the \( k \)th
position; this corresponds to removing an element from \( A \), which is the same behavior of the source functions in a cubical internal \( n \)-category. The target functions behave similarly, and analogous conclusions can be drawn for composition.

Furthermore, the functoriality of the many different source, target, composition, and identity functions in the \( n \)-fold internal category \( C \) is restated through Axioms 4 and 5 of the definition of cubical internal \( n \)-categories, though in the cubical definition these axioms have a somewhat different interpretation, explored in the next section.

5. Interpretation of cubical internal categories

We first explain the intuitive and visual meaning behind the axioms used in the above definition. Axioms 1 and 2 essentially just assert that the axioms defining an internal category hold true for \( \Theta \) for arrows of all types. Axiom 3 gives a nice visualization the shape of morphisms in \( \Theta \): a \( k \)-morphism \( u \) in \( \Theta \) is naturally encased in a particular \( k \)-cube built out of lower-dimensional morphisms arising as different possible strings of sources and targets of \( u \); the commutative diagram ensures that these all fit into a \( k \)-cube.

Axiom 4 can be thought of as making sure composition fits nicely with the cubical behavior. Essentially, an ordered pair of \( k \)-morphisms of the same type can be composed in \( k \) different ways, corresponding to gluing their corresponding \( k \)-cubes along a \( (k-1) \)-face in \( k \) different possible directions, since the number of \( (k-1) \)-faces of a \( k \)-cube is \( 2k \).

Finally, Axiom 5 gives “multi-associativity” for the different types of composition of morphisms of the same type and dimension. Visually, this will correspond to the fact that \( k \)-cubes can be glued into \( k \)-rectangular prisms, and they give the same answer when composed along all the directions in any order.

We now outline a particular procedure to visualize a cubical internal \( n \)-category \( \Theta \), assuming it is internal to \( \text{Set} \). We will visualize \( \Theta \) by treating its morphisms as objects embedded in \( \mathbb{R}^n \). First, objects / 0-cells (of which there is always only one type) in \( \Theta \) are visualized as vertices viewed as points in \( \mathbb{R}^n \).

Let \( \vec{e}_0, \ldots, \vec{e}_{n-1} \) denote the standard basis of \( \mathbb{R}^n \). Then, a \( k \)-morphism \( u \) of type \( A \) (so \( A \subseteq 2^n \) has cardinality \( k \)) in \( \Theta \) will be visually represented by an oriented \( k \)-planar \( k \)-cube-shaped polytope in \( \mathbb{R}^n \), lying in a \( k \)-plane parallel to that spanned by the vectors \( \{\vec{e}_a\}_{a \in A} \). The cubical \( k \)-polytope for \( u \) is also encased with all of its possible multi-fold sources and targets, which make up the full skeleton of the \( k \)-polytope for \( u \). The \( (k-1) \)-faces of \( u \), of which there are \( 2k \) in number, correspond to the different possible (single-fold) sources and targets of \( u \), along the \( k \) different directions given by \( \vec{e}_a \) with \( a \) ranging over \( A \).
For example, here is a view of the 3-cube of sources and targets for a 3-morphism in a cubical 3-category, visualized in $\mathbb{R}^3$:

To simplify the diagram, only the 2-morphisms in the three front faces have been drawn, though the 2-morphisms in the three back faces are exactly analogous. The arrow representing the 3-morphism could be drawn as a three-lined arrow stretching from the back upper left corner to the front lower right corner. Note that in a cubical 3-category, there are 3 types of 1-morphisms and 3 types of 2-morphisms, as shown in the different 1-morphisms and 2-morphisms above.

In our visualization, we see that two $k$-morphisms $u, v$ both of type $A$ can be composed, when compatible, in $k$ different possible ways (with the order of the two fixed): this corresponds to fitting together the $k$-cubes for $u$ and $v$ along the $k$ different possible directions $\{e_a\}_{a \in A}$ of $u$ and $v$ along a compatible $(k-1)$-face, which must correspond to a $(k-1)$-morphism which is a source of $u$ and a target of $v$ in the same choice of direction $e_a$.

In this visualization procedure, for a given integer $k$ such that $0 \leq k \leq n$, the $\binom{n}{k}$ different types of $k$-morphisms in $\Theta$ arise from the $\binom{n}{k}$ different possible choices of a $k$-plane spanned by the standard basis vectors of $\mathbb{R}^n$. And as explained before, multi-associativity (Axiom 5) in this visualization is the statement that when multiple $k$-cubes for $k$-morphisms of the same type are composed—via gluing along compatible $(k-1)$-faces—into a $k$-rectangular prism, the compositions in all the different directions involved can be done in any order and the final resulting $k$-morphism must be the same in all cases.

Acknowledgments

It is a great pleasure to thank my mentor Weinan Lin for his help and support in my journey of understanding higher categories—strict, weak, and internal—and algebraic topology, and the relations between the two subjects. I would also like to thank Professor J Peter May, for organizing the 2018 UChicago Mathematics REU, which gave me the opportunity to study and research higher categories, and for his assistance, advice, and support throughout the program.

(To make the diagrams in this paper I used Weinan Lin’s online tool for generating LaTeX diagrams, which can be found at his website.)
References

