AN EIGENVALUE APPROACH TO SPHERE RIGIDITY

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ABSTRACT. In this paper, we set out to prove a rigidity theorem for spheres from classic Riemannian geometry. After motivating this theorem by a "curvature implies topology" result, we shift gears to analytic techniques. In particular, we introduce the basic tools of eigenvalue comparison, which are essential for a number of profound geometric results. Ultimately, our rigidity theorem will follow from a set of surprising "curvature implies analysis" results, which are foundational in the world of eigenvalue comparison.

CONTENTS

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1. Introduction

Perhaps the most simple but useful example of non-Euclidean geometry is the round sphere. One justification for this assertion is the sphere's homogeneity. Like Euclidean space, the sphere looks the same, no matter your location. As a consequence, its curvature, an essential tool in differential geometry, is constant everywhere. Unlike Euclidean space, however, the sphere's curvature is nontrivial. This property is essential, as it allows us to do meaningful geometric work in a space that is still nice.

Given its desirability, then, we might try to find ways to discern if an arbitrary manifold is spherical. In this paper, we will study and prove a surprising result that achieves this goal. This result is a generalization of a theorem of Toponogov (see [8]) that was originally proved by Cheng (see [4]). We will state the theorem at the end of the introduction, and the proof can be found in §7.

First, we must establish some notation and conventions. We assume that the reader is familiar with Riemannian geometry, at the level of the first several chapters of [6]. In this paper, all Riemannian manifolds are connected. All manifolds are also equipped with the Levi-Civita connection (see [6]). Lastly, we use the averaging convention for Ricci curvature; for any orthonormal basis $\{e_i\}$ of $v^\perp$ (the $(n-1)$-dimensional space orthogonal to v), we have

$$Ric_p(v) = \frac{1}{n-1}\sum_{i=1}^{n-1} K_p(v, e_i),$$

where K_p is the sectional curvature at p.

Now that our notation is established, we state our main theorem.

Theorem 1.1. *Let M be a complete n-dimensional Riemannian manifold. Suppose that there is some $\kappa > 0$ such that*

$$Ric_p(v) \geq \kappa |v|^2$$

for all $(p, v) \in TM$. If $\operatorname{diam}(M) = \pi/\sqrt{\kappa}$, then M is isometric to the n-dimensional sphere of constant sectional curvature κ.

The modern approach to this result uses techniques in volume growth comparison (see [7]). However, Cheng's original proof used techniques in eigenvalue comparison. Unfortunately, the treatment of this approach in the literature is fairly limited. Moreover, many existing treatments omit some details that may be non-obvious to first-time readers. Thus, this paper is an attempt to treat Cheng's original approach by comprehensively filling in some important details.

We begin in §2 with a theorem of Bonnet-Myers that provides an interesting motivation for Theorem 1.1. In §3 through §6, we introduce the essential tools for Cheng's eigenvalue approach. Lastly, we prove Theorem 1.1 in §7.

2. The Theorem of Bonnet-Myers

A fascinating branch of differential geometry studies the ways in which curvature places restrictions on the topological properties of smooth manifolds. The classical example of this sort is the theorem of Gauss-Bonnet, which links the Euler characteristic of a surface to its Gaussian curvature (for details, see [5]).

There are many topological properties that we might wish to guarantee for a manifold; one of especial utility is compactness. In this section, we prove the following theorem of Bonnet-Myers, which, as desired, ties compactness to curvature.

Theorem 2.1. *Let M be a complete Riemannian manifold of dimension n. Suppose that there is some $\kappa > 0$ such that*

$$Ric_p(v) \geq \kappa |v|^2$$

for all $(p, v) \in TM$. Then M is compact and $diam(M) \leq \pi/\sqrt{\kappa}$.

Notice that Theorem 1.1, our main result, deals with the equality case of Theorem 2.1 (i.e. when $\operatorname{diam}(M) = \pi/\sqrt{\kappa}$). After proving Theorem 2.1, we will discuss this connection in more detail.

Before proving Bonnet-Myers, we must develop some tools from variational calculus.

2.1. **Variational Calculus on Manifolds.** Our goal in introducing tools from variational calculus is twofold. First, we want to establish a notion of the energy of a curve that is minimized by geodesics. Second, we want to compute the derivatives of this energy functional as we vary along a family of "neighboring" curves. These two results will be essential in our proof of Bonnet-Myers.

Recall that the length of a piecewise smooth curve $c : [0, a] \to M$ is given by

$$l(c) = \int_0^a \left\langle \frac{dc}{dt}, \frac{dc}{dt} \right\rangle^{1/2} dt.$$

In other words, to find the length of a curve, we evaluate its speed at every point.

Moreover, recall that the kinetic energy of a particle scales like speed squared. Naturally, then, we evaluate the speed squared at every point of a curve to find its energy.

Definition 2.2. Let $c : [0, a] \to M$ be a piecewise smooth curve. The *energy* of c is given by

$$E(c) = \int_0^a \left\langle \frac{dc}{dt}, \frac{dc}{dt} \right\rangle dt.$$

It follows from Cauchy-Schwarz that

$$l(c)^2 \leq aE(c), \tag{2.3}$$

and that equality holds if and only if $\left\langle \frac{dc}{dt}, \frac{dc}{dt} \right\rangle^{1/2}$ is a constant, i.e. if and only if c is parametrized proportional to arc-length. We now ensure that geodesics minimize this energy.

Lemma 2.4. *Let $\gamma : [0, a] \to M$ be a length-minimizing geodesic joining p to q. Then*

$$E(\gamma) \leq E(c)$$

for all piecewise smooth curves c joining p to q. Moreover, equality holds if and only if c is a length-minimizing geodesic.

The lemma follows directly from inequality (2.3) and the fact that geodesics are parametrized proportional to arc length. For a complete proof, see [6].

Next, we formalize what we mean by a family of "neighboring" curves.

Definition 2.5. Let $c : [0, a] \to M$ be a smooth curve. A *variation* of c is a smooth function

$$f : (-\epsilon, \epsilon) \times [0, a] \to M$$

such that

$$f(0, t) = c(t).$$

A variation of c is *proper* if

$$f(s, 0) = c(0), \quad f(s, a) = c(a).$$

Remark 2.6. For a discussion of variations of *piecewise* smooth curves, we refer the reader to [6].

Suppose $c : [0, a] \to M$ is a smooth curve joining p to q. Then a proper variation f of c provides a family of smooth curves

$$f_s(t) = f(s, t)$$

joining p to q, which vary smoothly with s. Hence, f yields a family of smooth curves neighboring c, as desired.

Next, we guarantee the existence of nontrivial variations.

Lemma 2.7. *Let $c : [0, a] \to M$ be a smooth curve, and let $V : [0, a] \to TM$ be a smooth vector field along c. Then there exists an $\epsilon > 0$ and a variation $f : (-\epsilon, \epsilon) \times [0, a] \to M$ of c such that*

$$\frac{\partial f}{\partial s}(0, t) = V(t)$$

for all $t \in [0, a]$. Moreover, if $V(0) = V(a) = 0$, then the above holds for a proper variation.

The proof of this result constructs the desired variation using the exponential map. We refer the reader to [6] for the complete proof.

We now return to our notion of energy. Equipped with a variation, we can study the energy of the resulting family of neighboring curves.

Definition 2.8. Let $f : (-\epsilon, \epsilon) \times [0, a] \to M$ be a variation of a smooth curve $c : [0, a] \to M$. The *energy* of f is a function

$$E : (-\epsilon, \epsilon) \to \mathbb{R}, \quad E(s) = E(f_s),$$

where f_s is the smooth curve $f_s(t) = f(s, t)$, and $E(f_s)$ is the energy of f_s.

We are now equipped to compute the derivatives of the energy of a variation, which will provide information about how the energy of neighboring curves varies. Since these are standard calculations, we refer the reader to [6] for the details.

Proposition 2.9. *Let $c : [0, a] \to M$ be a smooth curve. Let $f : (-\epsilon, \epsilon) \times [0, a] \to M$ be a variation of c. Let $E : (-\epsilon, \epsilon) \to \mathbb{R}$ be the energy of f. Then*

$$\frac{1}{2}E'(s) = \left\langle \frac{\partial f}{\partial s}(s, t), \frac{\partial f}{\partial t}(s, t) \right\rangle \Big|_{t=0}^{a} - \int_0^a \left\langle \frac{\partial f}{\partial s}(s, t), \frac{D}{\partial t}\frac{\partial f}{\partial t}(s, t) \right\rangle dt. \tag{2.10}$$

Remark 2.11. If f is a proper variation, then $\frac{\partial f}{\partial s}(s, 0) = \frac{\partial f}{\partial s}(s, a) = 0$ for all s, in which case the first term of (2.10) is zero. Moreover, if c is a geodesic, then the second term of (2.10) evaluated at $s = 0$ gives zero.

Using equation (2.10), we can now compute the second derivative of the energy. However, we only need the value of the second derivative at $s = 0$ in our proof of Bonnet-Myers, which is what we compute in the next proposition.

Proposition 2.12. *Let $c : [0, a] \to M$ be a smooth curve. Let $V : [0, a] \to M$ be a smooth vector field along c with $V(0) = V(a) = 0$. Let $f : (-\epsilon, \epsilon) \times [0, a] \to M$ be a proper variation of c with $\frac{\partial f}{\partial s}(0, t) = V(t)$. Let E be the energy of f. Then*

$$\frac{1}{2}E''(0) = -\int_0^a \left\langle V(t), \frac{D^2 V}{\partial t^2}(t) + R(\gamma'(t), V(t))\gamma'(t) \right\rangle dt \tag{2.13}$$

Remark 2.14. If c is a geodesic and V is a Jacobi field, then (2.13) implies that the second derivative of the energy is zero.

2.2. **Proof of the Theorem of Bonnet-Myers.**

Theorem 2.15. *Let M be a complete Riemannian manifold of dimension n. Suppose that there is some $\kappa > 0$ such that*

$$Ric_p(v) \geq \kappa |v|^2$$

for all $(p, v) \in TM$. Then M is compact and diam$(M) \leq \pi/\sqrt{\kappa}$.

Proof. Since M is complete, compactness follows from the diameter bound.

Let p and q be any two points in M. Let $\gamma : [0, 1] \to M$ be a length-minimizing geodesic joining p to q. Suppose, for the sake of contradiction, that $l(\gamma) > \pi/\sqrt{\kappa}$.

Let $\{X_i\}$ be an orthonormal basis for T_pM, with $X_n = \frac{\gamma'(0)}{|\gamma'(0)|}$. Let $X_i(t)$ be the vector field along γ obtained by parallel transport of X_i. Since parallel transport is an isometry, $\{X_i(t)\}$ forms an orthonormal basis for $T_{\gamma(t)}M$ for all $t \in [0, 1]$. Moreover, since $X_n(0) = \frac{\gamma'(0)}{|\gamma'(0)|}$ and γ is geodesic, $X_n(t) = \frac{\gamma'(t)}{|\gamma'(t)|}$ for all $t \in [0, 1]$.

Define a collection of vector fields J_i along γ by

$$J_i(t) = \sin(\pi t) X_i(t). \tag{2.16}$$

Let $f_i : (-\epsilon, \epsilon) \times [0, 1] \to M$ be a proper variation of γ with transverse velocity given by J_i. Letting $E_i(s)$ be the energy of f_i, we have by (2.13) that

$$\begin{aligned}\frac{1}{2}E_i''(0) &= -\int_0^1 \left\langle J_i(t), \frac{D^2 J_i}{dt^2}(t) + R(\gamma'(t), J_i(t))\gamma'(t) \right\rangle dt \\ &= \int_0^1 \sin^2(\pi t)(\pi^2 - l(\gamma)^2 K_{\gamma(t)}(E_n(t), E_i(t)))\, dt.\end{aligned}$$

Taking the average as i varies from 1 to $n - 1$, we get

$$\frac{1}{2(n-1)} \sum_{i=1}^{n-1} E_i''(0) = \int_0^1 \sin^2(\pi t)(\pi^2 - l(\gamma)^2 \mathrm{Ric}_{\gamma(t)}(E_n(t)))dt. \tag{2.17}$$

By assumption, we have that $\mathrm{Ric}_{\gamma(t)}(E_n(t)) \geq \kappa$ and that $l(\gamma) > \pi/\sqrt{\kappa}$, so

$$l(\gamma)^2 \mathrm{Ric}_{\gamma(t)}(E_n(t)) > \pi^2.$$

Then by (2.17), we have

$$\sum_{i=1}^{n-1} E_i''(0) < 0,$$

which implies that $E_i''(0) < 0$ for some i. Thus, γ does not minimize energy. Therefore, we have obtained a contradiction with Lemma 2.4, so we conclude that $l(\gamma) \leq \pi/\sqrt{\kappa}$. Since this holds for all $p, q \in M$, we have $\operatorname{diam}(M) \leq \pi/\sqrt{\kappa}$. □

2.3. **Motivation for the Theorem of Cheng-Toponogov.** Since Theorem 1.1 is the equality case of Theorem 2.15, we might ask if there are reasons to suspect that Theorem 1.1 holds based on the above proof.

In the proof of Theorem 2.15, the vector fields (2.16) along the geodesic γ are used to achieve the contradiction $E_i''(0) < 0$. Heuristically, we might then expect that those vector fields give $E_i''(0) = 0$ when the diameter of M is $\pi/\sqrt{\kappa}$. Thus, following the suggestion of Remark 2.14, we might ask that a manifold M satisfying the setup of Theorem 2.15 and having diameter $\pi/\sqrt{\kappa}$ has Jacobi fields given by (2.16). Suggestively, these vector fields span the $(n-1)$-dimensional vector space of Jacobi fields orthogonal to γ' with $J(0) = 0$ for the n-sphere of constant curvature κ (see §5).

3. Laplacian Eigenvalue Problems

To understand Cheng's eigenvalue approach to Theorem 1.1, we must first develop the basic tools. In the next few sections, we investigate general Laplacian eigenvalue problems, before returning in §6 to the specific setup of Theorem 1.1.

In what follows, we write ∂_i to mean $\frac{\partial}{\partial x_i}$ when we work in coordinates. Moreover, we write g_{ij} and g^{ij} for the entries of the metric and the entries of the inverse of the metric respectively. Lastly, we write g for the determinant of the metric.

3.1. **The Laplacian.** We begin by defining four tools familiar from multivariate calculus: gradient, divergence, Laplacian, and Hessian.

Carrying out the calculations of these tools in a coordinate chart would not provide useful insight for our ultimate goal. Moreover, these calculations are well treated in [3]; we refer the interested reader there for the details.

3.1.1. *Gradient.*

Definition 3.1. Let M be an n-dimensional Riemannian manifold. Let $f : M \to \mathbb{R}$ be a C^1 function. Then the *gradient* of f is the vector field $\mathrm{grad} f$ satisfying

$$\langle \mathrm{grad} f, X\rangle = X(f)$$

for all continuous vector fields X.

As noted above, we omit the calculation of the gradient in coordinates (see [3]). That calculation yields

$$\mathrm{grad} f = \sum_{i,j=1}^{n} (g^{ij}\partial_j f)\partial_i. \tag{3.2}$$

From (3.2), we get the following properties.

Proposition 3.3. *For C^1 functions $f, h : M \to \mathbb{R}$, we have*

$$\begin{aligned} \mathrm{grad}(f + h) &= \mathrm{grad} f + \mathrm{grad} h, \\ \mathrm{grad}(fh) &= h(\mathrm{grad} f) + f(\mathrm{grad} h). \end{aligned}$$

3.1.2. *Divergence.*

Definition 3.4. Let M be an n-dimensional Riemannian manifold. Let X be a C^1 vector field. The *divergence* of X is the function $\mathrm{div}X$ given by

$$\mathrm{div}X(p) = \mathrm{Tr}(v \mapsto \nabla_v X),$$

where $v \in T_pM$.

Again, we omit the calculation in coordinates (see [3]). For a vector field X given locally by

$$\sum_{i=1}^{n} a_i \partial_i,$$

we get

$$\mathrm{div}X = \frac{1}{\sqrt{g}} \sum_{j=1}^{n} \partial_j (a_j \sqrt{g}). \tag{3.5}$$

Equation (3.5) implies the following elementary properties of divergence.

Proposition 3.6. *For C^1 vector fields X, Y and C^1 function $f : M \to \mathbb{R}$, we have*

$$\begin{aligned} \mathrm{div}(X+Y) &= \mathrm{div}X + \mathrm{div}Y, \\ \mathrm{div}(fX) &= f(\mathrm{div}X) + \langle \mathrm{grad}f, X\rangle. \end{aligned}$$

3.1.3. *Laplacian.*

Definition 3.7. Let M be an n-dimensional Riemannian manifold. Let $f : M \to \mathbb{R}$ be a C^2 function. The *Laplacian* of f is the function Δf given by

$$\Delta f = \mathrm{div}\ \mathrm{grad}f.$$

Plugging (3.2) into (3.5), we get the following expression for the Laplacian in local coordinates:

$$\Delta f = \frac{1}{\sqrt{g}} \sum_{i,j=1}^{n} \partial_j (g^{ji} \sqrt{g} \partial_i f). \tag{3.8}$$

Using Propositions 3.3 and 3.6, the following properties hold for the Laplacian.

Proposition 3.9. *For any C^2 functions $f, h : M \to \mathbb{R}$, we have*

$$\begin{aligned} \Delta(f+h) &= \Delta f + \Delta h, \\ \mathrm{div}(h(\mathrm{grad}f)) &= h\Delta f + \langle \mathrm{grad}f, \mathrm{grad}h\rangle, \\ \Delta(fh) &= h\Delta f + f\Delta h + 2\langle \mathrm{grad}f, \mathrm{grad}h\rangle. \end{aligned}$$

3.1.4. *Hessian.*

Definition 3.10. Let M be an n-dimensional Riemannian manifold. Let $f : M \to \mathbb{R}$ be a C^2 function. The *Hessian* of f is an operator on vector fields given by

$$\mathrm{Hess}f(X) = \nabla_X \mathrm{grad}f.$$

We will make two remarks about the Hessian operator. First, by Definitions 3.1 and 3.4, we have the following identity:

$$\Delta f = \mathrm{Tr}(\mathrm{Hess}f). \tag{3.11}$$

Second, we need the Lichnerowicz formula:

$$\frac{1}{2}\Delta(|\mathrm{grad}f|^2) = |\mathrm{Hess}f|^2 + \langle \mathrm{grad}f, \mathrm{grad}\Delta f\rangle + Ric(\mathrm{grad}f), \tag{3.12}$$

where

$$|\mathrm{Hess}f|^2(p) = \sum_{i,j=1}^{n} \langle (\mathrm{Hess}f(p))e_i, e_j\rangle$$

for any orthonormal basis $\{e_i\}$ of T_pM. We refer the reader to [2] for the proof.

3.2. **Eigenvalue Problems.** Equipped with the essential tools, we can now consider eigenvalue problems for the Laplacian.

We concern ourselves with two cases. In the first case, we have a compact, closed manifold. In the second case, we have a normal domain, which we define below.

Definition 3.13. A *normal domain* is a manifold with compact closure and nonempty, piecewise smooth boundary.

3.2.1. *Closed Eigenvalue Problem.* Let M be a compact, closed n-dimensional Riemannian manifold. Then the Laplacian eigenvalue problem asks for a function $f : M \to \mathbb{R}$ (not identically zero) and a constant λ such that

$$\Delta f + \lambda f = 0.$$

This eigenvalue problem is called closed.

3.2.2. *Dirichlet Eigenvalue Problem.* Let M be a normal domain. Then we modify the closed eigenvalue problem by adding a boundary condition:

$$\Delta f + \lambda f = 0, \quad f\mid_{\partial M} = 0.$$

This eigenvalue problem is called Dirichlet.

3.3. **Admissible Functions.** Before proceeding, we must take care of some results from functional analysis. Since most of these results are orthogonal to our purpose, we refer the interested reader to [3] for the details.

Briefly, before discussing these results, it will be useful to have the divergence theorem on hand, which is a consequence of Stokes's theorem.

Theorem 3.14. *If X is a C^1 vector field on $\overline{M}$ with compact support on $\overline{M}$, then*

$$\int_M (\mathrm{div}X)\ dV = \int_{\partial M} \langle X, \nu\rangle\ dA,$$

where dA is the induced measure density on the boundary, and ν is the outward pointing unit normal vector field.

We begin by establishing the right function space. Recall that the L^2 inner product and norm on $C^0(M)$ are given by

$$(f,h) = \int_M fh\ dV, \quad \|f\|^2 = (f,f).$$

We then let $L^2(M)$ be the completion in the L^2 norm of the space of C^0 functions with bounded L^2 norm.

Within the L^2 framework, we can already discern some useful properties of the eigenvalues, eigenfunctions, and eigenspaces of the Laplacian. For a discussion of the following theorem, we refer the reader to [3].

Theorem 3.15. *In either eigenvalue problem above, the following claims hold:*

(1) *The set of eigenvalues (repeated with multiplicity equal to the dimension of the corresponding eigenspace) forms an increasing sequence*
$$0 \leq \lambda_1 \leq \lambda_2 \leq \ldots \to \infty.$$
(2) *The eigenspace associated to each distinct eigenvalue has finite dimension.*
(3) *Eigenspaces associated to distinct eigenvalues are orthogonal in $L^2(M)$.*
(4) *$L^2(M)$ is the direct sum of the eigenspaces.*
(5) *Every eigenfunction is smooth on $\overline{M}$.*

For the closed eigenvalue problem, any nonzero constant function is a solution with eigenvalue zero. It can also be shown that this eigenspace has dimension one. Thus, we are interested in the smallest *nonzero* eigenvalue of closed manifolds, which we denote $\lambda(M)$. On the other hand, the boundary condition in the Dirichlet eigenvalue problem prohibits functions with zero Laplacian, so the smallest eigenvalue of a normal domain D is positive and denoted $\lambda_1(D)$.

Using Theorem 3.15, we can construct a nice basis for $L^2(M)$. Let $\{\phi_i\}$ be a sequence of orthonormal eigenfunctions corresponding to each eigenvalue λ_i. Then for any function $f \in L^2(M)$, we have what are known as the Parseval identities:
$$f = \sum_{i=1}^{\infty}(f, \phi_i)\phi_i, \quad \|f\|^2 = \sum_{i=1}^{\infty}(f, \phi_i)^2.$$

Before refining our function space, we must first establish the right space of vector fields. Along the lines of L^2, we define an inner product and norm for continuous vector fields on M, called $\mathscr{L}^2$, given by
$$(X, Y) = \int_M \langle X, Y\rangle \; dV, \quad \|X\|^2 = (X, X).$$
We then let $\mathscr{L}^2(M)$ be the completion in the $\mathscr{L}^2$ norm of the space of C^0 vector fields with bounded $\mathscr{L}^2$ norm.

By Theorem 3.14 and Proposition 3.6, we obtain a useful property that holds for any C^1 vector field X with compact support and any C^1 function f:
$$(\mathrm{grad} f, X) = -(f, \mathrm{div} X). \tag{3.16}$$

In fact, we can maintain property (3.16) while passing to a Sobolev space as follows. We say that a vector field $Y \in \mathscr{L}^2(M)$ is the *weak gradient* of a function $f \in L^2(M)$ if
$$(Y, X) = -(f, \mathrm{div} X)$$
for every C^1 vector field X with compact support. We denote the space of functions in $L^2(M)$ possessing weak gradients by $\mathscr{H}(M)$.

It is shown in [3] that when a weak gradient exists, it is unique. Thus, we write the weak gradient with the same notation as the usual gradient. We can then equip $\mathscr{H}(M)$ with a norm given by
$$\|f\|^2_{\mathscr{H}} = \|f\|^2 + \|\mathrm{grad} f\|^2.$$
It is shown in [3] that $\mathscr{H}(M)$ is the completion in the $\mathscr{H}$ norm of the smooth functions on M with bounded $\mathscr{H}$ norm.

We must make one more refinement of our function space, which relates to our eigenvalue problem. We first define the *Dirichlet integral* on $\mathscr{H}(M)$ by

$$D[f, h] = (\mathrm{grad} f, \mathrm{grad} h).$$

Now, we limit ourselves to functions f satisfying

$$(\Delta\phi, f) = -D[\phi, f] \tag{3.17}$$

for any eigenfunction ϕ. We call the space of functions satisfying (3.17) *admissible*.

It turns out that we can explicitly characterize the space of admissible functions for both eigenvalue problems. For a proof of the following result, see [3].

Theorem 3.18. *For the closed eigenvalue problem, the space of admissible functions is $\mathscr{H}(M)$. For the Dirichlet eigenvalue problem, the space of admissible functions is the completion in the $\mathscr{H}$ norm of the smooth functions with bounded $\mathscr{H}$ norm and compact support on M.*

3.4. **Rayleigh Quotients.** We have refined our functional setting around equation (3.17). Now, we use (3.17) to establish an essential tool for eigenvalue comparison, namely, *Rayleigh quotients*.

Notice that for an eigenfunction f with eigenvalue λ, we have

$$D[f, f]/\|f\|^2 = -(\Delta f, f)/\|f\|^2 = \lambda. \tag{3.19}$$

The quotient on the left hand side is called a *Rayleigh quotient*.

Given that (3.19) holds for an eigenfunction, we might ask if the Rayleigh quotient of an arbitrary admissible function is useful. In fact, we can use the Rayleigh quotient of an admissible function to bound our eigenvalues.

Theorem 3.20. *Let M be a closed manifold with closed eigenvalue problem or a normal domain with the Dirichlet eigenvalue problem. Let*

$$0 \leq \lambda_1 \leq \lambda_2 \leq \dots$$

be the eigenvalues of the specified eigenvalue problem, repeated with multiplicity. Let $\{\phi_i\}$ be a complete orthonormal basis of $L^2(M)$ such that each ϕ_i is an eigenfunction of λ_i. Then for any nonzero admissible function f satisfying

$$(f, \phi_1) = \dots = (f, \phi_{k-1}) = 0$$

for some $k > 1$, we have

$$\lambda_k \leq D[f, f]/\|f\|^2.$$

Moreover, we have for every nonzero admissible function f that

$$\lambda_1 \leq D[f, f]/\|f\|^2.$$

Equality holds if and only if f is an eigenfunction of λ_k.

Theorem 3.20 follows from a standard calculation that is well-treated in [3]. More importantly, this result is a critical step in getting eigenvalue comparison off the ground, because Rayleigh quotients can be compared between different domains.

3.5. **Domain Monotonicity.** Domain monotonicity is the second critical step for eigenvalue comparison, as it equips us with a profoundly useful principle: smaller domains have larger eigenvalues. To "prove" such a principle, we compare Rayleigh quotients between domains, as suggested above.

We will repeatedly use the functions $\{\phi_i\}$ to denote our complete orthonormal eigenbasis for $L^2(M)$, as in the statement of Theorem 3.20.

Theorem 3.21. *Let M be a closed manifold with closed eigenvalue problem or a normal domain with Dirichlet eigenvalue problem. Let*

$$0 \le \lambda_1 \le \lambda_2 \le \dots$$

be the eigenvalues of the specified eigenvalue problem, repeated with multiplicity. Let $D_1, \dots, D_r$ be pairwise disjoint normal domains in M, each with the Dirichlet eigenvalue problem. Arrange the eigenvalues of all of the domains D_i in an increasing sequence

$$0 \le \nu_1 \le \nu_2 \le \dots,$$

repeated with multiplicity. Then for all $k \ge 1$, we have

$$\lambda_k \le \nu_k.$$

Proof. Let $\rho(i)$ be the index of the domain corresponding to eigenvalue ν_i. Then for each $i = 1, \dots, k$, we define a function $\psi_i : M \to \mathbb{R}$ given by

$$\psi_i(x) = \begin{cases} \Phi_i(x) & x \in D_{\rho(i)} \\ 0 & \text{else} \end{cases},$$

where Φ_i is a unit-norm eigenfunction on $D_{\rho(i)}$ with eigenvalue ν_i. We can choose the ψ_i to be orthonormal.

With some care, we can show that ψ_i is an admissible function on M; we refer the reader to [3] for the details. Since ψ_i is identically zero outside $D_{\rho(i)}$, we have for any admissible function f that

$$D[f, \psi_i] = \nu_i(f, \psi_i).$$

Let $f : M \to \mathbb{R}$ be given by

$$f = \sum_{i=1}^{k} \alpha_i \psi_i$$

for some coefficients α_i, such that

$$(f, \phi_1) = \dots = (f, \phi_{k-1}) = 0.$$

These conditions form a system of $k-1$ linear equations in k variables (i.e. the α_i). Thus, there is a solution with $\alpha_i \ne 0$ for some i. Fix the coefficients α_i for such a solution.

By Theorem 3.20, we have

$$\lambda_k \|f\|^2 \le D[f, f].$$

Moreoever, we have that

$$D[f, f] = \sum_{i=1}^{k} \alpha_i D[f, \psi_i] = \sum_{i=1}^{k} \nu_i \alpha_i^2 \le \nu_k \|f\|^2.$$

Thus, we have $\lambda_k \le \nu_k$. $\square$

Theorem 3.22. *Let M be a closed manifold with closed eigenvalue problem or a normal domain with Dirichlet eigenvalue problem. Let*

$$0 \leq \lambda_1 \leq \lambda_2 \leq \dots$$

be the eigenvalues of the specified eigenvalue problem, repeated with multiplicity. Let D be a normal domain in M with the Dirichlet eigenvalue problem. Let

$$0 \leq \nu_1 \leq \nu_2 \leq \dots,$$

be the eigenvalues of D, repeated with multiplicity. If $M \setminus \overline{D}$ is nonempty and open in M, then for all $k \geq 1$, we have

$$\lambda_k < \nu_k.$$

Proof. Let f be an eigenfunction of D with eigenvalue ν_k when restricted to D and zero elsewhere. Suppose for the sake of contradiction that $\lambda_k = \nu_k$. Then we have

$$D[f,f]/\|f\|^2 = \nu_k = \lambda_k,$$

which implies by Theorem 3.20 that f is an eigenfunction of M with eigenvalue λ_k. However, f is zero on a non-empty open set in M. Then by the unique continuation principle (see [1]), f must be identically zero, which yields a contradiction. □

In general, we would like to weaken regularity from normal domains to arbitrary open sets. However, we cannot apply the eigenvalue problem to any open set. To fulfill this need, we introduce the fundamental tone.

Definition 3.23. Let $U \subset M$ be an arbitrary open set in a closed manifold or normal domain. Then the *fundamental tone* of U is

$$\lambda^*(U) = \inf D[f,f]/\|f\|^2,$$

where f is nonzero, varying over the completion in the $\mathscr{H}$ norm of the smooth functions with bounded $\mathscr{H}$ norm and compact support on U. As in the normal domain case, we call this space of functions *admissible*.

Intuitively, the fundamental tone defines the "smallest Dirichlet eigenvalue" indirectly via Rayleigh quotients. In fact, the fundamental tone of a normal domain is the smallest Dirichlet eigenvalue.

With this tool, we can prove analogous monotonicity results for open sets.

Theorem 3.24. *Let M be a closed manifold with closed eigenvalue problem or a normal domain with Dirichlet eigenvalue problem. Let*

$$0 \leq \lambda_1 \leq \lambda_2 \leq \dots$$

be the eigenvalues of the specified eigenvalue problem, repeated with multiplicity. Let $U_1, \dots, U_r$ be pairwise disjoint open sets in M. Arrange the fundamental tones of each open set in an increasing sequence

$$0 \leq \nu_1 \leq \dots \leq \nu_r.$$

Then for all $k = 1, \dots, r$, we have

$$\lambda_k \leq \nu_k.$$

Proof. Let $\rho(i)$ be the index of the domain corresponding to tone ν_i. For any $\epsilon > 0$, there is a smooth, unit-norm function Φ_i with compact support on $U_{\rho(i)}$ such that

$$D[\Phi_i, \Phi_i] \leq \nu_i + \epsilon.$$

For each $i = 1, \ldots, k$, define a unit-norm function $\psi_i : M \to \mathbb{R}$ given by

$$\psi_i(x) = \begin{cases} \Phi_i(x) & x \in U_{\rho(i)} \\ 0 & \text{else} \end{cases}.$$

The functions ψ_i are orthonormal because the U_i are disjoint. In the same way as Theorem 3.21, we can show that ψ_i is an admissible function on M.

Let $f : M \to \mathbb{R}$ be given by

$$f = \sum_{i=1}^{k} \alpha_i \psi_i$$

for some coefficients α_i, such that

$$(f, \phi_1) = \ldots = (f, \phi_{k-1}) = 0.$$

These conditions form a system of $k-1$ linear equations in k variables (i.e. the α_i). Thus, there is a solution with $\alpha_i \neq 0$ for some i. Fix the coefficients α_i for such a solution.

By Theorem 3.20, we have

$$\lambda_k \|f\|^2 \leq D[f, f].$$

Since the U_i are disjoint and ψ_i is zero outside $U_{\rho(i)}$, we have

$$D[f, \psi_i] = \sum_{j=1}^{k} \alpha_j D[\psi_i, \psi_j] = \alpha_i D[\Phi_i, \Phi_i] \leq \alpha_i(\nu_i + \epsilon).$$

Consequently, we have

$$D[f, f] = \sum_{i=1}^{k} \alpha_i D[f, \psi_i] \leq \sum_{i=1}^{k} (\nu_i + \epsilon)\alpha_i^2 \leq (\nu_k + \epsilon)\|f\|^2.$$

Thus, we have $\lambda_k \leq \nu_k + \epsilon$. Since this holds for all $\epsilon > 0$, we can take the limit as ϵ tends to zero, yielding the desired inequality. □

Theorem 3.25. *Let U be an open set in a closed manifold or normal domain. Let D be a normal domain in U with the Dirichlet eigenvalue problem. If $U \setminus \overline{D}$ is nonempty and open in U, then*

$$\lambda^*(U) < \lambda_1(D).$$

Proof. Let $\epsilon > 0$. Let $\phi : D \to \mathbb{R}$ be an eigenfunction of D with eigenvalue $\lambda_1(D)$. Let $f : U \to \mathbb{R}$ be ϕ when restricted to D and zero elsewhere. Suppose for the sake of contradiction that $\lambda^*(U) = \lambda_1(D)$. Then we have

$$D[f, f]/\|f\|^2 = \lambda_1(D) = \lambda^*(U).$$

Then f achieves the fundamental tone of U, which implies that U is a normal domain with smallest Dirichlet eigenvalue equal to $\lambda^*(U)$. Thus, the proof proceeds identically to Theorem 3.22. □

4. Geodesic Spherical Coordinates

In general, little can be said about arbitrary coordinate charts. However, we can construct a particularly useful system of coordinates using the exponential map.

For any point $p \in M$, there is a $\delta > 0$ such that the geodesic ball $B(p, \delta)$ is a normal neighborhood. There, the exponential map is a diffeomorphism. To each unit vector $v \in \mathbb{R}^n$, we associate a normalized geodesic emanating from p given by

$$\gamma_v(t) = \exp_p(vt), \quad t \in [0, \delta).$$

Naturally, we can identify the set of unit vectors in $\mathbb{R}^n$ with the sphere $\mathbb{S}^{n-1}$. Then to each $q \in B(p, \delta) \setminus \{p\}$, there is a unique pair $(t, v) \in (0, \delta) \times \mathbb{S}^{n-1}$ such that $q = \exp_p(tv)$.

Let $v^{-1} : \mathbb{S}^{n-1} \to \mathbb{R}^{n-1}$ be a coordinate chart on $\mathbb{S}^{n-1}$ (see Remark 4.1 below for a caveat). Then we get the following chart, called *geodesic spherical coordinates*:

$$u^{-1} : B(p, \delta) \to \mathbb{R} \times \mathbb{R}^{n-1}, \quad u(t, x) = \exp_p(tv(x)).$$

Remark 4.1. A single chart on $\mathbb{S}^{n-1}$ does not exist. There *is* a chart on $\mathbb{S}^{n-1}$ minus one point, which gives a chart on $B(p, \delta) \setminus L$ for some line segment L. This segment has measure zero, so we can calculate as if we *effectively* have a chart on $\mathbb{S}^{n-1}$.

4.1. **Computing the Metric.**

In this section, we compute the metric in geodesic spherical coordinates, which will be critically important.

Let $(x_1, \ldots, x_{n-1})$ be the coordinates associated with the chart v on $\mathbb{S}^{n-1}$. We use the following notation for this chart:

$$\partial_i v := \partial_i(v(x_1, \ldots, x_{n-1})).$$

Similarly, we use the following notation for the chart u on $B(p, \delta)$:

$$\partial_t u := \partial_t(u(t, x)), \quad \partial_i u := \partial_i(u(t, x)).$$

Then we have

$$\partial_t u = d(\exp_p)_{tv}(v) = \gamma_v'(t), \tag{4.2}$$

$$\partial_i u = d(\exp_p)_{tv}(t\partial_i v). \tag{4.3}$$

Before computing the components of the metric, we must discuss Jacobi fields. Recall that the unique Jacobi field $J(t)$ along γ_v with initial conditions

$$J(0) = 0, \quad \frac{DJ}{dt}(0) = w$$

is given by

$$d(\exp_p)_{tv}(tw).$$

For notative ease, we define a linear transformation $J(t, v)$ by

$$J(t, v) : \gamma_v'(0)^\perp \to \gamma_v'(t)^\perp, \quad J(t, v)w = d(\exp_p)_{tv}(tw).$$

With this notation and the above observation, (4.3) becomes

$$\partial_i u = d(\exp_p)_{tv}(t\partial_i v) = J(t, v)\partial_i v. \tag{4.4}$$

We now return to our metric computation. Since γ_v is a normalized geodesic, $g_{tt} = 1$. By Gauss's lemma, the linearity of the differential, and the constant length of v, we have $g_{ti} = 0$. Lastly, we have

$$g_{ij} = \langle J(t, v)\partial_i v, J(t, v)\partial_j v\rangle. \tag{4.5}$$

We want to put (4.5) in a more useful form. To do this, we begin with some notation. Let

$$P(t,v) : \gamma_v'(0)^\perp \to \gamma_v'(t)^\perp$$

be given by parallel transport along γ_v from p to $\gamma_v(t)$. Next, we define a linear transformation $\mathscr{A}(t,v)$ by

$$\mathscr{A}(t,v) : \gamma_v'(0)^\perp \to \gamma_v'(0)^\perp, \quad \mathscr{A}(t,v) = P(t,v)^{-1}J(t,v).$$

Since parallel transport is an isometry, we have by (4.5) that

$$g_{ij} = \langle \mathscr{A}(t,v)\partial_i v, \mathscr{A}(t,v)\partial_j v\rangle. \tag{4.6}$$

Thus, we get the following formal expression for the metric on $B(p,\delta)$:

$$ds^2 = dt^2 + |\mathscr{A}\, dv|^2, \tag{4.7}$$

where $|\mathscr{A}\, dv|^2$ is the $(n-1)$-dimensional metric with entries given by (4.6).

We can now compute the density of the Riemannian volume measure. First, we define the function θ by

$$\theta : (0,\delta) \times \mathbb{S}^{n-1} \to \mathbb{R}, \quad \theta(t,v) = \det(\mathscr{A}(t,v)). \tag{4.8}$$

If we let dA be the density of the measure on the sphere $\mathbb{S}^{n-1}$, equation (4.7) gives

$$dV = \theta(t,v) dt\; dA. \tag{4.9}$$

4.2. **Connection to Curvature.** We now define two useful curvature-related linear transformations. First, we define the linear transformation $\mathbf{R}(t,v)$ by

$$\mathbf{R}(t,v) : \gamma_v'(t)^\perp \to \gamma_v'(t)^\perp, \quad \mathbf{R}(t,v)w = R(\gamma_v'(t), w)\gamma_v'(t).$$

Second, we define the linear transformation $\mathscr{R}(t,v)$ by

$$\mathscr{R}(t,v) : \gamma_v'(0)^\perp \to \gamma_v'(0)^\perp, \quad \mathscr{R}(t,v) = P(t,v)^{-1}\mathbf{R}(t,v)P(t,v).$$

Lemma 4.10. *The transformations* $\mathbf{R}$ *and* $\mathscr{R}$ *are self-adjoint with respect to the Riemannian metric.*

Proof. It is a known property of the curvature tensor that

$$\langle R(u,v)u, w\rangle = \langle v, R(u,w)u\rangle.$$

Thus, $\mathbf{R}$ is self-adjoint.

Additionally, $P(t,v)$ is an orthogonal transformation, so we have

$$\mathscr{R}^T = (P^{-1}\mathbf{R}P)^T = P^T\mathbf{R}(P^{-1})^T = P^{-1}\mathbf{R}P = \mathscr{R}.$$

Thus, $\mathscr{R}$ is also self-adjoint. □

Equipped with these curvature-related transformations, we can show the following lemma.

Lemma 4.11. *For a fixed value of v, the transformation $\mathscr{A}(t,v)$ satisfies*

$$\mathscr{A}'' + \mathscr{R}\mathscr{A} = 0$$

with the initial conditions

$$\mathscr{A}(0,v) = 0, \quad A'(0,v) = \mathrm{id}.$$

The result follows from direct computation of the derivatives of P and J. For more discussion, see [3].

Recall that the cut point along a normalized geodesic γ is $\gamma(t_0)$, where t_0 is the least time after which γ is not length minimizing. A consequence of the initial conditions in Lemma 4.11, and the fact that the cut point is not past the first point $J(t, v)$ is singular, is that the function $\theta(t, v)$ is positive before the cut point.

5. Manifolds of Constant Sectional Curvature

We now focus on the n-dimensional, complete, simply connected manifold of constant sectional curvature κ, which we denote $\mathbb{M}^n_\kappa$. Note that if $\kappa > 0$, then $\mathbb{M}^n_\kappa$ is the sphere $\mathbb{S}^n(1/\sqrt{\kappa})$ (see [6] for the proof).

In this setting, we compute the metric in geodesic spherical coordinates and make some observations about eigenvalues.

5.1. **Computing the Metric.**

In $\mathbb{M}^n_\kappa$, the Jacobi equation becomes

$$\frac{D^2 J}{dt^2} + \kappa J = 0.$$

In particular, we consider Jacobi fields orthogonal to the velocity field of the given geodesic with initial condition $J(0) = 0$.

Fix a geodesic γ in M with $\gamma(0) = p$. Let $\{e_i\}$ be an orthonormal basis for $\gamma'(0)^\perp \subset T_pM$. Then let

$$e_i(t) = P(t, \gamma'(0))e_i,$$

namely, the parallel transport of e_i along γ. Suppose a Jacobi field is given by

$$J_i(t) = f(t)e_i(t).$$

Then we have

$$f''(t) + \kappa f(t) = 0, \quad f(0) = 0.$$

To fix the scaling of the solution, we ask that $f'(0) = 1$. The solution to this differential equation is given by

$$S_\kappa(t) = \begin{cases} \frac{1}{\sqrt{\kappa}} \sin(\sqrt{\kappa} t) & \kappa > 0 \\ t & \kappa = 0 \\ \frac{1}{\sqrt{-\kappa}} \sinh(\sqrt{-\kappa} t) & \kappa < 0 \end{cases}. \tag{5.1}$$

We reuse this notation later. Since the space of Jacobi fields orthogonal to $\gamma'(t)$ with $J(0) = 0$ has dimension $n - 1$, a basis is formed by the fields

$$J_i(t) = S_\kappa(t)e_i(t). \tag{5.2}$$

If we take the covariant derivative of the vector fields J_i at $t = 0$, we get

$$\frac{DJ_i}{dt}(0) = S'_\kappa(0)e_i(0) = e_i.$$

Therefore, we have

$$\mathscr{A}(t, \gamma'(0))e_i = P(t, \gamma'(0))^{-1} J_i(t) = S_\kappa(t)e_i.$$

Thus,

$$\mathscr{A}(t, v) = S_\kappa(t)(\mathrm{id}), \tag{5.3}$$

and

$$\theta(t, \gamma'(0)) = \det(\mathscr{A}(t, \gamma'(0))) = S^{n-1}_\kappa(t). \tag{5.4}$$

5.2. **Computing Eigenfunctions and Eigenvalues.** We would like a better understanding of the eigenvalues and eigenfunctions of the Laplacian in $\mathbb{M}^n_\kappa$. Since the required computations are orthogonal to our purpose, we refer the reader to the thorough treatment of this case in [3]. The results are summarized below.

Theorem 5.5. *Let $B_\kappa(\delta)$ be the ball of radius $\delta > 0$ in $\mathbb{M}^n_\kappa$. Then the lowest Dirichlet eigenvalue $\lambda_\kappa(\delta)$ has eigenfunction F of the form*

$$F(\exp_p(tv)) = T(t), \quad (t, v) \in [0, \delta) \times \mathbb{S}^{n-1},$$

where T solves

$$(S_\kappa^{n-1}T')' + \lambda_\kappa(\delta)S_\kappa^{n-1}T = 0, \tag{5.6}$$

and

$$T'(0) = T(\delta) = 0, \quad T\,|_{[0,\delta)}\neq 0.$$

Since $T\,|_{[0,\delta)}\neq 0$, we always assume that $T\,|_{[0,\delta)}> 0$. Then we also have the following fact (see [3] for the proof).

Lemma 5.7. *Let T be as in Theorem 5.5, with $T\,|_{[0,\delta)}> 0$. Then $T'\,|_{(0,\delta]}< 0$.*

Lastly, we make note of one more useful calculation in this case. Let $\kappa > 0$ and $\delta = \pi/(2\sqrt{\kappa})$. If we let $T(t) = \cos(\sqrt{\kappa}t)$, then we have

$$(S_\kappa^{n-1}T')' + n\kappa S_\kappa^{n-1}T = 0.$$

Thus, T is an eigenfunction for the lowest Dirichlet eigenvalue, so we have

$$\lambda_\kappa(\pi/(2\sqrt{\kappa})) = n\kappa. \tag{5.8}$$

6. Comparison Theorems for Curvature Bounded from Below

We now return to the setup of our main theorem (Theorem 1.1), in which our manifold has Ricci curvature bounded from below. We use this curvature bound to make comparisons between the metric of our manifold M and the metric of $\mathbb{M}^n_\kappa$, which then enables volume and eigenvalue comparison.

We will frequently work with the geodesic ball $B(p, \delta)$. To simplify notation, we let $c(v)$ denote the distance from p to the first cut point along the direction $v \in \mathbb{S}^{n-1} \subset T_pM$. We then let $b(v)$ denote the minimum of $c(v)$ and the radius δ.

6.1. **Metric Comparison.** We begin with a theorem due to Bishop.

Theorem 6.1. *Let M be an n-dimensional complete Riemannian manifold. Suppose that there is some constant $\kappa \in \mathbb{R}$ such that*

$$Ric_p(v) \geq \kappa |v|^2$$

for all $(p, v) \in TM$. Fix a pair $(p, v) \in TM$ such that $|v| = 1$. If for some constant $\beta > 0$ we have that $\theta(t, v) > 0$ on $(0, \beta)$, then

$$\left(\frac{\theta(t, v)}{S_\kappa^{n-1}(t)}\right)' \leq 0 \tag{6.2}$$

on $(0, \beta)$, and

$$\theta(t, v) \leq S_\kappa^{n-1}(t) \tag{6.3}$$

on $(0, \beta]$.

Proof. We define a linear transformation $U(t,v)$ for $t \in (0,\beta)$ by

$$U(t,v) : v^\perp \to v^\perp, \quad U(t,v) = \mathscr{A}'(t,v)\mathscr{A}^{-1}(t,v).$$

We want to show that U is self-adjoint. First, we let

$$W := (\mathscr{A}')^T \mathscr{A} - \mathscr{A}^T \mathscr{A}'.$$

Then by Lemma 4.10 and Lemma 4.11, we have

$$W' = (\mathscr{A}'')^T \mathscr{A} + (\mathscr{A}')^T \mathscr{A}' - (\mathscr{A}')^T \mathscr{A}' - \mathscr{A}^T \mathscr{A}'' = 0.$$

Moreover, since $\mathscr{A}(0,v) = 0$, we have $W \equiv 0$ for all $t \in (0,\beta)$. Thus, we have

$$U^T - U = (\mathscr{A}^{-1})^T (\mathscr{A}')^T - \mathscr{A}'\mathscr{A}^{-1} = (\mathscr{A}^{-1})^T W A^{-1} = 0.$$

Next, we want to show that U satisfies

$$U' + U^2 + \mathscr{R} = 0. \tag{6.4}$$

By Lemma 4.11, we have

$$\begin{aligned} U' + U^2 + \mathscr{R} &= \mathscr{A}''\mathscr{A}^{-1} + \mathscr{A}'(\mathscr{A}^{-1})' + \mathscr{A}'\mathscr{A}^{-1}\mathscr{A}'\mathscr{A}^{-1} + \mathscr{R} \\ &= -\mathscr{R}\mathscr{A}\mathscr{A}^{-1} + \mathscr{A}'(-\mathscr{A}^{-1}\mathscr{A}'\mathscr{A}^{-1}) + \mathscr{A}'\mathscr{A}^{-1}\mathscr{A}'\mathscr{A}^{-1} + \mathscr{R} \\ &= 0. \end{aligned}$$

We are interested in the trace of equation (6.4). First, the linearity of the derivative operation implies that $\mathrm{Tr}(U') = \mathrm{Tr}(U)'$. Second, because U is self-adjoint, Cauchy-Schwarz implies that

$$(n-1)\mathrm{Tr}(U^2) \geq \mathrm{Tr}(U)^2,$$

with equality holding if and only if U is a scalar transformation. Lastly, we have by our assumption on curvature that

$$\mathrm{Tr}(\mathscr{R}) = \mathrm{Tr}(\mathbf{R}) = (n-1)Ric_p(v) \geq \kappa(n-1).$$

Therefore, equation (6.4) implies that

$$(\mathrm{Tr}(U))' + \mathrm{Tr}(U)^2/(n-1) + \kappa(n-1) \leq 0.$$

By Jacobi's formula for the derivative of the determinant, we have

$$\mathrm{Tr}(U) = (1/\theta)\mathrm{Tr}(\theta\mathscr{A}^{-1}\mathscr{A}') = \theta'/\theta.$$

For notative ease, we set $\phi(t) = \mathrm{Tr}(U)$.

We now compare θ with S_κ. We set

$$\psi(t) = (n-1)S_\kappa'/S_\kappa,$$

which is well-defined on $(0, \pi/\sqrt{\kappa})$ (where we set $\pi/\sqrt{\kappa} = +\infty$ if $\kappa \leq 0$). On this interval, ψ is strictly decreasing and bijective. Moreover, we have

$$\psi' + \psi^2/(n-1) + \kappa(n-1) = 0.$$

Let $\mu(t)$ be the function on $(0,\beta) \cap (0, \pi/\sqrt{\kappa})$ given by

$$\mu(t) = \psi^{-1}(\phi(t)), \quad \mu(0) = 0.$$

In other words, we have

$$\phi(t) = \psi(\mu(t)).$$

Differentiating, we get

$$\begin{aligned}\mu'(t) &= \frac{-\phi'(t)}{-\psi'(\mu(t))}\\ &\geq \frac{\phi(t)^2/(n-1)+\kappa(n-1)}{\psi(\mu(t))^2/(n-1)+\kappa(n-1)}\\ &= 1.\end{aligned}$$

Then $\mu(t) \geq t$, which implies

$$\phi(t) = \psi(\mu(t)) \leq \psi(t).$$

We now unpack the information contained in the above inequality. By definition, we have on $(0,\beta)\cap(0,\pi/\sqrt{\kappa})$ that

$$\frac{\theta'}{\theta} \leq (n-1)\frac{S_\kappa'}{S_\kappa}. \tag{6.5}$$

On this interval, θ and S_κ are positive, so we have

$$\left(\frac{\theta}{S_\kappa^{n-1}}\right)' = \frac{\theta'}{S_\kappa^{n-1}} - (n-1)\frac{\theta S_\kappa'}{S_\kappa^n} \leq 0,$$

which gives (6.2) on $(0,\beta)\cap(0,\pi/\sqrt{\kappa})$.

We also have by (6.5) that

$$\log(\theta)' \leq \log(S_\kappa^{n-1})'.$$

Since $\theta(0,v) = S_\kappa^{n-1}(0) = 0$, we have on $(0,\beta]\cap(0,\pi/\sqrt{\kappa}]$

$$\log(\theta) \leq \log(S_\kappa^{n-1}).$$

Lastly, since log is an increasing function, we have

$$\theta \leq S_\kappa^{n-1},$$

which gives (6.3) $(0,\beta]\cap(0,\pi/\sqrt{\kappa}]$.

Finally, we want to show that $\beta \leq \pi/\sqrt{\kappa}$. If $\kappa \leq 0$, then we defined $\pi/\sqrt{\kappa}$ to be $+\infty$, so the inequality is trivial. If $\kappa > 0$, then $S_\kappa(\pi/\sqrt{\kappa}) = 0$. Thus, inequality (6.3) implies that θ is nonpositive before $\pi/\sqrt{\kappa}$, which implies that $\beta \leq \pi/\sqrt{\kappa}$. $\square$

Now, we consider the equality case of Theorem 6.1. We use the same notation as the above proof.

Theorem 6.6. *Consider the setup of Theorem 6.1. If equality holds in (6.2) at $t_0 \in (0,\beta)$ or in (6.3) at $t_0 \in (0,\beta]$, then*

$$\mathscr{R}(t,v) = \kappa(\mathrm{id}), \quad \mathscr{A}(t,v) = S_\kappa(t)(\mathrm{id})$$

on $[0,t_0]$.

Proof. If equality holds in (6.3) at t_0, then equality holds in (6.3) for all $t \in (0,t_0]$, which implies that equality holds in (6.2) for all $t \in (0,t_0)$. Thus, we only need to consider when equality holds in (6.2) for $t_0 \in (0,\beta)$.

Equality in (6.2) at t_0 implies that $\phi(t) = \psi(t)$ for all $t \in [0,t_0]$. Then we must have that

$$\phi' + \phi^2/(n-1) + \kappa(n-1) = 0,$$

which requires that U is a scalar transformation and that $\mathrm{Tr}(\mathscr{R}) = Ric_p(v) = \kappa$.

Since U is scalar, we have

$$\mathscr{A}' = S_\kappa'/S_\kappa\mathscr{A}.$$

Differentiating, we get

$$\mathscr{A}'' = -\kappa \mathscr{A},$$

which implies by Lemma 4.11 that

$$\mathscr{R} = \kappa(\mathrm{id}).$$

Lastly, if we set $A(t) = S_\kappa(t)(\mathrm{id})$, we have

$$A'' = -\kappa A, \quad A(0) = 0, \quad A'(0) = \mathrm{id}.$$

Then by Lemma 4.11, we have $\mathscr{A} = S_\kappa(\mathrm{id})$, which concludes the proof. □

6.2. **Volume Comparison.** Applying the above metric comparison, we can now compare volumes in M and $\mathbb{M}^n_\kappa$.

Theorem 6.7. *Let M be a complete n-dimensional Riemannian manifold. Suppose that there is some $\kappa > 0$ such that*

$$Ric_p(v) \geq \kappa |v|^2$$

for all $(p, v) \in TM$. Then for any positive δ at most $\pi/\sqrt{\kappa}$ and any $p \in M$, we have

$$\mathrm{vol}(B(p, \delta)) \leq \mathrm{vol}(B_\kappa(\delta)).$$

If equality is achieved, then the two balls are isometric.

Proof. Because double counting occurs beyond the cut locus, we have

$$\mathrm{vol}(B(p, \delta)) = \int_{\mathbb{S}^{n-1}} \int_0^{b(v)} \theta(t, v)\ dt\ dA.$$

Then by Theorem 6.1, we have

$$\begin{aligned}\int_{\mathbb{S}^{n-1}} \int_0^{b(v)} \theta(t, v)\ dt\ dA &\leq \int_{\mathbb{S}^{n-1}} \int_0^{b(v)} S_\kappa^{n-1}(t)\ dt\ dA \\ &\leq \int_{\mathbb{S}^{n-1}} \int_0^{\delta} S_\kappa^{n-1}(t)\ dt\ dA \\ &= \mathrm{vol}(B_\kappa(\delta)),\end{aligned}$$

which gives the desired inequality.

Suppose equality holds. Then we have

$$\int_{\mathbb{S}^{n-1}} \int_0^{b(v)} S_\kappa^{n-1}(t)\ dt\ dA = \int_{\mathbb{S}^{n-1}} \int_0^{\delta} S_\kappa^{n-1}(t)\ dt\ dA.$$

Since $S_\kappa^{n-1}(t)$ is positive on $(0, \pi/\sqrt{\kappa})$, we have $b(v) = \delta$ for all v. Also, we have

$$\int_{\mathbb{S}^{n-1}} \int_0^{\delta} \theta(t, v)\ dt\ dA = \int_{\mathbb{S}^{n-1}} \int_0^{\delta} S_\kappa^{n-1}(t)\ dt\ dA.$$

Since $\theta(t, v) \leq S_\kappa^{n-1}(t)$ by Theorem 6.1, it must be the case that

$$\theta(t, v) = S_\kappa^{n-1}(t)$$

for all v and $t \in [0, \delta]$. Then by Theorem 6.6, we have $\mathscr{A} = S_\kappa(\mathrm{id})$. By metric calculations (4.7) and (5.3), $B(p, \delta)$ is isometric to $B_\kappa(\delta)$. □

6.3. **Eigenvalue Comparison.** Similarly, we can now compare eigenvalues in M and $\mathbb{M}^n_\kappa$. First, we compare the fundamental tones of balls of the same radius in each manifold.

Theorem 6.8. *Let M be a complete n-dimensional Riemannian manifold. Suppose that there is some $\kappa > 0$ such that*

$$Ric_p(v) \geq \kappa |v|^2$$

for all $(p, v) \in TM$. Then for any positive δ less than $\pi/\sqrt{\kappa}$ and any $p \in M$, we have

$$\lambda^*(B(p, \delta)) \leq \lambda_\kappa(\delta).$$

If equality is achieved, then the two balls are isometric.

Proof. Let $T(t)$ be the eigenfunction of the lowest Dirichlet eigenvalue of $B_\kappa(\delta)$ as in Theorem 5.5 with $T\mid_{[0,\delta)} > 0$. On $[0, b(v)] \times \mathbb{S}^{n-1}$, define on M the function

$$F(\exp_p(tv)) = T(t).$$

In fact, F is admissible on $B(p, \delta)$; we refer the reader to [3] for that argument.

We want to show that

$$D[F, F]/\|F\|^2 \leq \lambda_\kappa(\delta), \tag{6.9}$$

because

$$\lambda^*(B(p, \delta)) \leq D[F, F]/\|F\|^2.$$

Note that

$$D[F, F] = \int_{\mathbb{S}^{n-1}} \int_0^{b(v)} T'(t)^2 \theta(t, v)\ dt\ dA,$$

and

$$\|F\|^2 = \int_{\mathbb{S}^{n-1}} \int_0^{b(v)} T(t)^2 \theta(t, v)\ dt\ dA.$$

For any direction v, we have by integration by parts and Theorem 5.5 that

$$\int_0^{b(v)} T'^2 \theta\ dt = T(b(v))T'(b(v))\theta(b(v), v) - \int_0^{b(v)} T(T'\theta)'\ dt.$$

Since $T > 0$, $T' < 0$ (by Lemma 5.7), and $\theta > 0$ on $(0, b(v))$, the right hand side is less than or equal to

$$-\int_0^{b(v)} T(T'' + T'\theta'/\theta)\theta\ dt.$$

By Theorem 6.1, the above integral is less than or equal to

$$-\int_0^{b(v)} T(T'' + (n-1)S'_\kappa/S_\kappa T')\theta\ dt.$$

By Theorem 5.5, the above integral is equal to

$$\lambda_\kappa(\delta) \int_0^{b(v)} T^2 \theta\ dt,$$

which gives us the desired inequality.

If equality holds, then we must have equality in (6.9). Since we showed that for every direction v we have

$$\int_0^{b(v)} T'^2 \theta\ dt \leq \lambda_\kappa(\delta) \int_0^{b(v)} T^2 \theta\ dt,$$

equality must hold for every v. Then we have

$$-\int_0^{b(v)} TT'\theta(\theta'/\theta)\ dt = -\int_0^{b(v)} TT'\theta(n-1)S_\kappa'/S_\kappa\ dt.$$

By Theorem 6.1, we conclude that for all v and for all $t \in (0, b(v))$,

$$\theta'/\theta = (n-1)S_\kappa'/S_\kappa.$$

Then by Theorem 6.6, we have

$$\mathscr{A}(t,v) = S_\kappa(\mathrm{id}), \quad \theta(t,v) = S_\kappa^{n-1}.$$

If we can show that $b(v) = \delta$ for all v, then the two balls are isometric by the metric calculations (4.7) and (5.3).

The above work shows that the images of the set

$$\{(t,v) \in \mathbb{R} \times \mathbb{S}^{n-1}; 0 \le t < b(v)\} \subset \mathbb{R}^n$$

under the exponential map in M and $\mathbb{S}^n(1/\sqrt{\kappa})$ are isometric. We call this submanifold N. Recall that Jacobi fields are defined locally. Then on N, the Jacobi fields orthogonal to a geodesic emanating from p with value 0 at p are given by (5.2). Since $\delta < \pi/\sqrt{\kappa}$, no point in ∂N is a conjugate point of p. Moreover, the distance between distinct geodesics emanating from p is bounded from below by a positive number as they approach the boundary, so continuity prohibits two distinct geodesic emanating from p that intersect the same point in the boundary. Thus, no point in the boundary can be a cut point, so $b(v) = \delta$ for all v. □

Second, we compare the first nonzero closed eigenvalue of M to the Dirichlet eigenvalue of a particular ball in $\mathbb{M}_\kappa^n$, namely, $\lambda_\kappa(\pi/(2\sqrt{\kappa}))$ (which is $n\kappa$ by (5.8)).

Theorem 6.10. *Let M be a complete n-dimensional Riemannian manifold. Suppose that there is some $\kappa > 0$ such that*

$$Ric_p(v) \ge \kappa |v|^2$$

for all $(p,v) \in TM$. Then

$$\lambda(M) \ge n\kappa.$$

Proof. Let f be an eigenfunction of M with eigenvalue $\lambda(M)$. By (3.12), we have

$$\frac{1}{2}\Delta(|\mathrm{grad} f|^2) = |\mathrm{Hess} f|^2 + \langle \mathrm{grad} f, \mathrm{grad}\Delta f\rangle + (n-1)Ric_p(\mathrm{grad} f).$$

Moreover, we have by Cauchy-Schwarz and (3.11) that

$$|\mathrm{Hess} f|^2 \ge (\mathrm{Tr}(\mathrm{Hess} f))^2/n = \lambda^2 f^2/n.$$

Lastly, we have by assumption that

$$\langle \mathrm{grad} f, \mathrm{grad}\Delta f\rangle + (n-1)Ric_p(\mathrm{grad} f) \ge ((n-1)\kappa - \lambda)|\mathrm{grad} f|^2.$$

Therefore, Theorem 3.14 implies that

$$\begin{aligned} 0 &= \frac{1}{2}\int_M \Delta(|\mathrm{grad} f|^2)\ dV \\ &\ge \int_M (\lambda^2 f^2/n + ((n-1)\kappa - \lambda)|\mathrm{grad} f|^2)\ dV \\ &= [(n-1)\lambda \|f\|^2/n](n\kappa - \lambda). \end{aligned}$$

Since $(n-1)\lambda\|f\|^2/n > 0$, the desired inequality follows. □

7. The Theorem of Cheng-Toponogov

We now have the tools to prove the main theorem, which we restate here.

Theorem 7.1. *Let M be a complete n-dimensional Riemannian manifold. Suppose that there is some $\kappa > 0$ such that*

$$Ric_p(v) \geq \kappa |v|^2$$

for all $(p, v) \in TM$. If $\operatorname{diam}(M) = \pi/\sqrt{\kappa}$, then M is isometric to the n-dimensional sphere of constant sectional curvature κ.

Proof. Since M is compact by Theorem 2.15, we can choose two points p and q in M such that $d(p, q) = \pi/\sqrt{\kappa}$. Then if we let

$$B_1 = B(p, \pi/(2\sqrt{\kappa})), \quad B_2 = B(q, \pi/(2\sqrt{\kappa})),$$

we have $B_1 \cap B_2 = \varnothing$ because $\operatorname{diam}(M) = \pi/\sqrt{\kappa}$. By Theorem 6.10, we have

$$n\kappa \leq \lambda(M). \tag{7.2}$$

By Theorem 3.24, we have

$$\lambda(M) \leq \max_{i \in \{1,2\}} \lambda^*(B_i).$$

By Theorem 6.8, we have

$$\lambda^*(B_i) \leq \lambda_\kappa(\pi/(2\sqrt{\kappa})).$$

Lastly, by computation (5.8) in the case of constant curvature, we have

$$\lambda_\kappa(\pi/(2\sqrt{\kappa})) = n\kappa.$$

Thus, all of the above inequalities must be equalities.

We want to show two facts: first, that

$$\lambda^*(B_1) = \lambda^*(B_2) = n\kappa, \tag{7.3}$$

and second, that

$$M = \overline{B_1} \cup \overline{B_2}. \tag{7.4}$$

If the first fact holds, then by the equality case in Theorem 6.8, each B_i is isometric to the n-dimensional hemisphere of constant sectional curvature κ. Then if both facts hold, we conclude that M is isometric to the n-dimensional sphere of constant sectional curvature κ. We demonstrate these facts by ruling out the three cases in which they do not both hold.

First, suppose both (7.3) and (7.4) do not hold. Then without loss of generality, we can suppose that

$$\lambda^*(B_1) < \lambda^*(B_2) = n\kappa.$$

Let $B_2' = M \setminus \overline{B_1}$. Since (7.4) does not hold, B_2' contains B_2 and $B_2' \setminus \overline{B_2}$ is an open set. Then by Theorem 3.25, we have

$$\lambda^*(B_2') < \lambda^*(B_2) = n\kappa.$$

But then applying Theorem 3.24 to B_1 and B_2' implies that $\lambda(M) < n\kappa$, which contradicts the fact that equality holds in (7.2).

Second, suppose that (7.3) holds but (7.4) does not hold. Equation (7.3) implies that each B_i is isometric to the n-dimensional hemisphere of constant sectional curvature κ by Theorem 6.8. So we have

$$\operatorname{vol}(M) > \operatorname{vol}(\overline{B_1} \cup \overline{B_2}) = \operatorname{vol}(\mathbb{S}^n(1/\sqrt{\kappa})),$$

where $\mathbb{S}^n(r)$ is the n-sphere of radius r. But this contradicts the fact that

$$\mathrm{vol}(M) \leq \mathrm{vol}(\mathbb{S}^n(1/\sqrt{\kappa}))$$

from Theorem 6.7.

Finally, suppose that (7.3) does not hold and (7.4) holds. Assume without loss of generality that

$$\lambda^*(B_1) < \lambda^*(B_2) = n\kappa.$$

Suppose that there is some $\epsilon > 0$ such that for the modified ball

$$B_1' = B(p, \pi/(2\sqrt{\kappa}) - \epsilon),$$

we have

$$\lambda^*(B_1') < n\kappa.$$

Then letting $B_2' = B(q, \pi/(2\sqrt{\kappa}) + \epsilon)$, Theorem 3.25 implies

$$\lambda^*(B_2') < \lambda^*(B_2) = n\kappa.$$

Since $d(p,q) = \pi/\sqrt{\kappa}$, we still have that $B_1' \cap B_2' = \varnothing$. Then by Theorem 3.24, we have $\lambda(M) < n\kappa$, which contradicts the fact that equality holds in (7.2). Thus, we only need to guarantee the existence of such an ϵ, which we show in Lemma 7.5.

Therefore, we conclude that both (7.3) and (7.4) hold, which completes the proof. $\square$

Lemma 7.5. *Let M be as in Theorem 7.1. Let B_1 and B_2 be as in the proof of Theorem 7.1 (with centers p and q respectively). If $M = \overline{B_1} \cup \overline{B_2}$ and*

$$\lambda^*(B_1) < \lambda^*(B_2) = n\kappa,$$

then there is some $\epsilon > 0$ such that

$$\lambda^*(B(p, \pi/(2\sqrt{\kappa}) - \epsilon)) < n\kappa.$$

Proof. To begin, we show that no point in $\overline{B_1}$ is a cut point of p. This fact implies that B_1 is diffeomorphic to the ball $B(0,\delta) \subset \mathbb{R}^n$, which is necessary for the radial scaling we use below. By Theorem 6.8, B_2 is isometric to the n-dimensional hemisphere of curvature κ. We need to show four facts.

First, we show that no point in ∂B_2 is a cut point of q. This fact holds by the cut point argument from the proof of the equality case of Theorem 6.8.

Second, we show that $\partial B_2 \subset \partial B_1$. Suppose $x \in \partial B_2$ and $x \notin \partial B_1$. Then there is a neighborhood U of x such that $U \cap \overline{B_1} = \varnothing$. Since x is not a cut point of q, U must contain a point not in $\overline{B_2}$. But this fact contradicts that $M = \overline{B_1} \cup \overline{B_2}$.

Third, we show that no point in ∂B_2 is a cut point of p. Let $x \in \partial B_2$. Let y be the midpoint of the unique length-minimizing geodesic (up to parametrization) joining q to x. Since no point in $\overline{B_2}$ is a cut point of q, x is the unique point in ∂B_2 that achieves the minimal distance to y. Since $\partial B_2 \subset \partial B_1$, the distance from p to every point in ∂B_2 is the same. Since M is complete, there is a length-minimizing geodesic joining p to y. That geodesic must intersect x, so x is not a cut point of p. Thus, no point in ∂B_2 is a cut point of p.

Fourth, we show that no point in $\overline{B_1}$ is a cut point of p. Since no point in ∂B_2 is a cut point of p and B_2 is a hemisphere, no point in $\overline{B_2} \setminus \{q\}$ is a cut point of p. Since the cut locus is connected and q is in the cut locus, we conclude that q is the only cut point of p. Since q is not in $\overline{B_1}$, no point in $\overline{B_1}$ is a cut point of p.

Now, we demonstrate the existence of the desired ϵ. Let $\xi = n\kappa - \lambda^*(B_1)$. Note that ξ is a positive number. Let $B_1' = B(p,r)$ for some $0 < r < \pi/(2\sqrt{\kappa})$. Let

$$c = \pi/(2\sqrt{\kappa}r).$$

Fix some $\delta > 0$. Let

$$F(t,v) = F(\exp_p(tv))$$

be a compactly supported smooth function on B_1 such that

$$D[F,F]/\|F\|^2 < \lambda^*(B_1) + \delta = n\kappa - \xi + \delta. \tag{7.6}$$

Let $\phi : B_1' \to B_1$ be the diffeomorphism corresponding to the radial scaling in coordinates given by

$$(t,v) \mapsto (ct,v).$$

Then we have

$$\begin{aligned}
\int_{B_1'} |\mathrm{grad}(F\circ\phi)|^2 \, dV &= \int_{\mathbb{S}^{n-1}} \int_0^r |\mathrm{grad}(F(ct,v))|^2 \theta(t,v) \, dt \, dA \\
&= c^2 \int_{\mathbb{S}^{n-1}} \int_0^r |(\mathrm{grad}F)(ct,v)|^2 \theta(t,v) \, dt \, dA \\
&= c \int_{\mathbb{S}^{n-1}} \int_0^{\pi/(2\sqrt{\kappa})} |(\mathrm{grad}F)(s,v)|^2 \theta(s/c,v) \, ds \, dA.
\end{aligned} \tag{7.7}$$

Since $\mathrm{grad}F$ is continuous in s and θ is continuous in its first argument, there is a continuous error function $\epsilon_1(c)$ that tends to 0 as c tends 1 so that (7.7) equals

$$c\left(\int_{B_1} |\mathrm{grad}F|^2 \, dV + \epsilon_1(c)\right).$$

We also have

$$\begin{aligned}
\int_{B_1'} |F\circ\phi|^2 \, dV &= \int_{\mathbb{S}^{n-1}} \int_0^r |F(ct,v)|^2 \theta(t,v) \, dt \, dA \\
&= c^{-1} \int_{\mathbb{S}^{n-1}} \int_0^{\pi/(2\sqrt{\kappa})} |F(s,v)|^2 \theta(s/c,v) \, ds \, dA.
\end{aligned} \tag{7.8}$$

By the same continuity argument, there is a continuous error function $\epsilon_2(c)$ that tends to 0 as c tends to 1 so that (7.8) equals

$$c^{-1}\left(\int_{B_1} |F|^2 \, dV + \epsilon_2(c)\right).$$

Then we have

$$\frac{\int_{B_1'} |\mathrm{grad}(F\circ\phi)|^2 \, dV}{\int_{B_1'} |F\circ\phi|^2 \, dV} = c^2 \left(\frac{\int_{B_1} |\mathrm{grad}F|^2 \, dV + \epsilon_1(c)}{\int_{B_1} |F|^2 \, dV + \epsilon_2(c)}\right).$$

The right hand side is continuous in c and tends to $D[F,F]/\|F\|^2$ as c tends to 1. Given (7.6), we can find r large enough (i.e. c close enough to 1) so that

$$\frac{\int_{B_1'} |\mathrm{grad}(F\circ\phi)|^2 \, dV}{\int_{B_1'} |F\circ\phi|^2 \, dV} \le n\kappa - \xi + \delta.$$

Thus,

$$\lambda^*(B_1') \le n\kappa - \xi + \delta.$$

The result holds by taking the limit as δ tends to zero. $\square$

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