INTRODUCTION TO HODGE THEORY!

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Abstract. We introduce real and complex Hodge theory to study topological invariants using harmonic analysis. To do so, we review Riemannian and complex geometry, introduce de Rham cohomology, and give the basic theorems of real and complex Hodge theory. To conclude, we present an application of the complex Hodge decomposition for Kähler manifolds to topology by working out the example of the $2n$-torus $T^{2n} = \mathbb{C}^n / \mathbb{Z}^{2n}$.

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1. Introduction

Fix $M$, a closed Riemannian $n$-manifold. Let $R = C^\infty(M, \mathbb{R})$ be the (infinite-dimensional) $\mathbb{R}$-algebra of smooth functions on $M$ and consider the $R$-module $\Omega^k(M)$ of differential $k$-forms, where $\alpha \in \Omega^k(M)$ measures flux through infinitesimal $k$-parallelotopes on $M$. There is a map $d = d_k : \Omega^k(M) \to \Omega^{k+1}(M)$ called the exterior derivative such that

$$\int_{\partial S} \alpha = \int_S d\alpha,$$

for all $(k+1)$-dimensional compact submanifolds $S \subseteq M$, so that $d\alpha(x)$ measures the flux of $\alpha$ through the boundary of an infinitesimal $(k+1)$-parallelotope at $x \in M$. Recall that
\(d^2 = 0\), so we have the chain complex

\[
0 \to \Omega^0(M) \xrightarrow{d} \Omega^1(M) \xrightarrow{d} \cdots \xrightarrow{d} \Omega^n(M) \to 0,
\]

which gives the de Rham cohomology groups \(H^k_{\text{dR}}(M) := \ker(d_k)/\text{im}(d_{k-1})\). de Rham’s Theorem \([2]\) gives an isomorphism between these cohomology groups and the usual singular cohomology groups with real coefficients, telling us that studying differential forms on \(M\) is strongly connected with studying the topological features of \(M\) itself.

For closed, orientable, Riemannian \(n\)-manifolds, the Hodge Theorem gives a decomposition of \(R\)-modules

\[
\Omega^k(M) = d\left(\Omega^{k-1}(M)\right) \oplus d^*\left(\Omega^{k+1}(M)\right) \oplus \mathcal{H}^k(M),
\]

where \(\mathcal{H}^k(M) := \ker(\Delta) \cap \Omega^k(M)\) denotes the space of harmonic \(k\)-forms and \(d^*\) is the adjoint to \(d\) under the \(L^2\) inner product arising from the Riemannian volume form. In particular, the groups \(H^k_{\text{dR}}(M)\) are naturally isomorphic to \(\mathcal{H}^k(M)\), so each cohomology class \([\alpha] \in H^k_{\text{dR}}(M)\) has a unique harmonic representative \(\omega \in \mathcal{H}^k(M)\) up to scaling.

When \(M\) is a complex manifold, considering the complex \(k\)-forms \(\Omega^k(M) \otimes \mathbb{C}\) as modules of the ring \(R \otimes \mathbb{C} = C^\infty(M, \mathbb{C})\), we can use the additional holomorphic structure to decompose forms as follows:

\[
\Omega^k(M) \otimes \mathbb{C} = \bigoplus_{p+q=k} \bigwedge^p \Omega^{1,0}(M) \otimes \bigwedge^q \Omega^{0,1}(M),
\]

where \(\Omega^{1,0}(M)\) is generated in local coordinates by \(dz^i\) as an \(R \otimes \mathbb{C}\)-module and \(\Omega^{0,1}(M)\) is generated by the conjugates \(d\bar{z}^j\). On Kähler manifolds, which are complex manifolds with a compatible symplectic structure, the Laplacian respects this refinement \([4]\). Hence, we obtain a decomposition of the cohomology groups

\[
H^k(M; \mathbb{C}) = \bigoplus_{p+q=k} \mathcal{H}^{p,q}_{\Delta}(M).
\]

These theorems show us ways that studying the analytic properties of harmonic forms on \(M\) is deeply connected to studying the topological properties of \(M\) as a space. In particular, this decomposition into \((p, q)\)-forms gives us constraints on the cohomology of Kähler manifolds, such as the property that odd Betti numbers of Kähler manifolds are even. This leads to other important results, such as the Lefschetz Hyperplane Theorem: see \([5]\).
2. Bundles on Manifolds

2.1. Real Manifolds.

Recall that a topological space $M$ is a real manifold if every point $p \in M$ admits an open neighborhood $U \subseteq M$ homeomorphic to $\mathbb{R}^n$. We call the homeomorphism $\phi : U \to \mathbb{R}^n$ the associated coordinate chart, and the inverse images of the usual coordinates on $\mathbb{R}^n$ are referred to as local coordinates $x^i$ on $M$. We call a collection of charts covering $M$ an atlas.

Further recall that if $(U, \phi)$ and $(V, \psi)$ are two coordinate charts with nonempty intersection, we can then define a transition map $\tau = \psi \circ \phi^{-1} : \phi(U \cap V) \to \psi(U \cap V)$ on subdomains of $\mathbb{R}^n$. If the transition maps of the manifold are smooth in the usual sense for each pair of charts, we say that $M$ is a smooth manifold. Similarly, replacing $\mathbb{R}$ with $\mathbb{C}$ and "smooth" with "holomorphic", we can define the notion of a complex manifold.

Lastly, recall that a function $f : M \to N$ between smooth manifolds is said to be smooth if it induces smooth functions on subdomains of $\mathbb{R}^m$ into subdomains of $\mathbb{R}^n$ by passing to coordinate charts, where $m$ and $n$ stand for the (locally constant) dimensions of $M$ and $N$ respectively.

**Definition 2.1.** Let $M$ be a smooth $n$-manifold. We say a smooth manifold $E$ is a bundle over $M$ with fiber $F$ if $E$ is locally trivial, i.e., for every $p \in M$, there exists a neighborhood $U$ and homeomorphism $\varphi : \pi^{-1}(U) \to U \times F$ such that the following diagram commutes

$$
\begin{array}{ccc}
\pi^{-1}(U) & \xrightarrow{\pi} & U \\
\downarrow \pi & & \downarrow \text{proj}_1 \\
U & & \end{array}
$$

If $E = \pi^{-1}(M) \cong M \times F$, then $E$ is said to be trivial. We write $F_p := \pi^{-1}(p)$ as the fiber over $p$ in the bundle $E$. If each fiber $F_p$ has the structure of a vector space and $\phi$ acts by linear isomorphisms on fibers, we say $E$ is a vector bundle.

Furthermore, given a vector bundle $E$, we define a section on $E$ to be a map $s : M \to E$ such that $\pi \circ s = \text{id}_M$. If $M$ is a smooth manifold, $E$ also has a smooth manifold structure. We denote by $\Gamma(E)$ the (infinite-dimensional) real vector space of smooth sections of $E$.

**Remark 2.2.** For any bundle $E$, we always have the zero section, given by

$$
(2.1) \quad u : M \to E, \quad u(x) = (x, 0).
$$

To show that this is well-defined, consider trivializations $h, h' : \pi^{-1}(U) \to U \times F$. The map $h' \circ h^{-1}|_{F_p} \in \text{Aut}(F_p) \cong \text{GL}_n(\mathbb{R})$ preserves zero for each $p \in U$, so the map $u$ is well-defined.
**Remark 2.3.** Here are a few common bundles that are useful to consider.

1. \(TM\), the **tangent bundle**. Choose \(p \in M\) and define a **tangent vector at** \(p\) to be a map \(v : C^\infty(M, \mathbb{R}) \to \mathbb{R}\) with the properties
   - \((a)\) \(v(f) \in \mathbb{R}\) for all \(f \in C^\infty(M, \mathbb{R})\).
   - \((b)\) \(v(\alpha f + \beta g) = \alpha v(f) + \beta v(g)\) for all \(\alpha, \beta \in \mathbb{R}\) and \(f, g \in C^\infty(M, \mathbb{R})\).
   - \((c)\) \(v(fg)(p) = f(p)v(g) + v(f)g(p)\).

   There are many equivalent viewpoints for defining tangent vectors—velocities of curves, derivations at a point, quotients of maximal ideals, etc. We will often use local coordinates, wherein tangent vectors can be expressed in the basis

   \[
   v = a^i \left. \frac{\partial}{\partial x^i} \right|_p,
   \]

   where we use Einstein summation convention here and throughout.

   The **tangent space of** \(M\) **at** \(p\) is the collection of all such vectors, denoted as \(T_pM\). Recall that \(\dim(T_pM) = \dim(M)\) for all connected manifolds \(M\) and points \(p \in M\). Additionally, we define the **tangent bundle of** \(M\) be the set

   \[
   TM = \bigsqcup_{p \in M} T_pM = \{(p, v)| p \in M, v \in T_pM\}.
   \]

   We also use the charts \(\phi_\alpha : U_\alpha \to \mathbb{R}^n\) of \(M\) to give \(TM\) a manifold structure

   \[
   \tilde{\phi}_\alpha : \pi^{-1}(U_\alpha) \to \mathbb{R}^{2n}
   \]

   \[
   \tilde{\phi}_\alpha \left( x, a^i \left. \frac{\partial}{\partial x^i} \right|_p \right) = (\phi_\alpha(x), a^1, ..., a^n).
   \]

2. \(T^*M\), the **cotangent bundle**. For each \(p \in M\), define \(T^*_pM := (T_pM)^*\) as the dual space. In local coordinates, we define the dual basis \(\{dx^i\}\) by

   \[
   dx^i \left( \frac{\partial}{\partial x^j} \right) = \delta^i_j = \begin{cases} 1, & i = j \\ 0, & i \neq j \end{cases}
   \]

   As a set, we define the cotangent bundle

   \[
   T^*M = \bigsqcup_{p \in M} T^*_pM
   \]

   and give \(T^*M\) a manifold structure in a similar fashion as before.

3. **Exterior powers.** Let \(\pi : E \to M\) be an arbitrary vector bundle. We consider the set

   \[
   \bigwedge^k(E) := \bigsqcup_{p \in M} \bigwedge^k F_p.
   \]
We assume the reader is comfortable with the construction of exterior powers of vector spaces: see [6] for more detail. To give \( \bigwedge^k(E) \) a manifold structure, we trivialize \( E \) using local coordinates \( \phi_\alpha \) with \( e^i \) the \( C^\infty(M, \mathbb{R}) \)-basis of sections in the trivialization. The basis for the neighborhood \( \pi^{-1}(U_\alpha) \) in \( \bigwedge^k(E) \) consists of elements of the form

\[
e^I := e^{i_1} \wedge \cdots \wedge e^{i_k},
\]

where \( I = (i_1, \ldots, i_k) \) and \( 1 \leq i_1 < i_2 < \cdots < i_k \leq n \). Hence, we have

\[
dim \left( \bigwedge^k(E) \right) = \binom{n}{k},
\]

where \( n = \text{dim}(E) \). The coordinate charts to give \( \bigwedge^k(E) \) a manifold structure are constructed in a similar method as before.

We call sections of \( TM \) the \textit{vector fields} on \( M \), sections of \( T^*M \) the \textit{1-forms} on \( M \), and sections of \( \bigwedge^k(T^*M) \) the \textit{k-forms} on \( M \). Recall that, given \( X \in \Gamma(TM) \), \( \alpha \in \Gamma(T^*M) \), there is a pointwise pairing to obtain a smooth function \( \alpha(X) \in C^\infty(M, \mathbb{R}) \). We also write

\[
\Omega^k(M) := \Gamma \left( \bigwedge^k(T^*M) \right)
\]

for the \textit{space of k-forms} on \( M \), whose elements measure \( k \)-volumes on \( M \) in the sense that they take \( k \)-tuples of vector fields and produce real numbers in an alternating fashion.

**Definition 2.4.** Choose \( x, y \in M \) and open sets \( x \in U, y \in V \) such that \( U \cap V \neq \emptyset \). Let \( \tau \) be a transition map from \( U \) to \( V \) with \( D\tau \) being the Jacobian matrix of \( \phi \). If \( \det(D\tau) > 0 \), we say \( \tau \) is an \textit{orientation-preserving transformation}. If there exists an atlas of \( M \) whose transition functions are all orientation-preserving, then \( M \) is said to be \textit{orientable}.

There is another formulation of orientability using differential forms which we will mention after reviewing Riemannian geometry. For more details, see [7].

**Definition 2.5.** We define a \textit{Riemannian metric} to be a section \( g \in \Gamma(T^*M \otimes T^*M) \) such that \( g(X, X) \geq 0 \) and \( g(X, Y) = g(Y, X) \) for all \( X, Y \in \Gamma(TM) \). In local coordinates, such a metric takes the form

\[
g := g_{ij} dx^i \otimes dx^j,
\]

where \( g_{ij} = g_{ji} \) is a positive-definite symmetric matrix. If \( M \) is equipped with a specific Riemannian metric, we say that \( M \) is a \textit{Riemannian manifold}. Such a manifold is thereby equipped with a smoothly varying way of measuring angles.
Remark 2.6. Given a Riemannian metric \( g \) on an orientable \( n \)-manifold \( M \), we can construct the volume form \( \text{vol} \in \Omega^n(M) \) pointwise:

\[
\text{vol} = \sqrt{\det(g_{ij})} dx^1 \wedge \cdots \wedge dx^n.
\]

If \( M \) is compact, we can define the volume of \( M \) as \( |M| = \int_M \text{vol} \).

We now review complex manifolds with Hermitian metrics.

2.2. Complex Manifolds.

When dealing with complex manifolds, we need not require orientability since the holomorphic transition maps preserve the choice of \( i \) as a square root of \(-1\) and hence give a canonical orientation to \( M \).

Definition 2.7. Given a \( n \)-dimensional real vector space \( V \), we define the complexification of \( V \) as \( V \otimes \mathbb{C} := V \otimes_{\mathbb{R}} \mathbb{C} \). We can give the complexification of \( V \) a complex vector space structure by the following: given \( v \otimes z \in V \otimes \mathbb{C} \) and \( w \in \mathbb{C} \), define

\[
w(v \otimes z) := v \otimes (wz).
\]

Note that while the complexification of \( V \) is larger than the original vector space, it behaves in much the same way as \( V \), however with an inherited complex structure. Next, we will define the complexification of vector bundles as a special case of complex vector bundles—bundles whose fibers are complex vector spaces.

Given a vector bundle \( \pi : E \to M \), we define

\[
E \otimes \mathbb{C} := \bigsqcup_{p \in M} (F_p \otimes \mathbb{C}).
\]

If \( M \) is a real manifold, we can write

\[
TM \otimes \mathbb{C} = \bigsqcup_{p \in M} (T_p M \otimes \mathbb{C})
\]

and use the coordinate charts \( \phi_\alpha \) of \( M \) to give \( TM \otimes \mathbb{C} \) a manifold structure:

\[
\tilde{\phi}_\alpha : \pi^{-1}(U_\alpha) \to U_\alpha \times \mathbb{R}^{2n},
\]

\[
\tilde{\phi}_\alpha(p, a_j \left( \frac{\partial}{\partial x} \otimes 1_\mathbb{C} \right)) = (\phi_\alpha(p), \text{Re}(a_1), \text{Im}(a_1), \ldots, \text{Re}(a_n), \text{Im}(a_n)),
\]

for \( a_i \in \mathbb{C} \). In local coordinates, vector fields \( X \in \Gamma(TM \otimes \mathbb{C}) \) take the form

\[
X(p) = a^i(p) \left. \frac{\partial}{\partial x^i} \right|_{p},
\]

where \( a^i \in C^\infty(M, \mathbb{C}) \). We define \( T^*M \otimes \mathbb{C} \) and \( \Omega^k(M) \otimes \mathbb{C} \) in a similar way.
If $M$ is a complex manifold, we use the complex coordinates $z^j = x^j + iy^j$, $\overline{z}^j = x^j - iy^j$ and write the Wirtinger basis for $TM \otimes \mathbb{C}$

\begin{align}
\frac{\partial}{\partial z^j} &= \frac{\partial}{\partial x^j} - i \frac{\partial}{\partial y^j} \\
\frac{\partial}{\partial \overline{z}^j} &= \frac{\partial}{\partial x^j} + i \frac{\partial}{\partial y^j}.
\end{align}

We use this definition so that the local Wirtinger basis for $T^*M \otimes \mathbb{C}$ can be written as

\begin{align}
dz^j &= dx^j + idy^j \\
d\overline{z}^j &= dx^j - idy^j.
\end{align}

Much as we defined Riemannian metrics on real manifolds, there is an analogous notion for complex manifolds. To do this, we need a notion of conjugate bundles.

**Definition 2.8.** Given a complex vector bundle $\pi : E \to M$, we define the conjugate bundle $\overline{E}$ by conjugating the complex scalar action on fibers.

**Definition 2.9.** We define a Hermitian metric to be a section $h \in \Gamma(T^*M \otimes \overline{T^*M})$ such that $h(X, X) \geq 0$ for all $X \in \Gamma(TM)$ and $h(X, Y) = \overline{h(Y, X)}$ for all $X, Y \in \Gamma(TM)$. In local coordinates, such a metric takes the form

\begin{equation}
h := h_{ij} dz^i \otimes d\overline{z}^j,
\end{equation}

where $h_{ij} = \overline{h_{ji}}$ is a positive definite Hermitian matrix. This gives a smoothly varying Hermitian inner product on $M$. If $M$ is equipped with a specific Hermitian metric, we say that $M$ is a Hermitian manifold.

**Remark 2.10.** There always exists a Hermitian metric on a complex manifold $M$. Given a Hermitian metric $h$ on $M$, we can recover a Riemannian metric associated to the real structure of the complex manifold by taking

\begin{equation}
g = \text{Re}(h) = \frac{1}{2} (h + \overline{h}) .
\end{equation}

3. **Real Hodge Theory**

For this section, let $M$ be a smooth, connected, closed, orientable, Riemannian $n$-manifold.

3.1. **de Rham Cohomology.**
Suppose $\alpha \in \Omega^k(M)$ can be written as $\alpha = d\omega$, for some $\omega \in \Omega^{k-1}(M)$. Such a form is called exact. Then we automatically know that $d\alpha = 0$ because $\ker(d_k) \supseteq \text{im}(d_{k-1})$. However, we can now ask the question: if $d\alpha = 0$, i.e. if $\alpha$ is a closed form, is there an $\omega$ such that $\omega = d\alpha$? Said differently, does each element that gets sent to zero by the exterior derivative have a preimage in $\Omega^{k-1}(M)$?
If the answer is always yes, then \( \ker(d_k) = \text{im}(d_{k-1}) \), or \( \ker(d_k)/\text{im}(d_{k-1}) \) is the trivial group. If the answer is no, then we have nontrivial cosets (called cohomology classes) of forms whose derivative is zero, but who do not have an antiderivative; the de Rham Theorem shows how the existence of such forms without antiderivatives is related to the topology of the underlying manifolds. Later in this section, we will discover a nice way of representing these cohomology classes using harmonic analysis. To be precise, we define the de Rham cohomology groups.

**Definition 3.1.** Consider the chain complex

\[
0 \to \Omega^0(M) \xrightarrow{d_0} \Omega^1(M) \xrightarrow{d_1} \cdots \xrightarrow{d_n} \Omega^n(M) \to 0.
\]

We define the de Rham cohomology groups as

\[
H^k_{\text{dR}} := \frac{\ker\{d_k : \Omega^k(M) \to \Omega^{k+1}(M)\}}{\text{im}\{d_{k-1} : \Omega^{k-1}(M) \to \Omega^k(M)\}}.
\]

We will now give an example for a manifold with a nontrivial cohomology group for clarity.

**Example 3.2.** Let \( S^1 \subseteq \mathbb{R}^2 \) and consider the 1-form \( \omega = \frac{x dy - y dx}{x^2 + y^2} \). We can compute

\[
d\omega = \left( \frac{x^2 + y^2 - 2x^2}{x^2 + y^2} \right) dx \wedge dy - \left( \frac{(-x^2 + y^2) - 2y^2}{x^2 + y^2} \right) dy \wedge dx = 0,
\]

but it is impossible to find a 0-form \( f \) such that \( \omega = df \). To show this, we can compute

\[
\int_{S^1} \omega = 2\pi
\]

and note that \( \partial S^1 = \emptyset \). However if \( \omega = df \) for some \( f \in \Omega^0(S^1) \), we use Stokes’ Theorem to write

\[
2\pi = \int_{S^1} df = \int_{\partial S^1} f = \int_{\emptyset} f = 0,
\]

a contradiction.

Thus, \([\omega] \in H^1_{\text{dR}}(S^1)\) is nontrivial. It turns out that \( H^1_{\text{dR}}(S^1) \) is generated by \([\omega]\), or that \( H^1_{\text{dR}}(S^1) \cong \mathbb{R} \) by the isomorphism \([\omega] \mapsto 2\pi\) given by integration.

Before moving onto Hodge theory, we first give a result by de Rham that will link our study of differential forms to topological invariants of a manifold. In essence, it tells us that studying differential forms on manifolds is connected to studying the topological properties of said manifold.
Theorem 3.3 (de Rham). Let $M$ be a smooth, closed $n$–manifold. Then, there is an isomorphism $H^k_{dR}(M) \cong H^k(M; \mathbb{R})$ given by the map

\begin{equation}
I : H^k_{dR}(M) \to H_k(M; \mathbb{R})^* \cong H^k(M; \mathbb{R})
\end{equation}

\begin{equation}
I([\alpha])([c]) := \int_C \alpha = \sum_{i=1}^r a_i \int_{\Delta^k} \sigma_i^*(\alpha).
\end{equation}

Here, $H^k(M; \mathbb{R})$ is the usual singular cohomology with real coefficients, and $C = \sum_{i=1}^r a_i \sigma_i$ is a representative for an arbitrary singular chain $[c] \in H_k(M, \mathbb{R})$.

The most important takeaway of this theorem is that studying differential forms on manifolds is equivalent to studying their topological properties, which is worthy motivation as we continue our delve into harmonic analysis.

3.2. Other Differential Operators. There is a pointwise pairing given by the wedge product: $\wedge : \Omega^k(M) \times \Omega^{n-k}(M) \to \Omega^n(M)$, allowing us to create $n$-forms from a $k$-form and an $(n-k)$-form. However, we would like to have a natural way to create an $(n-k)$-form from a $k$-form via some nondegenerate pairing, which is where the notion of the Hodge star arises.

Note that a Riemannian metric on $M$ gives rise to a Riemannian metric on $\Omega^k(M)$, so we can make the following definition.

Definition 3.4. Let $M$ be a smooth, connected, closed, orientable, Riemannian $n$-manifold. We define the Hodge star $\star : \Omega^k(M) \to \Omega^{n-k}(M)$ by the pairing

\begin{equation}
\alpha \wedge \star \beta = g(\alpha, \beta)\text{vol}.
\end{equation}

In local coordinates around $x \in M$ with an orthonormal basis $e_1, \ldots, e_n$ of $\Omega^1(M)$ trivialized around $x$, and $\pi \in S_n$ a permutation, we can write

\begin{equation}
\star e_{\pi(1)} \wedge \cdots \wedge e_{\pi(k)} = \text{sgn}(\pi)e_{\pi(k+1)} \wedge \cdots \wedge e_{\pi(n)}.
\end{equation}

One can then check that $\star^2 = (-1)^{k(n-k)}$, so $\star$ is an isomorphism and is thereby invertible with inverse $\star^{-1}$. From this, we can define the Hodge inner product on $\Omega^k(M)$

\begin{equation}
(\alpha, \beta) := \int_M \alpha \wedge \star \beta = \int_M g(\alpha, \beta)\text{vol}.
\end{equation}

With this inner product, we have a notion of adjoint operators, namely for the exterior derivative.
By Stokes’ Theorem, the assumption of $M$ being closed means $\int_M d\omega = 0$ for all $\omega \in \Omega^{n-1}(M)$. Then, choose $\alpha \in \Omega^k(M)$ and $\beta \in \Omega^{k+1}(M)$:

\begin{align}
0 &= \int_M d(\alpha \wedge \star \beta) \\
&= \int_M d\alpha \wedge \star \beta - (-1)^{k+1} \alpha \wedge d \star \beta \\
&= \int_M d\alpha \wedge \star \beta - \int_M \alpha \wedge (\star^{-1} (-1)^{k+1} d \star \beta) \\
&= (d\alpha, \beta) - (\alpha, (-1)^{k+1} \star^{-1} d \star \beta) .
\end{align}

(3.11)

We define the exterior coderivative $\delta_k = (-1)^{k+1} \star^{-1} d_k \star$ so that $\delta_k = d_k^*$. Just as we say a form $\alpha$ is closed if $d\alpha = 0$, we say $\alpha$ is coclosed if $\delta \alpha = 0$.

Recalling de Rham’s Theorem, we now have a framework for analyzing the topological properties of $M$ using cohomology classes in $H^k_{\text{dR}}(M)$, but it would be nice to have individual elements be representatives of this group instead of referring to an entire class of forms. One way to find such representatives is to pick closed elements of least magnitude with respect to the $L^2$ norm. If we take an arbitrary element $\alpha \in [\alpha] \in H^k_{\text{dR}}(M)$ and a test form $\phi$, we have the expansion

\begin{equation}
||\alpha + t\phi||^2 = ||\alpha||^2 + 2t(\alpha, d\phi) + O(t^2),
\end{equation}

(3.12)

so to minimize this quantity, we need $(\alpha, d\phi) = (\delta \alpha, \phi) = 0$ for all $\phi$. Thus, $\alpha$ needs to also be coclosed. This motivates the definition of the Hodge Laplacian

\begin{equation}
\Delta_k = d_{k+1} \delta_{k+1} + \delta_k d_k : \Omega^k(M) \to \Omega^k(M).
\end{equation}

(3.13)

We will write $\Delta := d\delta + \delta d$ when indices are unimportant.

**Example 3.5.** For $f \in \Omega^0(\mathbb{R}^3)$, we have

\begin{align}
\Delta f &= (d\delta + \delta d)f \\
&= 0 + \delta \nabla f \cdot (dx, dy, dz) \\
&= -\star^{-1} d\nabla f \cdot (dy \wedge dz, dz \wedge dx, dx \wedge dy) \\
&= -\star^{-1} \nabla \cdot \nabla f dx \wedge dy \wedge dz \\
&= -\nabla \cdot \nabla f \\
&=: -\nabla^2 f,
\end{align}

(3.14)

the usual Laplacian.
**Proposition 3.6.** Let $M$ be a closed, oriented, Riemannian $n$-manifold. Then,

1. $\Delta$ is self-adjoint.
2. $\Delta$ is positive semidefinite.
3. $\omega \in \ker(\Delta)$ if and only if $\omega \in \ker(d)$ and $\omega \in \ker(\delta)$.

**Proof.** Note that for $\alpha, \beta \in \Omega^k(M)$,

\[
(\Delta \alpha, \beta) = (d\delta \alpha, \beta) + (\delta d \alpha, \beta) = (\delta \alpha, \delta \beta) + (d \alpha, d \beta).
\]

To prove item 1, we note that this is equal to

\[
(\alpha, d\delta \beta) + (\alpha, \delta d \beta) = (\alpha, \Delta \beta).
\]

To prove items 2 and 3, we take $\alpha = \beta$ to get

\[
(\Delta \alpha, \alpha) = ||\delta \alpha||^2 + ||d \alpha||^2 \geq 0,
\]

with equality if and only if $d \alpha = \delta \alpha = 0$. □

It turns out that a stronger property than item 2 holds, namely that $\Delta$ is **elliptic**. In this context, an operator being elliptic refers to it being everywhere nonsingular. However, the proof of this is above the scope of this paper, so we will proceed with the Hodge Decomposition on real manifolds instead.

### 3.3. Hodge Theorems on Real Manifolds.

We review three major theorems regarding Hodge Theory on real manifolds. The first deals with the internal structure of the vector space $\Omega^k(M)$ with respect to $d, \delta$, and $\Delta$.

**Theorem 3.7** (Hodge Decomposition). Let $M$ be a closed, orientable, Riemannian $n$-manifold. For every $0 \leq k \leq n$, there is a decomposition of vector spaces

\[
\Omega^k(M) = \text{im}(d_{k-1}) \oplus \text{im}(\delta_k) \oplus H^k_\Delta(M).
\]

The proof of this theorem uses the theory of elliptic operators and Sobolev spaces, which is beyond the scope of this paper: see [3]. Combined with Proposition 3.6, Theorem 3.7 gives the following.

**Theorem 3.8** (Hodge). Let $M$ be a closed, orientable, Riemannian $n$-manifold. Then there is a canonical isomorphism

\[
H^k_\Delta(M) \cong H^k_{\text{dr}}(M).
\]

The essence of this theorem is that each cohomology class has a **unique** harmonic representative, up to scaling, which gives us a concrete method of obtaining representatives of cohomology classes.
The major difficulties in proving these statements are that \( \Omega^k(M) \) is not a Hilbert space and that \( \Delta \) is discontinuous. To see this, consider the example \( S^1 = \mathbb{R}/\mathbb{Z} \).

\[
\frac{d^2}{dx^2} \sin(nx) = -n^2 \sin(nx),
\]

for \( n \in \mathbb{Z} \). This implies that \( \| \Delta \sin(nx) \| \to \infty \) as \( |n| \to \infty \), meaning that \( \Delta \) is an unbounded operator. To get around these issues, \[2\] uses the fact that \( e^{-\Delta} \) is compact. However, it is worth mentioning the following corollary due to the finite-dimensionality of kernels of elliptic operators.

**Corollary 3.9.** Let \( M \) be a closed, orientable, Riemannian \( n \)-manifold. Then for all \( k \geq 0 \),

\[
(3.21) \quad \dim(H^k_{dR}(M)) < \infty.
\]

**Remark 3.10.** Recall that \( \ker(d_k) \) and \( \im(d_{k-1}) \) are infinite-dimensional vector spaces created by unbounded operators. It is remarkable that we can obtain a finite-dimensional quotient in such a non-obvious way!

From here, we will move into studying Hodge Theory on complex manifolds.

### 4. Complex Hodge Theory

Let \( M \) be a closed Hermitian \( n \)-manifold. Recall that a 1-form \( \omega \in \Gamma(T^*M \otimes \mathbb{C}) \) can be written in local coordinates as

\[
(4.1) \quad \omega = f_i dz^i + g_j d\bar{z}^j.
\]

Thus, we have a decomposition \( \Omega^1(M) \otimes \mathbb{C} = \Omega^{1,0}(M) \oplus \Omega^{0,1}(M) \), where \( \Omega^{1,0}(M) \) is generated by the \( dz^i \) and \( \Omega^{0,1}(M) \) is generated by the \( d\bar{z}^j \) as \( C^\infty(M, \mathbb{C}) \)-modules. This makes sense because our transition maps are holomorphic, so this structure is preserved globally. Furthermore, we also write \( \Omega^{p,q}(M) = \bigwedge^p \Omega^{1,0}(M) \otimes \bigwedge^q \Omega^{0,1}(M) \) as the vector space of \((p, q)\)-type forms, and we note the decomposition

\[
(4.2) \quad \Omega^k(M) \otimes \mathbb{C} = \bigoplus_{p+q=k} \Omega^{p,q}(M).
\]

**Proposition 4.1.** On a smooth, closed, Hermitian \( n \)-manifold \( M \), \( \bigwedge^{p,q}(M) \cong \Omega^{p,q}(M) \).

**Proof.**

\[
(4.3) \quad \bigwedge^{p,q}(M) = \bigwedge^p \Omega^{1,0}(M) \otimes \bigwedge^q \Omega^{0,1}(M) = \bigwedge^p \Omega^{1,0}(M) \otimes \bigwedge^q \Omega^{0,1}(M) = \bigwedge^p \Omega^{0,1}(M) \otimes \bigwedge^q \Omega^{1,0}(M) \cong \Omega^{p,q}(M).
\]

\[\square\]
Now, we would like to redefine the notion of a derivative that respects this additional structure. To do so, we introduce the Dolbeaut derivatives

\[ \partial : \Omega^0(M) \to \Omega^{1,0}(M) \]  
\[ \partial f = \frac{\partial}{\partial z^j} f dz^j \]  
\[ \partial : \Omega^0(M) \to \Omega^{1,0}(M) \]  
\[ \bar{\partial} g = \frac{\partial}{\partial \bar{z}^j} gd\bar{z}^j. \]

Additionally, the Dolbeaut operators can be used to give us maps

\[ \partial : \Omega^{p,q}(M) \to \Omega^{p+1,q}(M); \]
\[ \bar{\partial} : \Omega^{p,q}(M) \to \Omega^{p,q+1}(M) \]

using similar axioms to the exterior derivatives. These operators give rise to cohomologies since \( \partial^2 = \bar{\partial}^2 = 0 \), but we will not dive into those. Additionally, note that the usual exterior derivative \( d \) can be written as \( d = \partial + \bar{\partial} \).

From here, the goal is to build up the relevant framework to understand the Hodge Theorem in the complex formulation. To do so, we need a Hodge star, coderivatives, and Laplacians.

We first construct the complex Hodge star by considering the Hermitian metric \( h \) on \( M \) and taking the Riemannian metric \( g = \text{Re}(h) \) on \( M \), now viewed as a \( 2n \)-dimensional real manifold. At each point, we can now define the Hodge star by

\[ \star : \Omega^{p,q}(M) \to \Omega^{n-q,n-p}(M) \]
\[ \alpha \wedge \star \beta = g(\alpha, \bar{\beta}) \text{vol}. \]

Now, we are ready to define the Dolbeaut coderivatives by

\[ \partial^* : \Omega^{p,q}(M) \to \Omega^{p+1,q}(M) \]
\[ \partial^* \omega = (-1)^k \star^{-1} \bar{\partial} \star \omega \]
\[ \bar{\partial}^* : \Omega^{p,q}(M) \to \Omega^{p,q+1}(M) \]
\[ \bar{\partial}^* \omega = (-1)^k \star^{-1} \partial \star \omega. \]

These in turn define two new Laplacians given by

\[ \Delta_\partial = \partial \partial^* + \partial^* \partial \]
\[ \Delta_{\bar{\partial}} = \bar{\partial} \bar{\partial}^* + \bar{\partial}^* \bar{\partial}, \]
and it would be really nice if $\Delta = \Delta_\partial + \Delta_{\bar{\partial}}$, since this would respect the refined structure of $\Omega^{p,q}(M)$. However, this is not correct generally. To make this true, we need to define a new class of manifolds called Kähler manifolds.

**Definition 4.2.** Consider the map\footnote{This is the map corresponding to multiplication by $i$ in the complex numbers.} $J : TM \to TM$ given in local coordinates by

$$ (4.18) \quad J \begin{pmatrix} \partial_x^j \\ \partial_y^j \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \partial_x^j \\ \partial_y^j \end{pmatrix} $$

such that $J^2 = -\text{id}$. Note that the Wirtinger basis diagonalizes $J$. We define a $(1,1)$-form

$$ (4.19) \quad \omega(X,Y) = g(JX,Y). $$

One can show $\omega$ is non-degenerate and $\text{vol} = \frac{1}{n!} \omega^n$. If $d\omega = 0$, we say $\omega$ is a symplectic form. A complex Riemannian manifold with such a compatible symplectic form is called Kähler.

The most important result about Kähler manifolds for our purposes is the following.

**Proposition 4.3.** Let $M$ be a Kähler manifold. Then $\Delta = \Delta_\partial + \Delta_{\bar{\partial}}$.

**Sketch of Proof.**

- Define $S : \Omega^{p,q}(M) \to \Omega^{p+1,q+1}(M)$ by $S\alpha = \omega \wedge \alpha$, for the symplectic form $\omega$.
- After a calculation, one can show $[S^*, \partial] = i\partial^*$ and $[S^*, \bar{\partial}] = i\bar{\partial}^*$.
- It follows that $\partial\partial^* + \partial^*\partial + \bar{\partial}\bar{\partial}^* + \bar{\partial}^*\bar{\partial} = 0$ and thus $\Delta_\partial = \Delta_{\bar{\partial}}$.
- Direct computation gives $\Delta = 2\Delta_\partial = 2\Delta_{\bar{\partial}}$. In particular, $\Delta = \Delta_\partial + \Delta_{\bar{\partial}}$.

□

Now, we have an honest-to-goodness operator $\Delta : \Omega^{p,q}(M) \to \Omega^{p,q}(M)$ and can define

$$ (4.20) \quad \mathcal{H}^{p,q}_\Delta(M; \mathbb{C}) := \ker(\Delta) \cap \Omega^{p,q}(M). $$

Letting $h^{p,q}(M) = \dim_{\mathbb{C}} \mathcal{H}^{p,q}_\Delta(M)$ as a refinement of the usual Betti numbers, we are ready to state the complex Hodge Theorem.

**Theorem 4.4** (Hodge). For $M$ a closed Kähler manifold, there is a canonical isomorphism

$$ (4.21) \quad H^k_{dR}(M) \otimes \mathbb{C} \cong \bigoplus_{p+q=k} \mathcal{H}^{p,q}_\Delta(M). $$

Furthermore, $\Delta$ being real and Proposition 4.1 imply

$$ (4.22) \quad \mathcal{H}^{p,q}(M) \cong \overline{\mathcal{H}^{q,p}(M)} \text{ and } \mathcal{H}^{p,q}(M) \cong \overline{\mathcal{H}^{n-q,n-p}(M)}. $$

In particular, we have the Hodge diamond:

$$ (4.23) \quad h^{p,q}(M) = h^{q,p}(M) = h^{n-p,n-q}(M). $$
5. COHOMOLOGY OF COMPLEX TORI

Let $\mathbb{T}^2 = \mathbb{C}/\mathbb{Z}^2$ be the standard 2-torus. We use the local coordinates $z = x + iy$ and the Hermitian inner product of complex vector fields

$$h(f_1 \partial_z + g_1 \partial_{\bar{z}}, f_2 \partial_z + g_2 \partial_{\bar{z}}) = f_1 \bar{f}_2 + g_1 g_2.$$  

Taking the real part, we have the Riemannian metric

$$g(f_1 \partial_x + g_1 \partial_y, f_2 \partial_x + g_2 \partial_y) = f_1 f_2 + g_1 g_2.$$  

We use the operator $J$ acting on $T\mathbb{C}$ by

$$J \begin{pmatrix} \partial_x \\ \partial_y \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \partial_x \\ \partial_y \end{pmatrix}.$$  

We have the symplectic form

$$\omega(f_1 \partial_x + g_1 \partial_y, f_2 \partial_x + g_2 \partial_y) = f_1 g_2 - g_1 f_2.$$  

The relevant Hodge star computations we need are

$$\star dz = id\bar{z} \quad \star d\bar{z} = -idz \quad \star (dz \wedge d\bar{z}) = -2i$$

We now can directly calculate $\Delta f$ for $f \in \Omega^0(\mathbb{T}^2)$.

$$\Delta f = \star d \star f = - (\partial_{xx} f + \partial_{yy} f).$$

Since the functions $z, \bar{z}$ are not well-defined on $\mathbb{T}^2$, the only solution to the Dirichlet problem $\Delta f = 0$ is $f = c$, some constant. Thus, we have shown

$$\dim_{\mathbb{C}}(\mathcal{H}_\Delta^0(\mathbb{T}^2)) = h^{0,0}(\mathbb{T}^2) = 1.$$  

Similarly, we can compute

$$\Delta(f dz + gd\bar{z}) = \Delta f dz + \Delta g d\bar{z}$$

and

$$\Delta(f dz \wedge d\bar{z}) = \Delta f dz \wedge d\bar{z},$$

leaving the details as an exercise to the reader. By the previous argument, we have shown

$$h^{0,0}(\mathbb{T}^2) = h^{0,1}(\mathbb{T}^2) = h^{1,1}(\mathbb{T}^2) = 1.$$  

We can rearrange this slightly to better visualize the Hodge Diamond.

$$\begin{array}{c}
 h^{1,1} = 1 \\
 h^{1,0} = 1 \\
 h^{0,1} = 1 \\
 h^{0,0} = 1 \\
\end{array}$$
Note that the torus is connected, that it contains two 1-dimensional holes, and that there is one 2-dimensional void, giving a notion of inside and outside. Readers familiar with topology realize these are the Betti numbers of the torus, topological invariants coming from the usual homotopy theory. In general, we can define the standard $2n$-torus $\mathbb{T}^{2n} = \mathbb{C}^n/\mathbb{Z}^{2n}$, obtaining that $h^{p,q}(\mathbb{T}^{2n}) = \binom{n}{p} \binom{n}{q}$ and

\begin{align}
\dim(H^k(\mathbb{T}^{2n}; \mathbb{C})) = \sum_{p+q=k} \dim(\mathcal{H}^{p,q}(\mathbb{T}^{2n})) = \sum_{p+q=k} \binom{n}{p} \binom{n}{q} = \binom{2n}{k}.
\end{align}

In general, the Hodge Diamond implies odd Betti numbers of Kähler manifolds are even.

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