BAYESIAN GAMES: GAMES OF INCOMPLETE INFORMATION

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Abstract. This paper is on Bayesian Games, which are games with incomplete information. We will start with a brief introduction into game theory, specifically about Nash equilibriums and mixed strategies. Then we will move to discuss the concept of common knowledge, various equilibrium concepts, and Bayesian games. Finally we will study the Chain Store paradox as an application of the Bayesian game model. This paper will assume familiarity with basic probability. I will briefly discuss introductory topics in game theory.

Contents

1. Introduction

Game theory is the study of interactions between players that will result in players getting a certain amount of reward. Bayesian games are games with incomplete information, which are, informally, games where players may not know all aspects of the game, such as the payoff functions for other players. For example, consider a game where the players will get different payoffs depending on whether they are agressive players or passive players. In this case, the exact payoff function of a player may not be known to the other players if they do not know whether they are agressive or passive.

Sections 5 and 6 will be about different equilibrium concepts, and section 7 will introduce Bayesian games and include the main theorem of this paper, which is that Bayesian games have Bayesian Nash equilibrium. Section 8 will illustrate Bayesian games using an example.

Date: November 23, 2018.

2. Nash Equilibrium

In games, there is a concept of an equilibrium point, which is, in a loose sense, a strategy profile where each player cannot get a better payoff by deviating from the action they take with the strategy. In this case, we use strategy to refer to what are now known as pure strategies, which dictate a specific action in the game for the player to take. For this section only, strategy may mean the same thing as action in the game.

We will begin with some definitions that will be necessary to the rest of the paper. For the rest of this section, we will use N to refer to the set of n players. We will also refer to the set of actions player i can take as A_i, a strategy profile as $a := (a_1, ..., a_n)$, and the set of strategy profiles as A. We will also use a_{-i} to refer to the strategy profile a without the ith strategy.

Definition 2.1. A utility function u_i is a function that assigns a value in $\mathbb{R}$ to each action a player i can take, so $u_i : A_i \to \mathbb{R}$.

Definition 2.2. A strategy profile a is an equilibrium point if for all i and for all $\hat{a}_i \in A_i$,

$$u_i(a) \geq u_i(a_{-i}, \hat{a}_i). \tag{2.3}$$

Nash proved that finite games have equilibrium points by using the concept of a mixed strategy.

Definition 2.4. A mixed strategy is a probability assignment over the set of pure strategies.

He used this notion of a mixed strategy to prove that every finite game has an equilibrium point.

Theorem 2.5. *Every finite game has an equilibrium point.*

We now refer to these equilibrium points as Nash equilibriums. Since this paper is focused on Bayesian games, we will not prove Nash's theorem, but a proof using fixed point theorems can be found in Nash's dissertation, which is cited at the end of this paper. However, we will illustrate an example to show the differences between pure strategies and mixed strategies and how mixed strategies can induce equilibriums in cases where pure strategies cannot.

Example 2.6. Suppose there are two players in a game, player 1 and player 2. Each player flips a fair coin. If their coins land on the same side, player 1 wins \$1 from player 2. Otherwise, player 1 pays player 2 \$1. The table for this game looks like this.

		Player 2	
		H	T
Player 1	H	$(1, -1)$	$(-1, 1)$
	T	$(-1, 1)$	$(1, -1)$

If we only look at pure strategies, this game has no Nash equilibrium. We can see this by simply looking at the four pure strategy profiles: $(H, H), (H, T), (T, H), (T, T)$.

If (H, H) is played, player 2 gets more by deviating to (H, T). If (H, T) is played, player 1 gets more by deviating to (T, T). The logic follows for the other two pure

strategy profiles. This shows that there is no Nash equilibrium in pure strategies for this case.

However, if we allow for mixed strategies, the strategy where each player chooses H with $\frac{1}{2}$ probability and chooses T with $\frac{1}{2}$ probability is a Nash equilbrium.

We end this section with a self-enforcing criterion on Nash equilibriums that we will refer to in later sections.

Definition 2.7. A Nash equilibrium s is self-enforcing if, when players are asked to play according to s, no player has incentives to deviate from it.

3. Knowledge

In this section, we will discuss knowledge, particularly what it means for something to be common knowledge.

Definition 3.1. An information model for a set of players $N = \{1, ..., n\}$ is a triple $I = (\Omega, \{\rho_i\}_{i\in N}, \{\mathcal{P}_i\}_{i\in N})$, whose elements are the following:

(1) A finite set Ω, with generic element ω.
(2) For all players $i \in N$ a probability measure ρ_i on Ω
(3) For all players $i \in N$, a partition $\mathcal{P}_i$ of Ω.

In a practical sense, Ω represents the set of all possible states of the world. The probability measures represent the prior beliefs of the different players about the state of the world that is going to be realized. In other words, the player i's probability measure represents what i believes about the state of the world. Partition $\mathcal{P}_i$ denotes the information partition of player i and is structured so that given a set P in $\mathcal{P}_i$, if the true state of the world is ω in P, then a player i only knows that the true state is some element of P but is not sure about which one. In other words, the elements of P are indistinguishable to player i.

Let $P_i(\omega)$ denote the set in the information partition $\mathcal{P}_i$ of player i which contains the true state of the world ω. Each subset E of Ω is an event.

Definition 3.2. An information model I has a common prior if for all $i, j \in N$, $\rho_i = \rho_j$. The prior beliefs in information models with a common prior are denoted using ρ instead of $\{\rho_i\}_{i\in N}$.

Let event $E \subset \Omega$ and ω be a state of the world in Ω, we say that player i knows event E at ω if $P_i(\omega)$ is a subset of E. Let $K_iE := \{\omega \in \Omega : P_i(\omega) \subset E\}$ denote the set of states of the world at which player i knows E. Then let $K_*E := \bigcap_{i\in N} K_iE$ denote the set of states of the world at which all the players know E.

For player 1 to know that player 2 knows an event $E \subset \Omega$ when the true state of the world is $a \in \Omega$, this means that for each $x \in P_1(a)$, the set $P_2(x) \subset E$.

Definition 3.3. An event $E \subset \Omega$ is common knowledge at $\omega \in \Omega$ if $\omega \in CKE$, where $CKE = K_*E \cap K_*K_*E \cap K_*K_*K_*E \cap \dots$.

In other words, an event is common knowledge if all the players know it, all the players know that all the players know it, all the players know that all the players know that all the players know it, ...

If $A \subset \Omega$ is an event, let $\underline{q}_i$ denote the posterior probability $p(A|\mathcal{P}_i)$ of A given $\mathcal{P}_i$. In other words, if $\omega \in \Omega$, then $\underline{q}_i(\omega) = \frac{p(A\cap P_i(\omega))}{p(P_i(\omega))}$.

Proposition 3.4. *Let $\omega \in \Omega$, and let q_1 and q_2 be numbers. If it is common knowledge at ω that $\underline{q}_1 = q_1$ and $\underline{q}_2 = q_2$, then $\underline{q}_1 = \underline{q}_2$. In other words, if two people have common prior beliefs about a certain event occurring, then they have the same posterior beliefs about the event.*

This theorem is not very relevent to the rest of this paper, but for the sake of understanding the mathematical concept of common knowledge, I will give the sketch of the proof.

Proof. Let P be the member of the finest common coarsening of $\mathcal{P}_1$ and $\mathcal{P}_2$ that contains ω. Since, for each i, P can be written as the union of disjoint members of $\mathcal{P}_i$, let $P = \cup_j P^j$, where the P^j are the disjoint members of $\mathcal{P}_i$. Since the set of nodes satisfying $\underline{q}_1 = q_1$, which we will denote E, is common knowledge, this means that $\omega \in CKE$, so we have that $\frac{p(A \cap P^j)}{p(P^j)} = q_1$ for all j. So $p(A \cap P^j) = q_1 p(P^j)$. Since the P^j are disjoint, we can sum over j to get $p(A \cap P) = q_1 p(P)$. Similarly, $p(A \cap P) = q_2 p(P)$ so $q_1 = q_2$. □

4. Extensive Games

Up to now, specifically in the Nash equilibrium section, we were dealing with static interactions between the players. In other words, we were only dealing with games where the players simultaneously make their decisions. We call these games strategic games. Now we move to discussing extensive games. Extensive games are a way to model games that have nonstatic interactions. We begin with an example to show the relationship between extensive games and strategic games.

Example 4.1. We will show an example of how a nonstatic interaction can be modeled in the way we have been thinking of games previously.

Suppose there are two players. First, player 1 chooses between L_1 and R_1. Then player 2, knowing what player 1 chose, can choose between L_2 and R_2 if player 1 chose L_1 and between l_2 and r_2 if player 2 chose R_1. Finally, if L_1 and L_2 were chosen, player 1 chooses between l_1 and r_1. Otherwise, the game ends with player 2's choice.

The strategy set for player 1 takes into account all the circumstances they will need to make a decision, even if it is impossible for the structure of this game. The strategy set for player 1 is $A_1 = \{L_1 l_1, L_1 r_1, R_1 l_1, R_1 r_1\}$ and the strategy set for player 2 is $A_2 = \{L_2 l_2, L_2 r_2, R_2 l_2, R_2 r_2\}$.

This example shows how we can model nonstatic interactions using a model where all players make their decisions simultaneously. However, by using this model, we neglect to account for certain circumstances. For example, we ignore the fact that the case where player 1 plays $R_1 l_1$ is impossible to reach given this game. Using the extensive games model, we can give a more detailed description of nonstatic games.

Definition 4.2. An n-player extensive game with the set of players N is a 7-tuple $\Gamma := (X, E, P, W, C, p, U)$ with elements:

(1) Game tree (X, E) is a finite tree such that
   (a) there exists a unique root node r such that there is no $x \in X$ where $(x, r) \in E$ and
   (b) for all $x \in X \setminus \{r\}$ there exists a unique path connecting r and x.

Let $F(x)$ denote the set of all nodes that come after x and let Z denote the set of terminal nodes of the tree.

(2) Player parition $P := \{P_0, P_1, ..., P_n\}$ is a partition of $X \setminus Z$ that indicates for each nonterminal node x, which player has to make a decision at x.
(3) Information partition $W := \{W_0, W_1, ..., W_n\}$ such that for all $i \in N$, W_i is a partition of P_i. Each $w \in W_i$ is a set that contains the nodes of P_i in which i has the same information about what has happened in the game up to that point. In other words, if $x, y \in w \in W_i$, player i does not know if they are at point x or point y.
(4) A choice for player $i \in N$ at information set $w \in W_i$ is a set that contains, for each $x \in w$, one alternative at x, represented by an edge from x to a node in $F(x)$ in E. C is the set of all possible choices of the different players at different information sets. Note that C is a partition of the set of edges starting at nodes outside P_0, and we call it the choice partition. For each $w \in W_i$, let C_w be the set of all choices at w. We say a node x comes after a choice c if one of the edges of c is on the path that connects r and x.
(5) p is a map that assigns to each $x \in P_0$ a probability distribution p_x, defined over the set of edges starting at x.
(6) $U := (U_1, ..., U_n)$ is the n-tuple of utility functions defined over Z.

The (non-terminal) nodes of an extensive game represent points in the game where a player makes a choice. Sometimes, a player may only have one choice.

Definition 4.3. An extensive game Γ has perfect information if for all players $i \in N$, each $w \in W_i$ contains exactly one node of $X \setminus Z$. If an extensive game does not have perfect information, then it has imperfect information.

We will model games of incomplete information as games of imperfect information, so it is important to note how they are different. The difference between imperfect and incomplete information is that in imperfect information games, not all the players will always know what choices were taken by other players on previous points of the game. In incomplete information games, the players do not know some aspects of the game. We will go into more detail about incomplete information games in a later section.

Definition 4.4. An extensive game Γ has perfect recall if, for all players $i \in N$ and each pair $w, \hat{w} \in W_i$, the game satisifies this property: if one node $x \in \hat{w}$ comes after choice $c \in C_w$, then every node $\hat{x} \in \hat{w}$ comes after c. In other words, a game has perfect recall if players recall what choices they made at previous points of the game. A game that does not have perfect recall has imperfect recall.

Now we discuss how strategies of extensive games are defined.

Definition 4.5. Let Γ be an extensive game with N set of players. A pure strategy a_i of player i is a map that assigns, to each $w \in W_i$, a choice $a_i(w) \in C_w$. A pure strategy profile of an extensive game is an n-tuple of pure strategies. We define A_i to be the set of pure strategies of player i, and A to be the set of pure strategy profiles.

As with games studied by Nash, we will define mixed strategies to be a lottery over the pure strategies of Γ. We will denote a mixed strategy of player i using s_i. We denote the n-tuple mixed strategy profile using s. We define S_i to be the set of mixed strategies of i and S to be the set of mixed strategy profiles.

Definition 4.6. Let Γ be an extensive game with N set of players. A behavior strategy b_i of player i is a map that assigns, to each $w \in W_i$, a lottery over C_w. For each $w \in W_i$ and each $c \in C_w$, $b_i(c)$ is the probability that P_i assigns to choice c when they are at information set w. A behavior strategy profile is an n-tuple of behavior strategies. We define B_i to be the set of behavior strategies of P_i and B to be the set of all behavior strategy profiles.

Let Γ be an extensive game. Let player $i \in N$. For each $w \in W_i$, defining our behavior strategy to assign probability 1 to a choice and 0 to all other choices at C_w. This shows that every pure strategy can be seen as a behavior strategy, so $A \subset B$.

Now for each $b \in B$ and each $x \in X$, let $p(x, b)$ be the probability that node x is reached if strategy b is played. For all players $i \in N$, i's payoff function is defined to be $u_i : B \to \mathbb{R}$ such that,

$$u_i(b) = \sum_{z \in Z} p(z, b) U_i(z). \tag{4.7}$$

Note that player i's payoff function is their expected utility.

Definition 4.8. Let Γ be an extensive game. Let player $i \in N$ and let $s_i \in S_i$ and $b_i \in B_i$. The strategies s_i and b_i are realization equivalent if for each $\hat{s}_{-i} \in S_{-i}$ and each $x \in X$, $p(x, (\hat{s}_{-i}, s_i)) = p(x, (\hat{s}_{-i}, b_i))$. In other words, s_i and b_i induce the same probabilities over X.

We continue using the notion of Nash equilibrium as before, this time extended to include behavior strategies in the analogous way.

We begin with a theorem to show the relationship between behavior strategies and mixed strategies.

Theorem 4.9 (Kuhn Theorem)**.** *Let Γ be an extensive game with perfect recall. Let $i \in N$, $s_i \in S_i$. Then there exists a behavior strategy $b_i \in B_i$ such that s_i and b_i are realization equivalent.*

Proof. Let $s \in S$ be a mixed strategy profile and let player $i \in N$. Let x be a node of information set $w \in W_i$ such that $p(x, s) > 0$. Let $c \in C_w$. Let $\overline{x}$ be the node that is reached when choice c is taken at x. We denote the conditional probability that c is chosen given that x has been reached by $p(c, x, s)$ Then

$$p(c, x, s) = p_x(\overline{x}, s) = \frac{p(\overline{x}, s)}{p(x, s)}. \tag{4.10}$$

First, we will show that $p(c, x, s)$ does not depend on s_{-i}. In other words, if $x, \hat{x} \in w \in W_i$, then $p(c, x, s) = p(c, \hat{x}, s)$. Then, we will use that fact to construct a b_i that is realization equivalent to s_i.

Given $a_i \in A_i$, we define $si(a_i|x)$ to be the probability that a_i is played given that x is reached. Given $\hat{x} \in X$, we define $A_i(\hat{x}) \subset A$ to be the set of pure strategies of player i that select the choices on the path to $\hat{x}$. Then,

$$s_i(a_i|x) = \begin{cases} \frac{s_i(a_i)}{\sum_{\hat{a}_i \in A_i(x)} s_i(\hat{a}_i)} & a_i \in A_i(x) \\ 0 & \text{otherwise} \end{cases} \tag{4.11}$$

So $p(c, x, s) = \sum_{\hat{a_i} \in A_i(\overline{x})} s_i(\hat{a}_i|x)$, which does not depend on s_{-i}.

Now let $x, \hat{x} \in w \in W_i$. Since Γ has perfect recall, the player i's choices on the path to x are the same as the choices of player i on the path to $\hat{x}$. In that case, for all $a_i \in A_i$, $s_i(a_i|x) = s_i(a_i|\hat{x})$ and $p(c, x, s) = p(c, \hat{x}, s)$. If information set $w \in W_i$ is reached, then we can define $p(c, w, s_i)$ as the conditional probability of i choosing choice c at w. Since $p(c, x, s)$ is independent of s_{-i}, we know that $p(c, w, s_i) = p(c, x, s)$, where x is any node in w.

Now we use the probabilities to construct the b_i that will be realization equivalent to s_i. Let $w \in W_i$. If there is $x \in w$ such that some strategy in $A_i(x)$ is selected by s_i with probability greater than zero, then for each $c \in C_w$, $b_i(c)$ will be defined to be

$$b_i(c) := p(c, w, s_i) \tag{4.12}$$

Now we assign arbitrary choices at information sets that will not be reached by s_i. Then any b_i satisfying these properties will be realization equivalent to s_i. □

Now we will use the Kuhn theorem to show that every extensive game with perfect recall has a Nash equilibrium.

Theorem 4.13. *Let Γ be an extensive game with perfect recall. Then Γ has at least one Nash equilibrium.*

Proof. We know that the mixed extension of the strategic game associated with Γ has at least one Nash equilibrium from Nash's theorem. Now let $s* \in S$ be that equilibrium point.

The main idea of this proof is to reduce this case to the original situation with mixed strategies then show that the behavior strategy that is realization equivalent is a Nash equilibrium. By the Kuhn theorem, there exists a behavior strategy $b* \in B$ such that $s*$ and $b*$ are realization equivalent. This is the most important part of this proof, which is why we need the Kuhn theorem. This must mean then that $s*$ and $b*$ will give the same payoff for all players (we call this being payoff equivalent). So $u(s*) = u(b*)$, where u is the utility function of Γ.

Now we want to show that $b*$ is a Nash equilibrium of Γ. Suppose $b*$ is not a Nash equilibrium of Γ. Then there exists a player $i \in N$ and a strategy of player i $b_i \in B_i$ such that $u_i(b*) < u_i(b^*_{-i}, b_i)$. Since $s*$ and $b*$ are payoff equivalent, this must mean that

$$u_i(s*) < u_i(b^*_{-i}, b_i) = u_i(s^*_{-i}, b_i) \tag{4.14}$$

So $s*$ is not a Nash equilibrium, which creates a contradiction. Therefore, $b*$ is a Nash equilibrium. □

However, there is a weakness in Nash equilibrium for extensive games. The weakness comes from the possible paths that are impossible given the game. We illustrate this with an example, which we will expand upon later in the paper.

Example 4.15. The following game is called the chain store game.

There is a market with two players, the monopolist and the entrant, which we will denote m and e. First, the entrant decides to stay out of the market (O), or to enter the market (E). If the entrant enters, the monopolist can either fight (F) or yield (Y).

If the entrant stays out, the payoff for the entrant is 0 and the payoff for the monopolist is $\alpha > 1$. If the entrant enters and the monopolist yields, the entrant's payoff is β where $0 < \beta < 1$ and the monopolist's payoff is 0. If the entrant enters

and the monopolist fights, the entrant's payoff is $\beta - 1$ and the monopolist's payoff is -1.

Both (O, F) an (E, Y) are Nash equilibriums. However, (O, F) is not a self-enforcing equilibrium. Fighting is the best reply for player 2 when the entrant stays out because player 2's information set is not reached. However, in a practical sense, if player 2's information set is reached, player 2 will choose to yield since they get a higher payoff by yielding. If player 1 realizes that player 2 will always yield, player 1 will always choose to enter.

Thus, the sensible Nash equilibrium is (E, Y).

Because of this weakness of how we have previously thought of equilibriums, we need a more sensible concept of equilibrium that is based on the condition of a player's information set being reached.

For all $b \in B$, player $i \in N$, and all $w \in W_i$ such that $p(w, b) > 0$, we will let $u_{iw}(b)$ denote the conditional expected utility of player i once w has been reached. Then $u_{iw}(b) := \sum_{z \in Z} p_w(z, b) u_i(z)$.

Definition 4.16. Let Γ be an extensive game and let $b \in B$ and $\overline{b}_i \in B_i$. Let $w \in W_i$ be such that $p(w, b) > 0$. Then $\overline{b}_i$ is a best reply of player i against b at w if

$$u_{iw}(b_{-i}, \overline{b}_i) = \max_{\hat{b}_i \in B_i} u_{iw}(b_{-i}, \hat{b}_i) \tag{4.17}$$

We will use this definition of best reply in later sections about different equilibrium concepts.

5. Subgame Perfect Equilibrium

Given the weakness in the Nash equilibrium concept for extensive games, we give another concept of equilibrium.

Given $\Gamma = (X, M, P, W, C, p, U)$ and node $x \in X$, $F(x)$ denotes the nodes that come after x in the tree of Γ including x.

Definition 5.1. Let Γ be an extensive game and let $x \in X \setminus Z$. The game Γ can be decomposed at x if there is no information set that contains both nodes from $F(x)$ and nodes from the set $X \setminus F(x)$.

Note that Γ cannot be decomposed at $x \in X$ that are contained in information sets that contain more than one node.

Definition 5.2. Let Γ be an extensive game that can be decomposed at $x \in X \setminus Z$. A subgame Γ_x of Γ is the game that Γ induces in the tree whose root node is x. We call a subgame of Γ a proper subgame if its root node is not r.

Now we give the definition of subgame perfect equilibrium.

Definition 5.3. Let Γ be an extensive game. A subgame perfect equilibrium of Γ is a strategy profile $b \in B$ such that for each subgame Γ_x of Γ, b_x is a Nash equilibrium of Γ_x.

Every subgame perfect equilibrium is a Nash equilibrium since every game is a subgame of itself. We will now illustrate subgame perfect equilibrium using the chain store example from before.

Example 5.4. In the chain store example, the only proper subgame is player 2 choosing to yield or to fight after player 1 chooses to enter. Player 2 will choose to yield in that case. So (E, Y) is the only subgame perfect equilibrium of the chain store game.

Now we move on to prove the existance of a subgame perfect equilibrium in games with perfect recall. We define the length of the extensive game Γ by the number of edges of the longest play of Γ. We denote the length by $L(\Gamma)$.

We also use $\Gamma^{b_{x_1,\ldots,x_m}}_{-(x_1,\ldots,x_m)}$ to denote the game Γ with subgames starting at $x_1, ..., x_m$ removed and defining, for each player $i \in N$ and for each $k \in \{1, ..., m\}$,

$$U_i^{\Gamma^{b_{x_1,\ldots,x_m}}_{-(x_1,\ldots,x_m)}}(x_k) := u_{ix_k}(b) \tag{5.5}$$

Theorem 5.6. *Every extensive game with perfect recall has at least one subgame perfect equilibrium.*

Proof. We show this by induction on $L(\Gamma)$.

We start with $L(\Gamma) = 1$. Then the Nash equilibrium that exists is a subgame perfect equilibrium since the game cannot be decomposed into subgames.

Now we assume that every extensive game with perfect recall of length $t-1$ has at least one subgame perfect equilibrium. Let Γ be an game with length t. If Γ has no proper subgames, then it is true that the Nash equilibrium is a subgame perfect equilibrium.

Suppose Γ has proper subgames. Then decompose Γ at nodes $x_1, ..., x_m \in X \setminus Z$ such that for any two $k, l \in \{1, ..., m\}$, x_k does not come after x_l and $\Gamma^{b_{x_1,\ldots,x_m}}_{-(x_1,\ldots,x_m)}$ does not have proper subgames. Then for each $k \in \{1, ..., m\}$, $L(\Gamma_{x_k}) \leq t-1$, so $L(\Gamma_{x_k})$ has a subgame perfect equilibrium $b^*_{x_k}$ of Γ_{x_k}. We know that the game $\Gamma^{b_{x_1,\ldots,x_m}}_{-(x_1,\ldots,x_m)}$ has a Nash equilibrium $b_{-(x_1,\ldots,x_m)}$.

Now we can construct a subgame perfect equilibrium. Let $b* = (b_{-(x_1,\ldots,x_m)}, b^*_{x_1}, ..., b^*_{x_1})$. Now $b*$ is a subgame perfect equilibrium. □

Subgame perfect equilibriums are sensible equilibriums for games with perfect information, because all information sets contain just one node, and so the game can be decomposed into proper subgames. However, if a game does not contain proper subgames, subgame perfect equlibrium does not refine Nash equilibriums to equlibriums that are self-enforcing.

6. Sequential Equilibriums

As we have seen in the previous section, the concept of subgame perfect equilibrium is flawed for games with imperfect information. To find a more sensible equilibrium concept, we want to find a definition of equilibrium that will maximize a player's expected payoff at all their information sets, not just at information sets with a node at which the game can be decomposed.

Definition 6.1. A system of beliefs over $X \setminus Z$ is a function $\mu : X \setminus Z \to [0, 1]$ such that for each information set w, $\sum_{x \in w} \mu(x) = 1$.

In a practical sense, the beliefs represent the probability a player assigns to begin at each node in an information set given that the information set has been reached.

Definition 6.2. Let Γ be an extensive game. An assessment is a pair (b, μ), where b is a behavior strategy profile and μ is a system of beliefs.

We denote the probability that terminal node z is reached given that w is reached, μ gives the probability of being at each node $x \in w$, and b is played thereafter as $\mu_w(z, b)$. More formally, we define $mu_w(z, b) := \sum_{x \in w} \mu(x) p_x(z, b)$.

We now let $u^{\mu}_{iw} := \sum_{z \in Z} \mu_w(z, b) u_i(z)$ denote the expected utility of player i given that w has been reached and that the player i's beliefs are given by μ.

Definition 6.3. Let Γ be an extensive game, (b, μ) an assessment, and $\overline{b}_i f \in B_i$. Let $w \in W_i$. Then $\overline{b}_i$ is a best reply of player i against (b, μ) at w if

$$u^{\mu}_{iw}(b_{-i}, \overline{b}_i) = \max_{\hat{b}_i \in B_i} u^{\mu}_{iw}(b_{-i}, \hat{b}_i) \tag{6.4}$$

Since subgame perfect equilibrium is flawed, we give definitions for two new equilibrium concepts: weak perfect Bayesian equilibrium and sequential equilbrium.

Definition 6.5. Let Γ be an extensive game. An assessment (b, μ) is sequentially rational if, for each $i \in N$ and each $w \in W_i$, b_i is a best reply of player i against (b, μ) at w.

Sequentially rational gives the condition for being an equilibrium at each information set.

Definition 6.6. Let Γ be an extensive game. An assessment (b, μ) is weakly consistent with Bayes rule if μ is derived by Bayes rule in the path of b. In other words, (b, μ) is weakly consistent with Bayes rule if, for each information set w such that $p(w, b) > 0$ and for each $x \in w$, $\mu(x) = p_w(x, b)$.

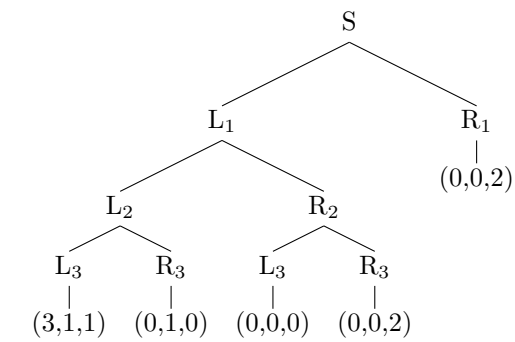
Being weakly consistent with Bayes rule gives a condition for updating the beliefs as the strategy plays out that does not depend on the existance of subgames in a game.

Definition 6.7. Let Γ be an extensive game. A weak perfect Bayesian equilibrium is an assessment that is sequentially rational and weakly consistent with Bayes rule.

If an assessment (b, μ) is weakly consistent with Bayes rule, then for all information sets w where $p(w, b) > 0$, we have that for each node $x \in w$, the probability of being at x, given that we are at w and using strategy b, is the belief of being at x. In other words, $p_w(x, b) = \mu(x)$. Thus, $u_{iw}(b) = u^{\mu}_{iw}(b)$. Then best replying against (b, μ) at w implies best replying against b at w. So a weak perfect Bayesian equilibrium is a Nash equilibrium.

However, a weak perfect Bayesian equilibrium does not address so-called "off-path" beliefs, and so does not imply a subgame perfect equilibrium. We can illustrate this with the following example.

Example 6.8. Consider the game given by the following tree. Three players take turns choosing left or right, starting with the first player.



All information sets are singletons except for the information set containing the node where the third player can make their choice after the second player chooses between left or right. In other words, the third player does not know if the second player chose left or right and must choose between L_3 and R_3.

Now consider the strategy profile $b = (R_1, L_2, R_3)$. Since player 3's information set will not be reached by b, any beliefs assigned at their information set will be, by definition, weakly consistent with Bayes rule. We can assign μ such that player 3 assigns probability 1 to player 2 choosing R_2. Then (b, μ) is weakly consistent with Bayes rule. However, b is not a subgame perfect equilibrium.

To find an equilibrium concept that will include subgame perfect equilibriums, we need another condition that will address "off-path" beliefs. In other words, we need to address assessments with strategies that are impossible to play.

First we begin with some definitions.

Definition 6.9. Strategy profile $b \in B$ is completely mixed if at each information set w, all the choices are taken with positive probability.

We let B^0 denote the set of all completely mixed behavior strategy profiles. If $b \in B^0$ is played, then all the nodes of the game can be reached with positive probability, so there is a unique system of beliefs that is associated with b and is consistent with Bayes rule. We let μ^b denote the system of beliefs that is associated with b and is derived using Bayes rule.

Definition 6.10. Let Γ be an extensive game. An assessment (b, μ) is consistent if there is a sequence $\{b^n\}_{n \in N} \subset B^0$, such that $(b, \mu) = \lim_{n \to \infty}(b^n, \mu^{b^n})$.

Definition 6.11. Let Γ be an extensive game. A sequential equilibrium is an assessment that is sequentially rational and consistent.

Proposition 6.12. *Let Γ be an extensive game and let (b, μ) be a sequential equilibrium of Γ. Then b is a subgame perfect equilibrium.*

Proof. If Γ does not have a proper subgame, then b is a subgame perfect equilibrium.

Suppose Γ has proper subgame(s). Let $x \in X$ be a node such that Γ_x is a proper subgame. Let w be the information set containing x. Then w must be a singleton (containing just one node) in order for Γ to be decomposed at x. So $\mu(x) = 1$. Therefore best replying against (b, μ) at w is the same as best replying against b at w. We know that (b, μ) is sequentially rational so all players best replying against (b, μ) at w. So all the players are best replying against b at w. Therefore b induces a Nash equilibrium in Γ_x. $\square$

7. Bayesian Games

As mentioned in an earlier section, we can use an extensive game with imperfect information to model games with incomplete information.

Definition 7.1. An n-player Bayesian game with set of players N is a 5-tuple $BG := (\Omega, \Theta, \rho, A, u)$ whose elements are:

(1) finite set Ω of the states of the world
(2) the types of the players are given by the finite set $\Theta := \prod_{i \in N} \Theta_i$
(3) the common prior probability distribution $\rho \in \Delta(\Theta \times \Omega)$
(4) the set of basic action profiles $A := \prod_{i=1}^{n} A_i$. A pure strategy of player i is a map $\hat{a} : \Theta \to A_i$. We denote the set of pure strategies of i as $\hat{A}_i$ and the set of pure strategy profiles as $\hat{A} := \prod_{i \in N}^{n} \hat{A}_i$. In other words, each player chooses an action based on the type that they have.
(5) the payoff function is defined to be $u := \prod_{i \in N}^{n} u_i$, where $u_i : \Omega \times \Theta_i \times A \to \mathbb{R}$. For each player i, their Bayesian payoff function $\hat{u}_i : \hat{A} \to \mathbb{R}$ is, for each $\hat{a} \in \hat{A}$,

$$\hat{u}_i(\hat{a}_i) := \sum_{(\theta,\omega) \in \Theta \times \Omega} \rho(\omega|\theta_i) u_i(\omega, \theta_i, \hat{a}(\theta)). \tag{7.2}$$

A Bayesian game is a way to represent a special type of extensive game. Therefore, we can define behavior and mixed strategies of Bayesian games. A mixed strategy s_i is, as before, defined to be a lottery over the pure strategies. A behavior strategy $b_i : \Theta_i \to \Delta A_i$ We will continue to use B to refer to the set of behavior strategy profiles and use S to refer to the set of mixed strategy profiles.

Let BG be a Bayesian game. Then we will denote an extensive game with imperfect information that corresponds to BG as Γ^{BG}.

Each strategy (pure, mixed, and behavior) in BG induces a strategy of the same type in Γ^{BG} and vice versa.

Definition 7.3. Let $BG = (\Omega, \Theta, \rho, A, u)$ be a Bayesian game. A Bayesian Nash equilibrium in pure strategies is a strategy profile $\hat{a}^* \in \hat{A}$ such that for each player $i \in N$ and each $\hat{a}_i \in A_i$, we have

$$\hat{u}_i(\hat{a}^*) \geq \hat{u}_i(\hat{a}^*_{-i}, \hat{a}_i). \tag{7.4}$$

Theorem 7.5. *Every Bayesian game has at least one Bayesian Nash equilibrium in behavior strategies.*

Proof. We have shown previously that an extensive game with perfect recall has a subgame perfect equilbrium. Since a Bayesian game is defined to be a game with perfect recall, we just need to show that a Nash equilibrium in the extensive game induces a Bayesian Nash equilibrium in the Bayesian game.

We will do this by showing that the Bayesian payoff and the payoff from before are equivalent. The utility functions are equivalent since there is a unique state of the world, type of the players, and strategy that will lead to a terminal node.

The probability of a player having a certain type corresponds to the probability that a certain terminal node will be reached using a strategy since the actions of the players are determined by the types that they have. □

8. The Chain Store Paradox

We will now study in depth a modified chain store problem.

There is a market with a monopolist and T number of entrants. The monopolist is either a strong monopolist (S) or a weak monopolist (W). Only the monopolist knows if they are strong or weak. Each entrant knows that the probability of the monopolist being strong is given by $q > 0$. Each entrant will decide whether or not to enter the market one-by-one.

The payoffs are given as before in the previous section, except the strong monopolist gets -1 if they yield and 0 if they fight entry.

Let A^0 denote empty history, as in the history at the start of the game. Let $h^t \in A^t$ denote the history of action profiles up to period t. Let $H := \bigcup_{t=1}^{T} A^{t-1}$ denote the set of all histories.

Each entrant can see everything that has happened before they make a decision to enter or stay out.

We will examine the extensive game corresponding to this Bayesian game, Γ^{CS}.

Let $q_1 := q$ and for all $t \in \{2, ..., T\}$, let $q_t(h_{t-1})$ be the probability that entrant t attaches to the monopolist being strong afeter having observed history h_{t-1}. We will use q_t to denote $q_t(h_{t-1})$.

We will give a way to compute the q_t beliefs:

(1) If there is no entry at $t-1$, then $q_t = q_{t-1}$.
(2) If there is entry at $t-1$, the monopolist fights it, and $q_{t-1} > 0$, then $q_t = \max\{\beta^{T-t+1}, q_{t-1}\}$.
(3) if there is entry at $t-1$, and either the monopolist yields or $q_{t-1} = 0$, then $q_t = 0$.

The following will be the strategy of the monopolist:

(1) If the monopolist is strong, the monopolist will always fight.
(2) If the monopolist is weak and entrant t enters, then if $t = T$, the monopolist yields, if $t < T$ and $q_t \geq \beta^{T-t}$, the monopolist fights, and if $t < T$ and $q_t < \beta^{T-t}$, then the monopolist fights with probability

$$\frac{(1-\beta^{T-t})q_t}{1-q_t)\beta^{T-t}}. \tag{8.1}$$

The following will be the strategy of the entrant:

(1) If $q_t > \beta^{T-t+1}$, entrant t stays out.
(2) If $q_t < \beta^{T-t+1}$, entrant t enters the market.
(3) If $q_t = \beta^{T-t+1}$, entrant t stays out with probability $\frac{1}{\alpha}$.

We will show that the strategies and beliefs outlined constitute a sequential equilibrium. For the sake of brevity, we will not give all details of the computations.

Lemma 8.2. *Let $t \in \{1, ..., T\}$. Then*

(1) If $q_t < \beta^{T-t+1}$, then if the players follow the strategies above, the total expected payoff of a weak monopolist from that point onwards is 0.
(2) If $q_t = \beta^{T-t+1}$, then if the players follow the strategies above, the total expected payoff of a weak monopolist from that point onwards is 1.

Proof. We will do backwards induction on t. Suppose $t = T$ and $q_t < \beta^{T-t+1}$. Then entrant T enters the market and the weak monopolist yields and gets 0.

Suppose $t = T$ and $q_t = \beta^{T-t+1}$. Then entrant T stays out with probability $\frac{1}{\alpha}$. If entrant T enters, the monopolist yields and gets 0. If entrant T stays out, the monopolist gets α. So the payoff is 1. So the base case for T is established for both 1 and 2.

Suppose both claims ar true for all $\bar{t} > t$, with $1 \leq t < T$. Suppose $q_t < \beta^{T-t+1}$. Then t enters the market according to the strategy. If the monopolist yields, they get 0 in the present period. Since $q_{t+1} = 0 < \beta^{T-t}$, by the induction hypothesis, the monopolist gets a total expected payoff of 0 from period $t+1$ onwards. If the monopolist fights, they get -1 in the present period. Since $q_{t+1} = \beta^{T-t}$, by the induction hypothesis, the monopolist gets a total expected payoff of 1 from period $t+1$ onwards. So their overall expected payoff by fighting is also 0.

This shows claim 1. Claim 2 is analogous to show, so for the sake of brevity, I will not be showing it here. □

Now we will use the lemma to show that the strategies and beliefs constitute a sequential equilibrium.

Theorem 8.3. *The above strategies form a sequential equilibrium of the chain store game with incomplete information.*

Proof. We will begin with satisfying the consistent condition. We will show this condition by showing that the beliefs are updated using Bayes rule and then by discussing "off-path" beliefs.

If there is no entry at period t, then nothing new is learned about the monopolist, so $q_{t+1} = q_t$.

Now suppose there is entry at t. If $q_t \geq \beta^{T-t}$, then the monopolist is supposed to fight regardless of whether they are weak or strong. Therefore, nothing new is learned, so $q_{t+1} = q_t$. If $q_t = 0$, the weak monopolist yields and once yield is played, Bayes rule implies that $q_{t+1} = q_t = 0$.

Now let $q_t \in (0, \beta^{T-t})$. In this case, the monopolist either yields or fights based on a specific probability. If the monopolist yields, since the strong monopollist would not yield, Bayes rule implies that $q_{t+1} = 0$.

For the case where the monopolist fights, we will use S to denote the event that the monopolist is strong, W to denote the event the monopolist is weak, and F^t the event that the monopolist fights at period t. Then by Bayes,

$$q_{t+1} = p(S|F^t) = \frac{p(S \cup F^t)}{p(F^t)} = \frac{p(F^t|S)p(S)}{p(F^t|S)p(S) + p(F^t|W)p(W)} \tag{8.4}$$

And this reduces

$$q_{t+1} = \frac{q_t}{q_t + \frac{(1-\beta^{T-t})q_t}{(1-q_t)\beta^{T-t}}(1-q_t)} = \beta^{T-t}, \tag{8.5}$$

which is consistent with the way we compute the beliefs.

Now we move on to "off-path" beliefs. There are two situations where Bayes rule may not apply.

(1) $q_t \geq \beta^{T-t}$ and the monopolist yields after entry.
(2) $q_t = 0$ and the monopolist fights.

In both cases, the beliefs $q_{t+1} = 0$. We refer to the choices where the player is not mixing between two choices as mistakes. It suffices to specify the probabilities

of the mistakes in a way that converge to 0 and lead to the beliefs we outlined. It does not matter how we define the probability of the entrants' mistakes because they do not impact the way the entrants compute their beliefs. Therefore, we will define the beliefs of μ^{b^n} to be such that for all $n \in \mathbb{N}$, each mistake of the weak monopolist is made with probability $\frac{1}{n}$ and each mistake of the strong monopolist is made with probability $\frac{1}{n^2}$. For the sake of brevity, I will not be detailing the computations of the beliefs.

Next, we will satisfy the sequentially rational condition. We will show this condition by showing that the way the players make decisions are their best reply against the strategy and beliefs.

We start with the incentives of the entrants:

(1) If $q_t \geq \beta^{T-t}$, entrant t expects entry to be fought and so stays out.
(2) If $\beta^{T-t+1} < q_t < \beta^{T-t}$, entrant t expects entry to be fought with probability $p(F|S)p(S)+p(F|W)p(W)$, which is greater than β, so the expected payoff after entry is less than 0 so entrant t stays out.
(3) If $q_t = \beta^{T-t+1}$, entrant t expects entry to be fought with probability β and so the expected payoff is 0, so entrant t is indifferent and can choose either entry or staying out.
(4) If $q_t < \beta^{T-t+1}$, then the expected payoff of entrant t by entering is positive so entrant t enters.

Now we talk about the incentives of the monopolist. The strong monopolist will always fight since yielding at any point will increase the number of entrants in the future, so the strong monopolist is always better off fighting entry. Now suppose the monopolist is weak. If entrant t enters and the monopolist yields, the monopolist gets 0. If the monopolist fights, the monopolist gets -1 in the present period and, using our lemma, gets, from this point onwards:

(1) 0 if $q_t = 0$
(2) 1 if $0 < q_t < \beta^{T-t}$
(3) greater than 1 if $q_t > \beta^{T-t}$.

So the above strategies and beliefs form a sequentially rational assessment, and therefore they form a sequential equilibrium.

□

Acknowledgments

I would like to thank my mentor, Santiago Chaves, for meeting with me consistently through this process and mentoring me through this project. I would also like to thank the REU program director, Peter May, for organizing this program and giving me the opportunity to participate in this program and write this paper.

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