HYPERBOLIC GEOMETRY, FUCHSIAN GROUPS, AND TILING SPACES

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Abstract. Expository paper on hyperbolic geometry and Fuchsian groups. Exposition is based on utilizing the group action of hyperbolic isometries to discover facts about the space across models. Fuchsian groups are characterized, and applied to construct tilings of hyperbolic space.

Contents

1. Introduction

One of the goals of geometry is to characterize what a space "looks like." This task is difficult because a space is just a non-empty set endowed with an axiomatic structure—a list of rules for its points to follow. But, *a priori*, there is nothing that a list of axioms tells us about the curvature of a space, or what a circle, triangle, or other classical shapes would look like in a space. Moreover, the ways geometry manifests itself vary wildly according to changes in the axioms. As an example, consider the following: There are two spaces, with two different rules. In one space, planets are round, and in the other, planets are flat.

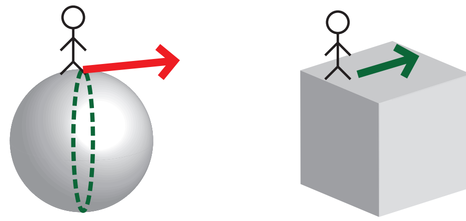

FIGURE 1. Walking in a straight line according to the flat planet is very difficult in the round planet.

If we define a straight line as the path carved out by moving in a single direction and never changing trajectory, we see that straight lines are both circles and squares, according to what planet we live in. Everything, including straight lines, follows rules: the axioms of a space.

In this paper, we have the goal of understanding what a particular space, called *hyperbolic space*, “looks like”, given its axioms. In a seemingly counter-intuitive way, we will learn everything about hyperbolic space and its geometry by studying the space itself as little as possible. Instead, we will study the internal symmetries of the space, encoded as groups of functions. Section 2 is a brief historical overview of hyperbolic geometry, along with the presentation of its axiomatic structure. In Section 3, we will explore how the symmetries of hyperbolic space makes its geometry accessible to us. We show in Section 4 that we can translate between models while preserving their geometry. In Section 5, we examine the structure present in the groups of symmetries to classify them. Finally, we construct Fuchsian groups and tile hyperbolic space in Section 6.

2. The Origin of Hyperbolic Geometry

Hyperbolic geometry began with a curious observation regarding Euclidean geometry. In his *Elements*, Euclid posed the following axioms for his space:

(1) A straight line segment can be drawn between any two points
(2) Any straight line segment can be extended indefinitely in a straight line
(3) Given any straight line segment, a circle can be drawn having its radius as the line segment, and one endpoint as the center
(4) All right angles are congruent
(5) If two lines are drawn which intersect a third in such a way that the sum of the inner angles on one side is less than two right angles, then the two lines inevitably must intersect each other on that side if extended far enough

The fifth statement is known as the *parallel postulate*, and mathematicians wondered if it was *independent* of the other four, that is, if one could deduce the parallel postulate from the axioms already given. Mathematicians failed to disprove the postulate's independence, reslting in the discovery that the postulate could not be deduced from other axioms. In 1823, Janos Bolyai and Nicolai Lobachevsky confirmed that there exist *non-Euclidean geometries* where the parallel postulate does not hold. Hyperbolic geometry is one such non-Euclidean geometry. To better understand the difference between Euclidean and hyperbolic geometry, note that the parallel postulate is equivalent to the following statement: For any line and any point not on that line, there exists a unique line through that point parallel to the original line. The fifth postulate of hyperbolic geometry is instead

"For any line, and a point not on the line, there exists a continuum of lines that intersect that point and are parallel to the original line."

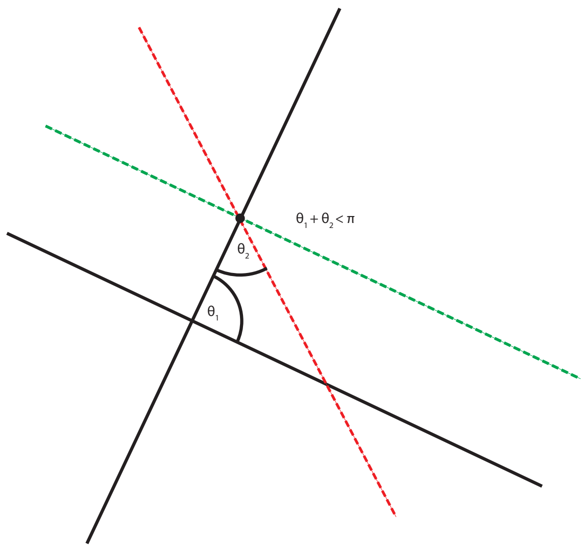

FIGURE 2. Illustration of the parallel postulate and Playfair's axiom. From this picture it can be seen that each concept is equivalent to the other, as only parallel lines can have the sum of angle equal to 180 (or π radians). In hyperbolic geometry, the green line is not unique.

In 1868, Eugenio Beltrami developed the first *models* of hyperbolic space—collections of mathematical objects that capture the axioms of hyperbolic geometry. By studying models of hyperbolic space, we will adopt the position of an astronaut. We will "drop into" these models and figure out what the spaces look like using a variety of techniques, which we will see almost all use symmetry.

3. The Poincaré Disk Model of 2D Hyperbolic Space

The Poincaré disk model of hyperbolic space has an ambient space and a collection of functions and variables that capture the hyperbolic axioms. We can think of the ambient space as a planet and the functions as the physical laws that apply to it, like gravity. Starting with these basic mathematical constructions, it is our task to figure out how basic concepts, like "length" or "straight lines", will behave in this planet.

Definition 3.1. *The Poincaré disk is a set $\mathbb{D}^2$ defined by*

$$\mathbb{D}^2 := \{z \in \mathbb{C} \mid |z| < 1\}.$$

The boundary of this space is the complex 1-sphere:

$$\mathbb{S}^1 := \{z \in \mathbb{C} \mid |z| = 1\} = \overline{\mathbb{D}^2} \backslash \mathbb{D}^2.$$

The space itself has no hyperbolic geometry. In fact, $\mathbb{D}^2$ by itself is a subset of $\mathbb{C}$, which is isomorphic to the Euclidean plane $\mathbb{R}^2$. In order for the model to be hyperbolic, we need to "morph" the set accordingly through a transformation known as the *hyperbolic kernel*.

Definition 3.2. *The* hyperbolic kernel *is a function $h : \mathbb{D}^2 \to \mathbb{D}^2$ defined by*

$$h(z) = \frac{2}{1 - |z|^2}.$$

Notice that as $|z| \to 1$, the kernel $h(z) \to \infty$. This will be important in allowing unbounded behavior in this model, even though its ambient space is a bounded set.

The hyperbolic kernel serves as a bridge between distance in the Euclidean sense and in the hyperbolic sense, allowing us to deduce the metric for this space.

3.1. **Length in the Poincaré Disk.**

Definition 3.3. *Let $u, v \in \mathbb{D}^2$. Let $\gamma : [0,1] \to \mathbb{D}^2$ be given by*

$$\gamma(t) = \alpha(t) + i\beta(t)$$

where both $\alpha, \beta \in \mathcal{C}^1$. We say that $C = \gamma([0,1])$ is a curve joining u and v if $\gamma(0) = u$ and $\gamma(1) = v$.

Now that we have a curve joining our two points of interest, we simply need to compose it with the hyperbolic kernel to get a curve joining the two points in hyperbolic space. To calculate the length of the now hyperbolic curve, we just integrate.

Definition 3.4. *Let $u, v \in \mathbb{D}^2$ and let γ be as above. Then, the length of the curve joining u, v in Hyperbolic Space is*

$$L(C) = \int_C h(z)\,|dz|$$

where z is given by $\gamma(t)$ for some t. Note that

$$\begin{aligned} \int_C h(z)\,|dz| &= \int_C \frac{2\,|\,dz|}{1 - |z|^2} \\ &= \int_0^1 \frac{2\sqrt{\alpha'(t)^2 + \beta'(t)^2}\,dt}{1 - (\alpha(t)^2 + \beta(t)^2)}. \end{aligned}$$

Your heart should have sunk a little because our definitions mean that in order to calculate distance in hyperbolic space, you need to work with some truly ugly integrals. But, there is a better way. Hyperbolic space has a very deep relationship to a group of functions, and by exploiting this relationship, we can make our calculations act at least half-civilized.

3.2. **Groups of Isometries in Hyperbolic Space.** Remember that we are astronauts in the middle of the Poincaré disk. With that analogy, we can think of isometries as movements we can make without worry. As an example, consider the translation invariance of the Euclidean plane. No matter how far a point is translated in a given direction, the Euclidean plane will still look like the Euclidean plane, which means in our astronaut analogy that we can walk as far as we'd like in a given direction knowing our space won't turn into a lake of fire, or a tar pit. In a sense, groups of isometries measure the *regularity* of the space. We can make this notion precise with metrics.

Definition 3.5. *Let (X, d) be a metric space. We say that the map $f : X \to X$ is an* isometry *if it is bijective and $d(u, v) = d(f(u), f(v))$ for any u, v in a space.*

The isometries of the Poincaré disk form a group. I will characterize this group now, and we will see that almost everything we know about hyperbolic space will be derived using it.

Definition 3.6. *The isometries of the Poincaré disk are given by*

$$\mathrm{con}(1) := \left\{ g : \mathbb{D}^2 \to \mathbb{D}^2 \mid \text{ for some } a, c \text{ with } |a|^2 - |c|^2 = 1,\ g(z) = \frac{az + \overline{c}}{cz - \overline{a}} \text{ or } g(z) = \frac{a\overline{z} + \overline{c}}{c\overline{z} - \overline{a}} \right\}.$$

The function with conjugate variables corresponds to parity-reversing isometries. The parity-preserving elements, denoted $\mathrm{con}^+(1)$, form a subgroup of the general Möbius group, which are functions of complex coefficients:

$$\text{Möb}^+ := \left\{ g : \hat{\mathbb{C}} \to \hat{\mathbb{C}} : z \mapsto \frac{az + b}{cz + d} \mid ad - bc \neq 0 \right\}.$$

Note that we adopt some conventions for this set of maps. The first convention is that for a Möbius map g, $g(\infty) = a/c$ if $c \neq 0$, and $g(\infty) = \infty$ otherwise. The second convention is that $g(0) = b/d$ if $c \neq 0$ and $g(0) = \infty$ otherwise.

From complex analysis, we know that Möbius maps are *conformal*, meaning they preserve angles locally. The group of isometries of $\mathbb{D}^2$ can be thought of as all the conformal automorphisms of the disk. Of course, we should prove that $\mathrm{con}(1)$ is actually the group of isometries for $\mathbb{D}^2$.

Theorem 3.7. $\mathrm{con}(1)$ *is a subset of the group of isometries of* $\mathbb{D}^2$, *or,*

$$d_{\mathbb{D}^2}(u, v) = d_{\mathbb{D}^2}(g(u), g(v))$$

for any $g \in \mathrm{con}(1)$ *and any* $u, v \in \mathbb{D}^2$.

Proof. Let $g \in \mathrm{con}^+(1)$ and let C be a $\mathcal{C}^1$ curve joining $u, v \in \mathbb{D}^2$. Using the substitution $z = g(t)$ and the previous lemma we have

$$L(g(C)) = \int_{g(C)} \frac{2\,|dz|}{1 - |z|^2} = \int_C \frac{2}{1 - |g(w)|^2} \left|g'(w)\right| |dw| .$$

It is an easy check to prove the following identity:

$$\left|g'(z)\right| = \frac{1 - |g(z)|^2}{1 - |z|^2}.$$

When we apply the identity to our work, the result follows, as

$$\begin{aligned}
\int_C \frac{2}{1-|g(w)|^2}\left|g'(w)\right||dw| &= \int_C \frac{2}{1-|g(w)|^2}\frac{1-|g(w)|^2}{1-|w|^2}|dw| \\
&= \int_C \frac{2}{1-|w|^2}|dw| \\
&= L(C).
\end{aligned}$$

It follows from the chain rule that $g \in \mathrm{con}^+(1)$ maps $\mathcal{C}^1$ curves to $\mathcal{C}^1$ curves joining $g(v)$ and $g(u)$ $(C \mapsto g(C))$. Therefore

$$d_{\mathbb{D}^2}(u,v) = \inf_C \{L(C)\} = \inf_C \{L(g(C))\} = \inf_{g(C)} \{L(g(C))\} = d_{\mathbb{D}^2}(g(u), g(v))$$

thereby showing that elements in $\mathrm{con}^+(1)$ are isometries of $\mathbb{D}^2$. Note that elements of $\mathrm{con}(1) \setminus \mathrm{con}^+(1)$ can be obtained from elements in $\mathrm{con}^+(1)$ composed with the conjugation map $z \mapsto \overline{z}$, which is an isometry. We can then conclude our result in full generality. ☐

We have actually shown that con(1) is a subset of the group of isometries. Anderson [3] fills in the details.

Since we have dropped inside the Poincaré disk, we figured out a reasonable notion of distance given in terms of the length of a curve. But now that we have an understanding of the group of isometries, we are going to learn what curves *look like* in the model. Notice that the phrase is not in scare quotes anymore. This exploration will be rigorous: we will not just find out what paths look like in $\mathbb{D}^2$, we are going to prove it.

3.3. **Hyperbolic Geodesics.** We started the paper by considering how straight lines might look in different spaces. Geodesics are the mathematical equivalent of straight lines, and they are crucial to the understanding of any space. Just as straight lines are the shortest paths between two different points, geodesics are the "length minimizing" paths, though they are not necessarily straight in a Euclidean sense. Because of this, the length of a geodesic curve is the metric value for its endpoints. Using our knowledge of con(1), we will find a slick way to calculate these values in a special case.

Lemma 3.8. *The metric value between the origin and any point in the Poincaré disk is given by the identity*

$$d_{\mathbb{D}^2}(0,z) = \log\frac{1+|z|^2}{1-|z|^2}.$$

Proof. Write z in polar form

$$z = |z|e^{i\theta}$$

and apply $g \in \mathrm{con}^+(1)$ defined by

$$g(w) = we^{-i\theta}.$$

This map corresponds to a rotation of the complex plane. Intuitively, we see that g is a conformal isometry, and we can put it into standard form of an element of con(1) by choosing $a = \exp(-i\theta/2)$ and $c = 0$

$$g(w) = we^{-i\theta} = \frac{\exp(-i\theta/2)w + 0}{0 \cdot w + \overline{\exp(-i\theta/2)}}.$$

Since g is an isometry of $\mathbb{D}^2$,

$$d_{\mathbb{D}^2}(0,z) = d_{\mathbb{D}^2}(g(0), g(w)) = d_{\mathbb{D}^2}(0, |z|).$$

Suppose there is a function $\gamma : [0,1] \to \mathbb{D}^2$ defined by

$$\gamma(t) = \alpha(t) + i\beta(t).$$

Then

$$\begin{aligned} L(C) = \int_0^1 \frac{\sqrt{\alpha'(t)^2 + \beta'(t)^2}}{1 - (\alpha(t)^2 + \beta(t)^2)}\, dt \geqslant \int_0^1 \frac{\sqrt{\alpha'(t)^2}}{1 - (\alpha(t)^2)}\, dt \\ = \int_0^1 \frac{2\alpha'(t)}{1 - \alpha^2}\, dt \\ = \int_0^{|z|} \frac{2}{1 - u^2}\, du \\ = \left[\log \frac{1+u}{1-u}\right]_0^{|z|} \\ = \log \frac{1+|z|}{1-|z|}. \end{aligned}$$

Showing one direction of the result. If we look at the work above, we see that equality only holds if $\beta = 0$, which corresponds to the Euclidean straight line between u and v. Therefore,

$$d_{\mathbb{D}^2}(0,z) = \inf L(\gamma([0,1])) = \log \frac{1+|z|}{1-|z|}.$$

□

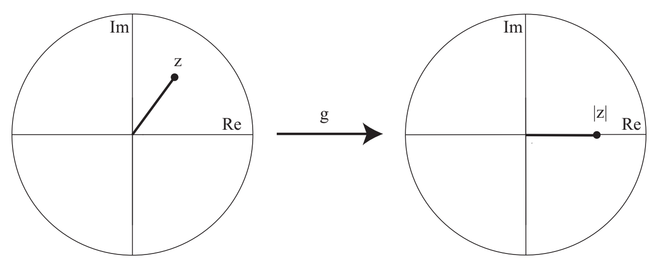

FIGURE 3. We can make an arbitrary point of the Poincaré disk fall under our Euclidean line identity using the appropriate rotation.

Theorem 3.9. *Let $v \in \mathbb{D}^2$ Then, $\exists g_v \in \mathrm{con}(1)$ such that the image of g_v at v is 0.*

Proof. Consider the function

$$g_v(w) = \frac{\frac{i}{\sqrt{1-|v|^2}} w + \frac{-iv}{\sqrt{1-|v|^2}}}{\frac{i\overline{v}}{\sqrt{1-|v|^2}} w + \frac{-i}{\sqrt{1-|v|^2}}}.$$

It is evident that $g_v(v) = 0$, and it remains to show $g_v \in \mathrm{con}(1)$, which is an easy check that

$$\left|\frac{i}{\sqrt{1-|v|^2}}\right|^2 - \left|\frac{iv}{\sqrt{1-|v|^2}}\right|^2 = \left(\frac{1-|v|}{1-|v|}\right)^2 = 1.$$

□

This lemma is incredibly useful because it allows us to make any hyperbolic geodesic into the Euclidean line we considered in Lemma 3.9. Now, to calculate the distance between two points in hyperbolic space, we simply apply the isometry above, apply a rotation that brings the other point to an axis, and use the formula we derived. We can always find the appropriate isometries due to intrinsic symmetries of the hyperbolic plane. Not all spaces are so well behaved.

The action of $\mathrm{con}^+(1)$ on $\mathbb{D}^2$ allowed us to start with a hyperbolic geodesic, pass it to a straight line, and measure distance. Now, using the same group, we will take the straight line, use the inverses of the functions above to pass to an arbitrary hyperbolic geodesic, and utilize properties of the Möbius group to classify the resulting curve.

Theorem 3.10. *The image of a doubly infinite straight line or a circle under a Möbius map is either a doubly infinite straight line or a circle.*

Proof. Let $g \in$ Möb, then it comes in the form

$$g(z) = \frac{az+b}{cz+d}.$$

This expression is a bit complicated, so we are going to decompose g into simpler forms. First, suppose that $c = 0$. In this case,

$$\frac{az+b}{cz+d} = \frac{az+b}{d} = \frac{a}{d}z + \frac{b}{d}.$$

This expression is the composition of a rotation, a dilation, and a translation. Each of which send straight lines to straight lines and circles to circles. Supposing that $c \neq 0$, I claim that any such transformation in Möb is the composition of the following:

- $z \mapsto z + \frac{d}{c}$ (a translation)
- $z \mapsto \frac{1}{z}$ (a circle inversion)
- $z \mapsto \frac{bc-ad}{c^2}z$ (a rotation composed with a dilation)
- $z \mapsto z + \frac{a}{c}$ (a translation)

To check this, just compose the functions in the order they appear. It will suffice to show that a circle inversion preserves lines and circles. To do this, we will break both lines and circles into a common algebraic form, and show that the form is invariant under a circle inversion. We will start with a line. Note that the points $x + iy$ in a straight line always satisfy

$$ax + by + c = 0$$

for some $a, b, c \in \mathbb{R}$. We can re-write this as

$$\frac{a}{2}(z + \overline{z}) + \frac{b}{2i}(z - \overline{z}) + c = 0.$$

Setting $B = a/2 + b/2i$ and $C = c$, the above formula reduces to

$$Bz + \overline{B}z + C = 0.$$

Examining a circle, we see that points on a circle centered at u with radius r satisfy

$$|z - r|^2 = (z - u)\overline{(z - u)} = r^2.$$

This yields

$$z\overline{z} - z\overline{u} - u\overline{z} + |u|^2 = r^2.$$

Setting $A = 1$, $B = -\overline{u}$ and $C = |u|^2 - r^2$, the equation reduces to

$$Az\overline{z} + Bz + \overline{B}\overline{z} + C = 0.$$

We have arrived at a general equation for a circle or a straight line, where B is complex, and C is real. For any line or circle, the A value for its equation will be either 1 or 0. A 0 value corresponds to lines and 1 value corresponds to circles. Now, let z be on the line or circle corresponding to A, B, C. Dividing by $z\overline{z}$ yields

$$C\frac{1}{z\overline{z}} + B\frac{1}{\overline{z}} + \overline{B}\frac{1}{z} + A = 0.$$

If $C = 0$, then $1/z$ lies on the line determined by $B = \overline{B}$ and $C = \overline{A}$. If $C \neq 0$, then further dividing by C shows that $1/z$ lies on the circle $A = \overline{1}$, $B = \overline{B/C}$, $C = A/C$. We can therefore deduce that $z \mapsto 1/z$ sends lines to lines and circles to circles. $\square$

With this result, classifying Poincaré disk geodesics is easy.

Theorem 3.11. *Hyperbolic geodesics exist and are unique between any two points in $\mathbb{D}^2$. Moreover, they either lie on Euclidean straight lines through the origin or on circles in $\mathbb{D}^2$ orthogonal to S^1.*

Proof. We already know that geodesics between 0 and u in $\mathbb{D}^2$ exist and are unique. Consider arbitrary points $u, v \in \mathbb{D}^2$. We have shown that given such a pair, there exists a $g_u \in \text{con}^+(1)$ such that $g_u(u) = 0$. Because of the group structure of $\text{con}^+(1)$ we immediately know that there is a unique geodesic between u, v by the image of g_u^{-1} under the Euclidean straight line geodesic between $0, v$ (if there were two, they would map to two different geodesics between 0 , v which we have shown to be unique). Since this geodesic is the image of a straight line emanating from the origin under some $g \in \text{con}^+(1)$, the geodesic between u, v lies in a circle orthogonal to S^1 or is the Euclidean straight line passing through the origin.

$\square$

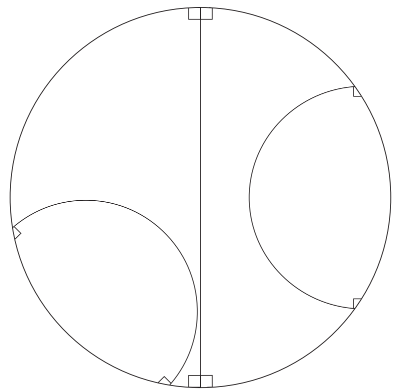

FIGURE 4. Geodesics in $\mathbb{D}^2$

Almost everything we discovered about $\mathbb{D}^2$, we did so using the action of con(1). The group behaved exactly as it should have intuitively: it allowed us to make transformations that made our calculations easier with the added assurance that we were not losing any of the information we were looking for. But, the power of groups does not end here. We can also utilize group structures to understand hyperbolic geometry across models.

4. The Half-Plane Model of Hyperbolic Space

Having several models of hyperbolic space is incredibly useful. Just as switching to polar coordinates can make would-be impossible integrals straightforward, considering hyperbolic space in different models can make solving certain problems, visualizing difficult concepts, and performing computations more accessible to us. But we have to earn these benefits; we have to re-learn hyperbolic space in the context of this new model.

Definition 4.1. *The* Upper Half-Plane *is the set of complex numbers*

$$\mathbb{H}^2 := \{z \in \mathbb{C} \mid \operatorname{Im}(z) > 0\}.$$

The boundary is the 1-point compactification of the real line

$$\hat{\mathbb{R}} = \mathbb{R} \cup \{\infty\}.$$

At first, introducing a new model of hyperbolic space might seem daunting. Nothing we know about the Poincaré disk holds in a different model. Luckily, group actions come to our rescue again. Both the Poincaré disk and the half-plane are subsets of $\hat{\mathbb{C}}$. This space has its own set of symmetry operations, and it turns that there is a transformation mapping the half-plane to the Poincaré disk.

Definition 4.2. *The* Cayley map $\phi : \mathbb{H}^2 \to \mathbb{D}^2$ *is a map defined by*

$$\phi(z) = \frac{z - i}{z + i}.$$

This map will allow us to uncover everything about the half-plane, from the metric to the geodesic curves, in terms of what we know about the Poincaré disk.

Lemma 4.3. *For the Cayley map ϕ we have that*

(1) $\phi(\mathbb{H}^2) = \mathbb{D}^2$

(2) $\phi \in \text{Möb}^+$
(3) $\phi(\mathbb{R}) = S^1 \backslash \{1\}$ *and* $\phi(\infty) = 1$
(4) $\phi^{-1} \in \text{Möb}^+$ *and is defined by*

$$\phi^{-1} = -i\frac{z+1}{z-1}$$

Proof. We first show that $\phi(\mathbb{H}^2) = \mathbb{D}^2$. Consider an arbitrary point $z \in \mathbb{H}^2$. Evaluating ϕ at z gives

$$\begin{aligned}\phi(z) &= \frac{z-i}{z+i} \\ &= \frac{z-i}{z+i} \cdot \frac{\overline{z}-i}{\overline{z}-i} \\ &= \frac{|z|^2 - zi + zi - 1}{|z|^2 + 1} \\ &= \frac{|z|^2 - 2\,\text{Im}(z) - 1}{|z|^2 + 1}.\end{aligned}$$

Because $z \in \mathbb{H}^2$, we know $\text{Im}(z) > 0$, which implies

$$\frac{|z|^2 - 2\,\text{Im}(z) - 1}{|z|^2 + 1} < 1 \implies \phi(\mathbb{H}^2) \subset \mathbb{D}^2.$$

To show the other inclusion, pick an arbitrary $z \in \mathbb{C}$. I claim that

$$\exists w = i\frac{z+1}{1-z} \mid \phi(w) = z.$$

An easy calculation proves this:

$$\begin{aligned}\frac{i\frac{z+1}{1-z} - i}{i\frac{z+1}{1-z} + i} &= \frac{zi + 1 - i + zi}{zi + 1 + i - zi} \cdot \frac{1-z}{1-z} \\ &= \frac{2zi}{2i} \\ &= z.\end{aligned}$$

Supposing $z_0 \in \mathbb{D}^2$, we know $|z_0| < 1$, implying

$$\text{Im}\left(w_{z_0}\right) = \text{Im}\left(i\frac{z+1}{1-z}\right) > 0.$$

This means that for all z in the Poincaré disk, there is a w in the half-plane such that $\phi(w) = z$, proving that $\mathbb{D}^2 \subset \phi(\mathbb{H}^2)$. Now we want to show that $\phi \in$ Möb, which simply requires that $i - (-i) = 2i \neq 0$. For (3), let $z \in \mathbb{R}$. Note that $\phi(z) \neq 1$ because $z = x + 0i$ and for any such z, $\text{Im}(\phi(z)) \neq 0$, and $\text{Im}(1) = 0$. Taking

$z = x + 0 \in \mathbb{R}$,

$$\begin{aligned} |\phi(z)| = \left|\frac{x-i}{x+i}\right| &= \left|\frac{x-i}{x+i} \cdot \frac{x-i}{x-i}\right| \\ &= \left|\frac{x^2-1-2ix}{x^2+1}\right| \\ &= \left(\frac{x^4-2x^2+1+4x^2}{x^2+2x^2+1}\right)^{1/2} \\ &= \left(\frac{x^2+2x^2+1}{x^2+2x^2+1}\right)^{1/2} \\ &= (1)^{1/2} = 1. \end{aligned}$$

Note that for the Cayley map, the c coeffficient is not 0, so we follow the convention $\phi(\infty) = a/c = 1/1 = 1$. It is left to the reader to check that $\phi(\phi^{-1}(z)) = z$ and vice versa through a straightforward computation. $\square$

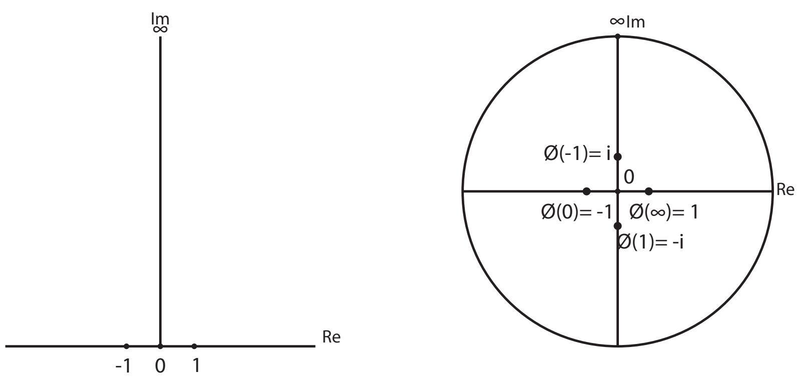

FIGURE 5. Comparison between the half-plane and Poincaré disk as the image and domain of the Cayley map.

The proof above involves pushing around a lot of symbols, but it is important to contextualize the work as "there is a map in the group of symmetry operations of $\hat{\mathbb{C}}$ that allows us to pass between the Poincaré disk and the half-plane." However, we have only considered the two models as sets. We really care about the rich algebraic structure of their isometries, and luckily for us, the Cayley map preserves this structure—allowing us to discover the isometries of the half-plane model without directly studying it at all.

Theorem 4.4. *Let* $\mathrm{Isom}(\mathbb{H}^2)$ *be the isometry group of the half-plane. Then,*

$$\mathrm{Isom}(\mathbb{H}^2) = \phi^{-1}\,\mathrm{con}(1)\,\phi.$$

Proof. First, suppose that $g \in \mathrm{con}(1)$. Then, by definition,

$$\begin{aligned} d_{\mathbb{H}^2}(z,w) &= d_{\mathbb{D}^2}(\phi(z),\phi(w)) \\ &= d_{\mathbb{D}^2}(g(\phi(z)),g(\phi(w))) \\ &= d_{\mathbb{H}^2}(\phi^{-1}(g(\phi(z))),\phi^{-1}(g(\phi(w)))) \\ &\implies \phi g \phi^{-1} \in \mathrm{Isom}(\mathbb{H}^2). \end{aligned}$$

Showing $\phi\,\mathrm{con}(1)\,\phi^{-1} \subset \mathrm{Isom}(\mathbb{H}^2)$. Now, let $g \in \mathrm{Isom}(\mathbb{H}^2)$ then

$$\begin{aligned} d_{\mathbb{D}^2}(z,w) &= d_{\mathbb{H}^2}(\phi^{-1}(z),\phi^{-1}(w)) \\ &= d_{\mathbb{H}^2}(g(\phi^{-1}(z)),g(\phi^{-1}(w))) \\ &= d_{\mathbb{D}^2}(\phi(g(\phi^{-1}(z))),\phi(g(\phi^{-1}(w)))) \\ &\implies \phi g \phi^{-1} \in \mathrm{con}(1) \end{aligned}$$

and this implies that $g \in \phi^{-1}\,\mathrm{con}(1)\,\phi$ as desired. (Note that $\phi^{-1}\,\mathrm{con}(1)\,\phi = \phi\,\mathrm{con}(1)\,\phi^{-1}$.) □

A more intuitive way to understand the proof above is to simply examine what the expression $\phi^{-1}\,\mathrm{con}(1)\,\phi$ means. Supposing g is a half-plane isometry, we know it is in the form $\phi h \phi^{-1}$ where h is Poincaré disk isometry. This translates to morphing the half-plane to the Poincaré disk, performing an isometry, and returning to the half-plane. The relation between the two isometries is called *algebraic conjugacy.* We will talk more about it later. For future reference, the isometry group of the half-plane model will be denoted $\mathrm{PSL}_2(\mathbb{R})$. This is the projective special linear group, defined by

$$\mathrm{PSL}_2(\mathbb{R}) := \left\{ g : \mathbb{H}^2 \to \mathbb{H}^2 \mid \text{ for some } a,b,c,d \in \mathbb{R} \text{ with } ad-bc=1 \ g(z) = \frac{az+b}{cz+d} \text{ for all } z \in \mathbb{H}^2 \right\}.$$

We can also uncover the metric using the relationship between the two models given by the Cayley map.

Theorem 4.5. *For all $u, v \in \mathbb{H}^2$,*

$$d_{\mathbb{H}^2}(u,v) = \inf\left\{ \int_C \frac{|dz|}{\mathrm{Im}(z)} \mid \ C \in \mathcal{C}^1 \textit{ joins } u \textit{ and } v \right\}.$$

Proof.

$$\begin{aligned} d_{\mathbb{H}^2}(u,v) &= d_{\mathbb{D}^2}(\phi(u),\phi(v)) \\ &= \inf\left\{ \int_C \frac{2\,|\phi'(z)|\,|dz|}{1-|\phi(z)|^2} \right\} \\ &= \inf\left\{ \int_C \frac{|dz|}{\mathrm{Im}(z)} \mid \ C \in \mathcal{C}^1 \text{ joins } u \text{ and } v \right\} \end{aligned}$$

where the final equality is obtained through the quotient rule and writing $z = x+iy$ to deduce that

$$\frac{2\,|\phi'(z)|}{1-|\phi(z)|^2} = \frac{1}{\mathrm{Im}(z)}.$$

□

We have shown that the isometry group of the half-plane model is a subset of the Möbius transformations, so we know that the image of a circle or a line under any map in the isometry group of the half-plane will be either a line or a circle, allowing us to classify the half-plane geodesics. By studying the images of the special points of the Cayley map and the conformality of the Möbius maps, we can break the geodesic between any two points in the half-plane into two cases: when neither of the points are 1 and when one of the points is 1. Under the two cases, it is easy to prove a characterization of hyperbolic geodesics.

Theorem 4.6. *Hyberbolic geodesics in the half-plane model exist and are unique. Moreover, they lie on straight, vertical Euclidean lines, or on semicircles that meet* $\mathbb{R}$ *at right angles.*

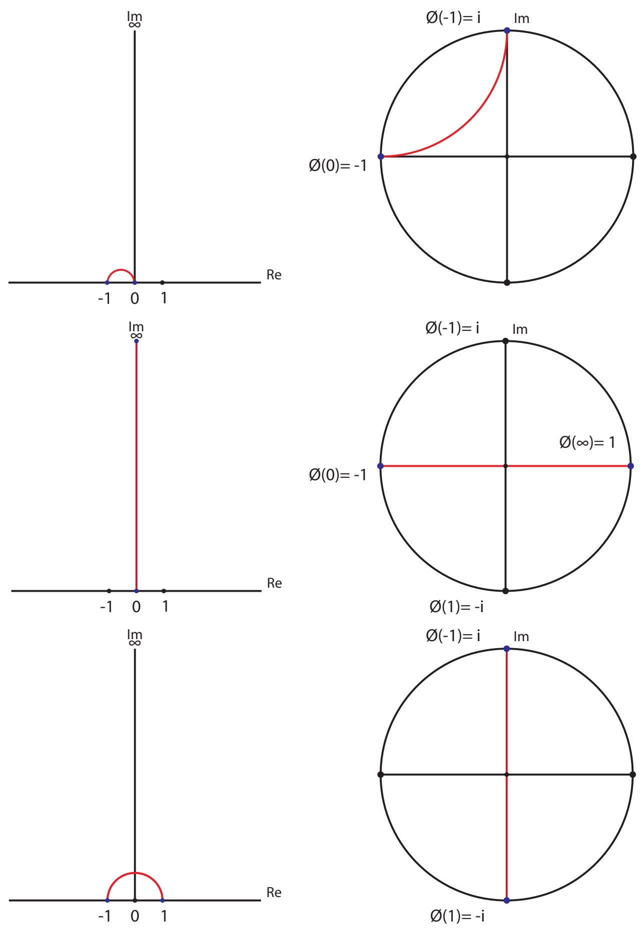

Figure 6. Each pair of models show what a geodesic in one model would look like in the other.

5. The Classification of Hyperbolic Isometries

This section addresses the concept of "sameness." The equal sign is the first statement of sameness we learn, but across the disciplines of mathematics and types of mathematical objects, the concept of sameness takes different forms, each having differing interpretations of what sameness means. In Section 4, we saw that using the Cayley map we could morph the half-plane into the Poincaré disk and express half-plane isometries in terms of con(1). This morphism demonstrates a degree of sameness between the two models because even though we could not write an equals sign between the two spaces, we could easily make one into the other. The algebraic conjugacy between the half-plane isometry group and the Poincaré disk isometry group is also a statement of sameness, and in this section we will learn how useful conjugacy can be by using it to classify all possible hyperbolic isometries, regardless of model.

Definition 5.1. *Let G be a group and let $g_1, g_2 \in G$. We say that g_1, g_2 are* conjugate *if there exists an $h \in G$ such that $g_1 = hg_2h^{-1}$.*

Using algebraic conjugacy, we can classify every possible hyperbolic isometry. Our mathematical work will be in the half-plane, but we can visualize the transformations in either model.

5.1. **The Three Classes of Isometries.** Hyperbolic isometries are grouped into three types:

(1) Hyperbolic elements
(2) Parabolic elements
(3) Elliptic elements

Definition 5.2. *Let $g \in \mathrm{PSL}_2(\mathbb{R})$. Define the* trace *of g by $a + d$ where*

$$g(z) = \frac{az + b}{cz + d}.$$

We say

- *g is hyperbolic if $\mathrm{Tr}(g)^2 > 4$*
- *g is parabolic if $\mathrm{Tr}(g)^2 = 4$*
- *g is elliptic if $\mathrm{Tr}(g)^2 < 4$*

The definition above leads to the question of why 4 is so special. For any non-negative x, the $\mathrm{Tr}(g)^2$ value of any hyperbolic isometry will either be larger, smaller, or equal to x. Isomorphisms (another way of measuring "sameness") are responsible this fact. The half-plane is a subset of the complex numbers, and it is well known that the complex numbers are isomorphic to the Euclidean plane. This connects our group of isometries to a subgroup of the linear transformations of $\mathbb{R}^2$. Accordingly, our seemingly arbitrary definition of the trace of an isometry becomes the trace of a linear transformation, and we can use what we know about spectral theory to understand the unique relationship our choice of 4 has to the fixed points of our hyperbolic isometries.

Theorem 5.3. *Let $g \in \mathrm{PSL}_2(\mathbb{R})$ be a non-identity map. Then the following are true.*

(1) If g is a hyperbolic element, then it has two fixed points, both in $\hat{\mathbb{R}}$
(2) If g is a parabolic element, then it has one fixed point in $\hat{\mathbb{R}}$
(3) If g is elliptic, then g has precisely two fixed points in $\hat{C}$, one of which is in $\mathbb{H}^2$ and the other of which is in $\overline{\mathbb{H}^2} := \{\overline{z} \mid z \in \mathbb{H}^2\}$

Proof. Let $g \in \mathrm{PSL}_2(\mathbb{R})$ be defined by

$$g(z) = \frac{az + b}{cz + d}.$$

We will again deal with two cases:

(1) $c = 0$
(2) $c \neq 0$

We begin with case (1). Supposing we have a fixed point z_0, then

$$g(z_0) = \frac{az_0 + b}{d} = z_0.$$

The only two cases are then $z_0 = \infty$ or $z_0 = b/(d - a)$. Recalling that $ad - bc = 1$, we know that $b \neq 0$ because $c = 0$. The map g is not the identity, meaning $ad = 1 \implies d = 1/a$ and

$$\mathrm{Tr}(g)^2 = (a + d)^2 = (a + \frac{1}{a})^2 = a^2 + 2 + \frac{1}{a^2} = 4 + (a^2 - 2 + \frac{1}{a^2}) = 4 + (a - \frac{1}{a^2}) \geqslant 4.$$

In fact, $\mathrm{Tr}(g)^2 = 4$ if and only if $a = 1/a = d$, which only occurs if ∞ is the only fixed point.

For case (2) we can simplify the problem by writing the set of fixed points of g as the solution set to the following polynomial equation:

$$z^2 + \left(\frac{d - a}{c}\right)^2 z - \frac{b}{c} = 0.$$

The solutions are given by

$$\left(\frac{a - d}{2c}\right) \pm \frac{1}{c}\sqrt{\left(\frac{a - d}{2}\right)^2 + bc}$$

where the expression under the radical sign is the discriminant Δ. I claim some clever algebraic manipulations lead to the identity

$$\Delta = \frac{\mathrm{Tr}(g)^2}{4} - 1.$$

As with any polynomial of degree 2, there now only 3 cases to consider:

(1) A positive Δ value means the polynomial has two real solutions, proving that g is hyperbolic and has two fixed points in $\mathbb{R} \cup \{\infty\}$
(2) A zero Δ value means the polynomial equation has a single solution with multiplicity 2, confirming that parabolic elements have one fixed point in $\mathbb{R} \cup \{\infty\}$
(3) A negative Δ value demonstrates that there are two, conjugate fixed points of elliptic elements. One has $\mathrm{Im}(z_0) > 0 \implies z_0 \in \mathbb{H}^2$, and the other has $\mathrm{Im}(z_0) < 0$

$\square$

Now we can solve the mystery of 4, thanks to our isomorphism. By re-writing the fixed points as solutions to that polynomial equation, we recognized that the fixed points of half-plane isometries correspond to eigenvalues of the linear transformations that they map to under isomorphism. We could then use the discriminant of characteristic polynomial to determine whether the transformation would have real or complex eigenvalues, allowing us to find where in $\mathbb{H}^2$ they would be.

We are still not satisfied. Just because we can characterize all of the possible fixed points a given hyperbolic isometry could have does not mean we know what all of the possible hyperbolic isometries "look like." Using algebraic conjugacy, we will discover that every isometry is the same as one of three base transformations.

5.2. **Hyperbolic Elements.** The standard form of a hyperbolic element of $\mathrm{PSL}_2(\mathbb{R})$ is the map

$$z \mapsto \alpha z$$

where α is a positive real not equal to 1. The standard form corresponds to a non-trivial dilation with fixed points at 0 and ∞. We can prove that if there is a map $g \in \mathrm{PSL}_2(\mathbb{R})$ that fixes ∞ and 0, then it necessarily must be of this form. This is because only 0 and ∞ are invariant under non-trivial scalings. Moreover, we can show that *all* hyperbolic elements are of this form up to conjugacy. This means that they both fix two points at the boundary of the half plane, and we can "change the coordinates" so that these two points become 0 and ∞. Suppose $g \in \mathrm{PSL}_2(\mathbb{R})$ is a hyperbolic element with fixed points $u, v \in \mathbb{R} \cup \{\infty\}$. Then there is a map $h_{u,v}$ so that

$$h_{u,v}(u) = 0 \quad \text{and} \quad h_{u,v}(v) = \infty.$$

Note that we can swap 0 and ∞ with the map $z \mapsto -1/z \in \mathrm{PSL}_2(\mathbb{R})$. We can therefore assume WLOG that $u < v$. We can send u to 0 with the map $z \mapsto z - u \in \mathrm{PSL}_2(\mathbb{R})$. If $v = \infty$ we are done, otherwise, $v \mapsto v - u > 0$ and we can send $v - u$ to ∞ with the map

$$z \mapsto \frac{-z}{z - (v - u)}$$

and $h_{u,v}$ comes from the composition of these maps. We can conjugate g by $h_{u,v} \in \mathrm{PSL}_2(\mathbb{R})$ (check that the two maps are in $\mathrm{PSL}_2(\mathbb{R})$) which fixes 0 and ∞.

5.2.1. *Hyperbolic Fixed Points.* Hyperbolic elements can be characterized by their *attracting* and *repelling* fixed points. Let $g \in \mathrm{PSL}_2(\mathbb{R})$ be put into standard form $z \mapsto \alpha z$. If $\alpha < 1$, then $\alpha z < z \implies \alpha z$ will be closer to 0, making it the attracting fixed point (and ∞ the repelling fixed point). If $\alpha > 1$, then ∞ is the attracting fixed point.

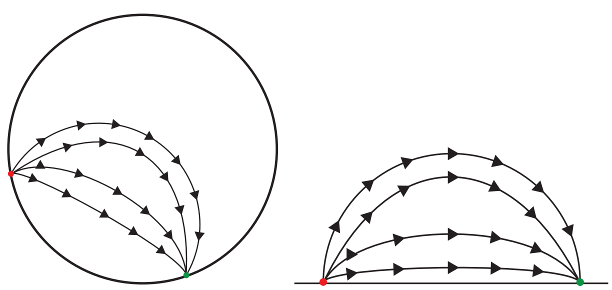

FIGURE 7. The attractive and repulsive action of a hyperbolic element on both models of hyperbolic space. The attracting fixed point is green and the repelling fixed point is red.

5.3. **Parabolic Elements.** The standard form of a parabolic element of $\mathrm{PSL}_2(\mathbb{R})$ is the map

$$z \mapsto z + \beta$$

corresponding to a translation of the half-plane. Since this map displaces every point in the set, the only fixed point is ∞, since it was infinitely far away to begin with. Again, all parabolic maps can be made equal to the standard form by performing a sort of change of coordinates. Let g be parabolic with fixed point u. We can send u to ∞ with the map h_u defined by

$$z \mapsto \frac{-1}{z-u}.$$

Conjugating g with h_u will yield the standard form.

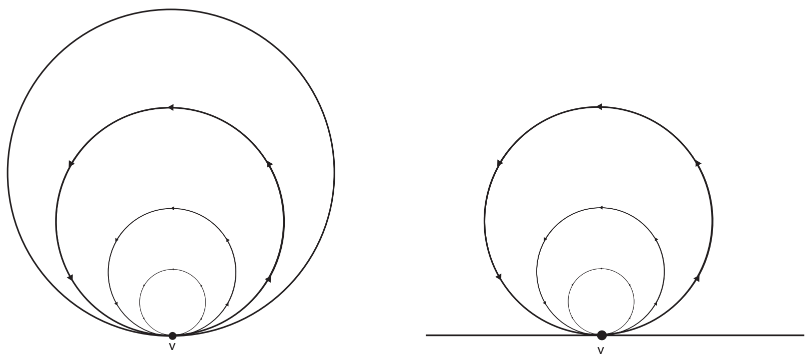

FIGURE 8. Parabolic element action on hyperbolic space. The arrows trace out the geodesic paths the transformation follows.

5.4. **Elliptic Elements.** The standard form of an elliptic element is the map

$$z \mapsto \frac{\cos(\theta)z + \sin(\theta)}{-\sin(\theta)z + \cos(\theta)}.$$

We can think of this as a "hyperbolic rotation" of the half-plane. The fact that this map is elliptic is due to

$$\mathrm{Tr}(g)^2 = (2\cos(\theta))^2 = 4\cos(\theta) < 4.$$

This standard form has the conjugate pair of fixed points $\pm i$. It is not trivial to see that if there is a map with the same fixed points, then it must be of the same standard form.

Lemma 5.4. *If $g \in \mathrm{PSL}_2(\mathbb{R})$ fixes i, Then it is in the form*

$$z \mapsto \frac{\cos(\theta)z + \sin(\theta)}{-\sin(\theta)z + \cos(\theta)}$$

for some $\theta \in [0, \pi]$.

Proof. Note that

$$g(i) = \frac{ai+b}{ci+d} = i \implies -c + di = ai + b \implies b = -c,\ d = a.$$

Plugging these identities into $ad - bc = 1$ implies that $a^2 + b^2 = 1$ and $a^2 + d^2 = 1$ which implies $d = a$, $b \pm \sqrt{1 - a^2}$, and $c \mp \sqrt{1 - a^2}$. Plug these identities into $ad - bc = 1$ again and notice that b and c must have opposite signs. In the case $b > 0$ then setting $a = \cos\theta$ will yield the desired result. Otherwise, setting $a = \cos(-\theta)$ will yield the result, when one recalls that cos is even and sin is odd. □

Like the other hyperbolic isometries, any elliptic element of $\mathrm{PSL}_2(\mathbb{R})$ is conjugate to a standard form element. In this case, we take the single (in the half-plane, not its boundary) to i and then apply the above lemma. However, it is important to note that elliptic elements fix a point *in* the half-plane, not the boundary.

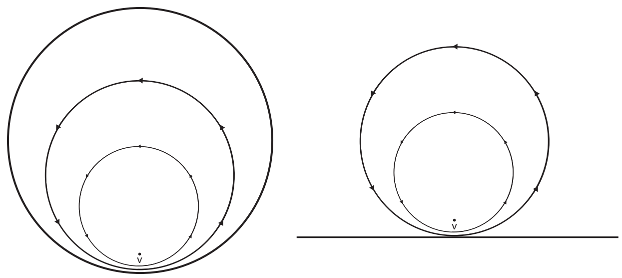

FIGURE 9. Elliptic element action on hyperbolic space. Note that v is not on the boundary.

6. Fuchsian Groups

Consider a parabolic element g. We know it is either in the form or conjugate to a map of the form

$$z \mapsto z + \beta.$$

This map translates points of hyperbolic space by some unique value. However, the map $z \mapsto z + \beta + 1/2$ translates the points relatively close to where g translated the same points. In fact, we can actually write

$$g = \sup \left\{ g_n : z \mapsto z + \beta + 1/n \mid n \in \mathbb{N} \right\}.$$

Every element of this set is an isometry of hyperbolic space, and as $n \to \infty$ the difference between $g_n(z)$ and $g(z)$ becomes arbitrarily small. In this sense, we can approximate g as well as we would like by choosing n large enough. Note that β is arbitrary, so for any parabolic element, there are other isometries of the half-plane that approximate it to arbitrary accuracy. Moreover, this phenomenon holds for all of the different kinds of elements in the isometries of hyperbolic space, as any rotation and dilation can also be approximated arbitrarily well in the same fashion. With these considerations, it should be easy to convince yourself that the following map is continuous:

$$\varphi : \mathrm{PSL}_2(\mathbb{R}) \times \mathbb{H}^2 \to \mathbb{H}^2 : (g, z) \mapsto g(z).$$

We have made the spectacular discovery that $\mathrm{PSL}_2(\mathbb{R})$ is a *Lie Group*. But this discovery is just as much a curse as it is a blessing. Everything we have learned about hyperbolic space we learned through capitalizing on a useful transformation or observing what a transformation does to a space, but we will not learn anything if our transformations are essentially imperceptible from each other! This section is going to explore what we can learn about hyperbolic space when our transformations are *different enough*, and bring us to the best answer to our original question: what hyperbolic space looks like.

6.1. **Defining Fuchsian Groups.** Let us make our notion of *different enough* precise.

Definition 6.1. *Let (X, d) be a metric space. We say that a subset $S \subset X$ is* discrete *if*

$$\forall s \in S \, \exists r \mid B_r(s) \cap S = \{s\}.$$

In words, discreteness means that there is a non-trivial amount of space between a point and other points in a space. When we apply this concept to $\mathrm{PSL}_2(\mathbb{R})$, discreteness means that there is a non-trivial difference between an isometry and its closest neighbor: we can tell the difference between them. Discreteness is then the property we want to impose on our group of isometries. Since this notion comes about only in the context of a metric space, we need to introduce a metric on our group that will give us a reasonable measure of how different the transformations are from each other. This is done by considering the general form of an isometry of hyperbolic space g defined by

$$g(z) = \frac{az + b}{cz + d}.$$

We can define the norm of this function to be the Euclidean norm ($\|\cdot\|_2$) as

$$\|g\|_2 = \left\| \frac{az + b}{cz + d} \right\|_2 = \sqrt{a^2 + b^2 + c^2 + d^2}$$

and define the metric

$$\rho : \mathrm{PSL}_2(\mathbb{R}) \times \mathrm{PSL}_2(\mathbb{R}) \to \mathbb{R} : (g_1, g_2) \mapsto \|g_1 - g_2\|_2 .$$

If we endow $\mathrm{PSL}_2(\mathbb{R})$ with this metric, then we can find its discrete subgroups.

Definition 6.2. *A* Fuchsian Group, *Γ of $\mathbb{H}^2$ is a discrete subgroup of the space $(\mathrm{PSL}_2(\mathbb{R}), \rho)$.*

6.2. **Fuchsian Group Action.** We have defined Fuchsian groups specifically as those with isometries that are sufficiently different from each other. Since the elements of such a group are functions, the way in which we make observations on them is by observing their images on the domain. This means that functions are only different if all the points on their images are different. This matters to hyperbolic geometry because a function's image on the domain is precisely the action of a Fuchsian group on hyperbolic space. The action of a Fuchsian group on hyperbolic space is very elegant, and leads to great visualizations of either model.

Definition 6.3. *Let $G \circlearrowright X$. An* orbit *of G on $x \in X$ is the set*

$$G(x) := \{g(x) \mid g \in G\}.$$

Definition 6.4. *Let $E \subset X$ where X is a metric space. We say that E is* locally finite *if for all compact $K \subset X$, the set $K \cap X$ is finite.*

We intuitively conceptualize continuity as being able to draw a curve without having to lift one's pencil. Looking at the definition below, we see that applying local finiteness to the orbit of a group on a set X will force us to lift our pencil. We call this action *properly discontinuous.*

Definition 6.5. *Let G be a group and X be a metric space. We say that G acts on X properly discontinuously if $\forall x \in X$, the orbit*

$$G(x) := \{g(x) \mid g \in G\}$$

is locally finite.

We have previously defined Fuchsian groups to be discrete subgroups of $(\mathrm{PSL}_2(\mathbb{R}), \rho)$. However, we can show that the discreteness property and a properly discontinuous action are equivalent statements.

Theorem 6.6. *A subgroup $\Gamma \leqslant \mathrm{PSL}_2(\mathbb{R})$ is Fuchsian if and only if it acts properly discontinuously on the half plane.*

Proof. ($\Longrightarrow$)
Let $\Gamma \leqslant \mathrm{PSL}_2(\mathbb{R})$ be Fuchsian and let $K \subset \mathbb{H}^2$ be compact. It would suffice to show that

$$|\Gamma(x) \cap K| \leqslant |\Gamma \cap \{g \in \mathrm{PSL}_2(\mathbb{R}) \mid g(x) \in K\}| < \infty.$$

The first inequality occurs because there could be more than one g which corresponds to some $g(x)$ in the orbit $\Gamma(x)$. The intersection of a closed discrete subset with a compact set is at most finite, and it is an easy check to show that Γ is closed. Therefore, it will suffice to show that G is compact. To show closure, consider the sequence

$$\{g_n\}_{n=1}^{\infty} \mid \mathrm{Image}(g_n) \in K \; \forall n \in \mathbb{N}$$

and suppose that $g_n \to g \in \mathrm{PSL}_2(\mathbb{R})$. Then,

$$\forall x \in K, g(x) = \lim_{n\to\infty} g_n(x).$$

Since K is closed, this limit will be an element of K so that $g(x)$ will be an element of K for all x and $g \in G$. To show the set is bounded, it will suffice to show that there is a uniform bound on a, b, c, d satisfying

$$\frac{az+b}{cz+d} \in K.$$

We know that K is bounded in the half plane, implying there is a $C > 1$ such that for all $w \in K$ we have $|w| \leqslant C$ and $\mathrm{Im}(w) > 1/C$. Our functions will then satisfy

$$|az+b| < C|cz+d|.$$

We can conclude then that a and b are bounded. To show that c and d are also bounded, observe that

$$1/C < \mathrm{Im}\left(\frac{az+b}{cz+d}\right) = \mathrm{Im}\left(\frac{(az+b)(c\overline{z}+d)}{|cz+d|^2}\right) = \frac{adz+bc\overline{z}}{|cz+d|^2} = \frac{\mathrm{Im}(z)(ad-bc)}{|cz+d|^2} = \frac{\mathrm{Im}(z)}{|cz+d|^2}$$

therefore

$$|cz+d| \leqslant \sqrt{C\,\mathrm{Im}(z)}$$

showing that c and d are bounded as well. Choose the largest such bound of the four to find the uniform bound, and conclude that a, b, c, d are all bounded. The proof is complete, as we have shown that G is compact.
($\Longleftarrow$)
Suppose $\Gamma \leqslant \mathrm{PSL}_2(\mathbb{R})$ acts properly discontinuously on the half-plane. Suppose that Γ is not discrete, meaning we can find an element $g \in \Gamma$ and a sequence $\{g_n\}_{n=1}^{\infty}$ such that $g_n \to g$. Let $h_n = g^{-1}g_n \in \Gamma$. For no one n does $h_n = \mathrm{Id}$, but $h_n \to \mathrm{Id}$. There are two cases, one of which must hold:

(1) Infinitely many of the h_n fix i In this case, we can find an infinite sequence of elliptic elements which all fix i and converge to the identity. If you consider the orbit $\Gamma(2i)$ under this sequence, you see that as $n \to \infty$ the distance between $2i$ and $h_n(2i)$ converges to 0. Thus, $2i$ is an accumulation point of this orbit, so Γ cannot act locally discontinuously.
(2) Infinitely many of the h_n do not fix i. In this case, i is an accumulation point of $\Gamma(i)$ by the same argument as above.

The subgroup Γ must be discrete, therefore Fuchsian. □

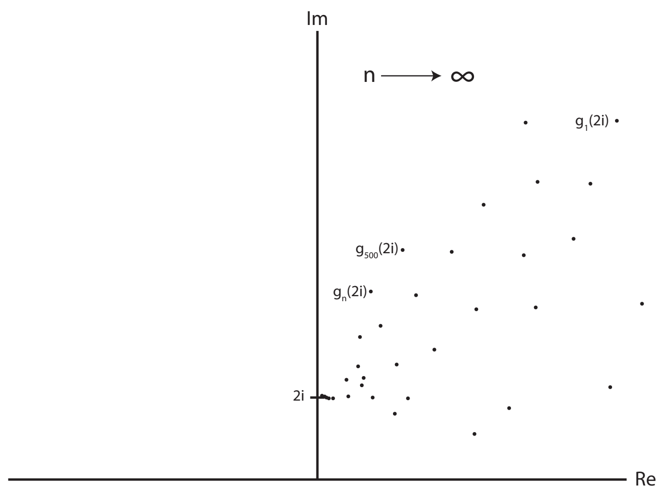

FIGURE 10. Visualization of the second half of Theorem 6.6. The sequence of functions converges to the identity, so action of the sequence will converge to the original point, making that point an accumulation point.

We are going to use the action of Fuchsian groups to pull off one of the most brilliant tricks in geometry: tiling a space. Remember that we are still astronauts in a new world. We have been able to use the group of isometries of our space to learn what movements we can make, and we have even learned what geodesics look like in a space. Sounds impressive, but we do not really have a lot of information. How much of the earth would we understand if we could only move in straight lines? Even worse, we have no idea what is going to happen if we take even a single step out of the geodesic paths and allowed isometric transformations. What do we do? Send a probe. With our knowledge of hyperbolic space, we know what hyperbolic polygons look like, so if we construct one out of the geodesics, we can just throw it in whatever direction we like and see what happens to it. This is the genius of tilings. Send out tiles we do understand to cover a space that we do not understand. Observing their deformations throughout will tell us how the intrinsic properties of the space are morphing our well understood shapes. Now we have a map of our space, and we know what to expect, no matter what path we take. We are going to find a tile for hyperbolic space, and then prove that we can use it. We will call these tiles *fundamental domains*.

Definition 6.7. *A* Fundamental Domain *of the half-plane is an open subset* $F \subset \mathbb{H}^2$ *such that*

(1) The whole plane is "tiled", that is, $\mathbb{H}^2 = \bigcup_{g \in \Gamma} g(\overline{F})$
(2) None of the tiles overlap: $\forall g, h \in \Gamma \mid g \neq h,\ g(\overline{F} \cap h(\overline{F})) = \varnothing$

We have created a new mathematical object, so we should check that it exists. In this case, we will show that the fundamental domain for a Fuchsian group is its Dirichlet region.

Definition 6.8. *Let* $\Gamma \leqslant \mathrm{PSL}_2(\mathbb{R})$ *be Fuchsian and let* $z \in \mathbb{H}^2$*. The* Dirichlet region *of* Γ *with respect to* z *is given by*

$$D_z(\Gamma) := \bigcap_{g \in \Gamma \setminus \{\mathrm{Id}\}} \left\{ w \in \mathbb{H}^2 \mid d_{\mathbb{H}^2}(w, z) < d_{\mathbb{H}^2}(w, g(z)) \right\}.$$

Theorem 6.9. *Let* $\Gamma \leqslant \mathrm{PSL}_2(\mathbb{R})$ *be Fuchsian. Any Dirichlet region for* Γ *is a connected, convex fundamental domain of the half-plane.*

Proof. Note that $D_z(\Gamma)$ is the intersection of convex, connected sets and is therefore connected and convex. The intersection of infinitely many open sets is not necessarily open, but in the case of the Dirichlet region, the action of Γ on the half-plane ensures the openness of $D_z(\Gamma)$. Let $B(w, r)$ be the (closed) ball centered at w with radius r. Since Γ acts properly discontinuously, we know that $|\Gamma(z) \cap B(w, 5d_{\mathbb{H}^2}(w, z))| < \infty$ and so for all but finitely many $g \in \Gamma$ we have

$$B(w, d_{\mathbb{H}^2}(w, z)) \cap \left\{ w \in \mathbb{H}^2 \mid d_{\mathbb{H}^2}(w, z) < d_{\mathbb{H}^2}(w, g(z)) \right\} = B(w, d_{\mathbb{H}^2}(w, z)).$$

For any w we can find a neighborhood of it inside $D_z(\Gamma)$ by taking the *finite* intersection of the half-spaces corresponding to the $g \in \Gamma$ that do not fit the description above. Then our Dirichlet region is open. To show that $D_z(\Gamma)$ is a fundamental domain we consider the following:

(1) Let $w \in \mathbb{H}^2$ be given. Since $\Gamma(w)$ is a closed, discrete set, there exists a $h \in \Gamma$ that minimizes the distance from $\Gamma(w)$ to z, that is,

$$\forall g \in \Gamma, d_{\mathbb{H}^2}(h(w), z) < d_{\mathbb{H}^2}(g(w), z)$$

and we can manipulate the inequality as shown:

$$d_{\mathbb{H}^2}(h(w), z) < d_{\mathbb{H}^2}(g(w), z) = d_{\mathbb{H}^2}(w, g^{-1}(z)) = d_{\mathbb{H}^2}(h(w), h(g^{-1}(z))).$$

Since arbitrary elements of Γ can be written in the form $h \circ g^{-1}$ we see that

$$h(w) \in \overline{D_z(\Gamma)}$$

and

$$w \in h^{-1}(\overline{D_z(\Gamma)}) \subset \bigcup_{g \in \Gamma} g(\overline{D_z(\Gamma)}).$$

(2) Let $w_1, w_2 \in \Gamma(v)$ for some $v \in \mathbb{H}^2$ be such that $w_1 \neq w_2$ and assume that $w_1 \in D_z(\Gamma)$. We will show that $w_2 \notin D_z(\Gamma)$ and use that to prove the other criterion. We can write w_1 and w_2 in the form $g_1(v), g_2(v)$ where $g_1 \neq g_2$. Since $w_1 \in D_z(\Gamma)$,

$$d_{\mathbb{H}^2}(w, z) < d_{\mathbb{H}^2}(w, g(z))$$

for all $g \in \Gamma \setminus \{\mathrm{Id}\}$. By choosing $g = g_1 \circ g_2^{-1}$we can see that

$$d_{\mathbb{H}^2}(w_1, z) < d_{\mathbb{H}^2}(w_1, g_1(g_2^{-1}(z))) = d_{\mathbb{H}^2}(g_2(g_1^{-1}(w_1)), z) = d_{\mathbb{H}^2}(w_2, z).$$

Also note that

$$d_{\mathbb{H}^2}(w_1, z) = d_{\mathbb{H}^2}(g_1(v), z) = d_{\mathbb{H}^2}(v, g_1^{-1}(z)) = d_{\mathbb{H}^2}(g_2(v), g_2(g_1^{-1}(z))) = d_{\mathbb{H}^2}(w_2, g_2(g_1^{-1}(z)))$$

so

$$d_{\mathbb{H}^2}(w_2, g_2(g_1^{-1}(z))) < d_{\mathbb{H}^2}(w_2, z) \implies w_2 \notin D_z(\Gamma).$$

To complete the proof suppose that $v \in g_1(D_z(\Gamma) \cap g_2(D_z(\Gamma)))$ for some $g_1, g_2 \in \Gamma$ where $g_1 \neq g_2$. Since $g_1(D_z(\Gamma) \cap g_2(D_z(\Gamma)))$ is open we can find a neighborhood $B(v, r) \subset g_1(D_z(\Gamma) \cap g_2(D_z(\Gamma)))$. Then for any point $u \in B(v, r)$, $g_1^{-1}(u)$ and $g_2^{-1}(u) \in D_z(\Gamma)$ which by the above argument implies $g_1 = g_2$ on the open set $B(v, r)$ which implies $g_1 = g_2$.

□

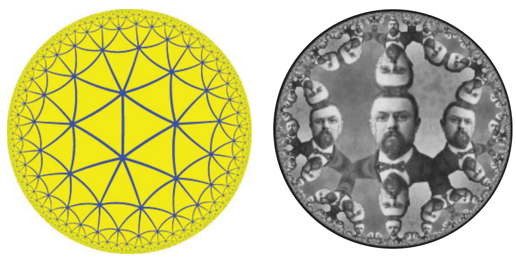

FIGURE 11. The image on the left is a tiling of the Poincaré disk using the (2,3,7) triangle group [5]. On the right is a similar tiling using an image of Poincaré himself. To find the original image, and to create a tiling of your own, see this site [6].

The proof demonstrates that the action of Γ was necessary for the creation of a good tile. From now on, we will assume that any $D_z(\Gamma)$ is polygonal, meaning a convex hyperbolic polygon with a finite number of vertices in $\overline{\mathbb{H}^2}$ and a finite number of edges, each of which is a hyperbolic geodesic. We used the transformations of a Fuchsian group to construct a fundamental domain, but that does not capture the richness of the connections between the two objects. Using a fundamental domain, we can actually construct the Fuchsian group that it comes from using the geometry of the polygonal tile. Essentially, Fuchsian groups and fundamental domains define each other.

Lemma 6.10. *Let $D_z(\Gamma)$ be a fundamental domain for a given $\Gamma \leqslant \mathrm{PSL}_2(\mathbb{R})$ and suppose that $D_z(\Gamma)$ is polygonal. Label each of its edges by e_g where g is the element of Γ such that e_g is part of the perpendicular bisector of the geodesic between z and $g(z)$. Then,*

$$g^{-1}(e_g) = e_{g^{-1}} \text{ and } g(e_{g^{-1}}) = e_g.$$

Proof. Let

$$H_g := \left\{ w \in \mathbb{H}^2 \mid d_{\mathbb{H}^2}(w, z) = d_{\mathbb{H}^2}(w, g(z)) \right\}$$

be the geodesic ray containing e_g. Observe that

$$w \in H_g \implies d_{\mathbb{H}^2}(w,z) = d_{\mathbb{H}^2}(w,g(z))$$
$$\implies d_{\mathbb{H}^2}(g^{-1}(w),g^{-1}(z)) = d_{\mathbb{H}^2}(g^{-1}(w),z) \implies g^{-1}(z) \in H_{g^{-1}}.$$

This shows that $g^{-1}(H_g) = H_{g^{-1}}$. Some more work shows that $g^{-1}(e_g) = e_{g^{-1}}$. $\square$

The above lemma implies that the sides of a polygonal fundamental domain come in pairs that are mapped to each other by members of Γ. These elements of Γ are called side pairing transformations. The one exception is when there is an elliptic fixed point in one of the sides. In this case we see that the elliptic element will interchange the two halves of the edge. We think of the fixed point as a vertex and each half of the edge as it own edge and our implication from the lemma holds.

Lemma 6.11. *The side-pairing transformations of a polygonal Dirichlet fundamental domain form a generating set for its corresponding Fuchsian group.*

Proof. Let D be a fundamental domain for Γ and let $H \leqslant \Gamma$ be the group generated by the side-pairing transformations. Because $H \leqslant \Gamma$, it will suffice to show that $\Gamma \subset H$. Let $g \in \Gamma$ and let $h \in H$. Note that if $g(D)$ and $h(D)$ are adjacent (share a side), then g is necessarily an element of H. This is because if $g(D)$ and $h(D)$ are adjacent, then $h^{-1}g(D)$ and $h^{-1}h(D)$ are adjacent. Because they are adjacent, $h^{-1}g$ must be a side pairing transformation, and the fact that H is a group allows the manipulation $h \cdot h^{-1} \cdot g = g$ to show g is an element of H. Now let $g \in \Gamma$. We want to show that $g \in H$. Note that $\Gamma(\overline{D})$ is a tiling of hyperbolic space, so we can find a sequence $g_1(D), g_2(D), \ldots, g_n(D)$ where $g_1 = Id$, $g_n = g$ and for each k, $g_k(D)$ is adjacent to $g_{k+1}(D)$ for $k \in \{1, \ldots n-1\}$ meaning that $g_k \in H \implies g_{k+1} \in H$. Our result follows from induction. $\square$

This lemma shows that the edges of the $D_z(\Gamma)$ fundamental domain for a given Γ are in a bijective correspondence with the generators for Γ. As it turns out, the vertices of a polygonal $D_z(\Gamma)$ fundamental domain are in a bijective correspondence with the set of relations for the presentation of Γ. This means that $D_z(\Gamma)$ encodes all of the information needed to figure out what its corresponding Γ looks like in terms of a presentation. The following *side-pairing algorithm* demonstrates how to do so.

Let E denote the finite set of edges and V the finite set of vertices in a polygonal fundamental domain. For each $v \in V \cap \mathbb{H}^2$, we can associate a relation as follows:

Step 1. Let $v_1 \in V$ and $e_{g_1} \in E$ be one of the edges adjacent to v_1. Let $\alpha(v_1)$ be the angle at v_1. We know that $g_1^{-1}(e_g) = e_{g^{-1}}$ and $g^{-1}(v_1) \in V$ is a vertex adjacent to $e_{g^{-1}}$.

Step 2. Set $v_2 = g^{-1}(v_1) \in V$ and let $\alpha(v_2)$ be the angle at v_1. There is a unique edge $e_{g_2} \in E$ different from $e_{g_1^{-1}}$ adjacent to v_2. Therefore, $g_2^{-1}(e_{g_2}) = e_{g_2^{-1}}$ and $g_2^{-1}(v_1) \in V$ is a vertex adjacent to $e_{g_2^{-1}}$.

Step 3. Set $v_3 = g^{-1}(v_2) \in V$ and let $\alpha(v_3)$ be the angle at v_3. There is a unique edge $e_{g_3} \in E$ different from $e_{g_2^{-1}}$ adjacent to v_2. Therefore, $g_3^{-1}(e_{g_3}) = e_{g_3^{-1}}$ and $g_3^{-1}(v_2) \in V$ is a vertex adjacent to $e_{g_2^{-1}}$.

$$\vdots$$

6.2.1. *Step $k(v)$.* Because V is finite, we know this process must terminate, meaning we must reach a vertex we have seen before. Note that each step in the algorithm described above utilizes both of the edges for every vertex except v_1, forcing the process to terminate at v_1 and

$$v_1 = g_{kv_1}^{-1}(g_{kv_1-1}^{-1}(\cdots g_2^{-1}(g_1^{-1}(v_1))\cdots))$$

Since $v_1 \in V$ was arbitrary, we can apply this to any such $v \in V$. The resulting relation is called a *cycle transformation* and we see that it either fixes v or is an elliptic element of finite order, i.e. $(g_1 \ldots g_{k(v)})^{n(v)} = \mathrm{Id}$ (otherwise the group would not act properly discontinuously). If the cycle transformation is elliptic, then the orbit of the fundamental domain under this map would tile a neighborhood of v_1. This algorithm almost proves the theorem below.

Theorem 6.12. *Suppose that Γ is a Fuchsian group and let $D_z(\Gamma)$ be a polygonal fundamental domain with finite vertex and edge sets. Let ε be the set of all side-pairing transformations, and g_v be the associated cycle transformation for each $v \in V$. Then,*

$$\Gamma \cong \langle \varepsilon : g_n^{n(v)}, v \in V \rangle.$$

We did the hard work of showing the Fuchsian group is homomorphic to the group with the above presentation. The little details can be found in Katok [4].

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