# FOURIER ANALYSIS AND THE WAVE EQUATION

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Abstract. This paper presents a study of the wave equation. We begin by considering a beaded string to derive the wave equation. We then seek to derive a closed form solution for the wave equation in one dimension. From there, we introduce the Fourier transform on $\mathbb{R}$, and then generalize to the Fourier transform on $\mathbb{R}^d$. After developing this theory, this paper seeks to prove the existence of the solution to the Cauchy problem for the wave equation. We then introduce the concept of energy in order to prove the uniqueness of this solution. Finally, we derive closed form solutions to the Cauchy problem for the wave equation in 3 and then 2 dimensions, and finish with a remark on the qualitative difference between these solutions. In several places, we closely follow the treatment in Fourier Analysis: An Introduction (see [1]).

## Contents



## 1. Derivation of the Wave Equation

This paper will begin with a physical motivation for the study of the wave equation. First, we will consider a transverse wave traveling along a beaded string to derive the wave equation in $\mathbb{R} \times \mathbb{R}$. From this, we will make the generalization to the wave equation in $\mathbb{R}^d \times \mathbb{R}$.

Consider a string of length $L$ in the $x$-$y$ plane connecting $N$ beads spaced equally far apart. Define the distance between adjacent beads to be $d := \frac{L}{(N-1)}$, and let the first bead be located at some position $(0, y_1(t))$, so that the last bead is located at position $(L, y_N(t))$. Then the $i^{th}$ bead has an $x$-coordinate given by $x_i = (i-1)d = \frac{(i-1)L}{(N-1)}$. We will consider the function $u(x,t)$ which gives the

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position of a point on this string at a time $t$ after it begins to vibrate. We make the following three assumptions:

(i) The only forces acting on each bead are from the tension in the string created by the relative positions of the two adjacent beads.
(ii) The string has constant mass density $\rho$, which is the mass per unit length, and constant coefficient of tension $\tau$, which describes the strength of the tension force caused by an elongation of the string.
(iii) The beads travel only in the transverse direction, so the beads' $x$-coordinates are fixed while their $y$-coordinates vary in time.

We will consider this system where the number of beads $N$ goes to $\infty$. First, note that the position of the $i^{th}$ bead is given by $u(x_i, t) = y_i(t)$. By Newton's third law, we have that $F = ma$. By assumption (iii), $a_i = y_i''(t) = \frac{\partial^2}{\partial t^2} u(x_i, t)$. Using assumption (ii), we can say that the mass of each bead, $m$, is given by $m = d\rho$. Hence,

$$F_i = d\rho \frac{\partial^2}{\partial t^2} u(x_i, t), \tag{1.1}$$

where $F_i$ is the force acting on the $i^{th}$ bead.

Let $F_{T_i^-}$ denote the transverse force on the $i^{th}$ bead caused by the tension in the string connecting beads $(i-1)$ and $i$, and define $F_{T_i^+}$ analogously. Then

$$F_{T_i^-} = \tau \left( \frac{y_{i-1} - y_i}{d} \right),$$

and

$$F_{T_i^+} = \tau \left( \frac{y_{i+1} - y_i}{d} \right).$$

Assumptions (i) and (iii) imply

$$F_i = \tau \left( \frac{y_{i+1} + y_{i-1} - 2y_i}{d} \right). \tag{1.2}$$

Combining (1.1) with (1.2), and recalling that $y_i(t) = u(x_i, t)$, we see that

$$\begin{aligned} \frac{\partial^2}{\partial t^2} u(x_i, t) &= \frac{\tau}{\rho} \left( \frac{u(x_{i+1}, t) + u(x_{i-1}, t) - 2u(x_i, t)}{d^2} \right) \\ &= \frac{\tau}{\rho} \left( \frac{u(x_i + d, t) + u(x_i - d, t) - 2u(x_i, t)}{d^2} \right). \end{aligned}$$

Now, note that

$$\frac{\partial^2}{\partial x^2} u(x_i, t) = \lim_{d \to 0} \frac{u(x_i + d, t) + u(x_i - d, t) - 2u(x_i, t)}{d^2}.$$

Finally, recall that we are considering the case where $N \to \infty$, so $d \to 0$. Therefore,

$$\frac{\partial^2}{\partial t^2} u(x_i, t) = \frac{\tau}{\rho} \frac{\partial^2}{\partial x^2} u(x_i, t).$$

Recognizing that the position $x_i$ was arbitrary, and defining $c := \sqrt{\frac{\tau}{\rho}}$, we have

$$\frac{\partial^2}{\partial t^2} u(x, t) = c^2 \frac{\partial^2}{\partial x^2} u(x, t). \tag{1.3}$$

Note that (1.3) is not unitless. Thus, we can rescale these units so that $c = 1$.

**Definition 1.4.** The wave equation in $\mathbb{R} \times \mathbb{R}$ is given by

$$\frac{\partial^2}{\partial x^2} u - \frac{\partial^2}{\partial t^2} u = 0. \tag{1.5}$$

This is the form of the wave equation that we will consider throughout this paper.

If $x \in \mathbb{R}^d$, then the natural generalization of this equation is given by

$$\frac{\partial^2}{\partial x_1^2} u(x,t) + \cdots + \frac{\partial^2}{\partial x_d^2} u(x,t) - \frac{\partial^2}{\partial t^2} u(x,t) = 0. \tag{1.6}$$

**Definition 1.7.** The Laplacian of a function $u(x,t)$ for $x \in \mathbb{R}^d$, denoted $\Delta u(x,t)$, is defined by

$$\Delta u(x,t) := \frac{\partial^2}{\partial x_1^2} u(x,t) + \cdots + \frac{\partial^2}{\partial x_d^2} u(x,t).$$

Combining (1.6) with Definition 1.7, we see that the wave equation in $\mathbb{R}^d \times \mathbb{R}$, for $d \geq 1$, is given by

$$\Delta u - \frac{\partial^2}{\partial t^2} u = 0. \tag{1.8}$$

## 2. A Solution to the Wave Equation in $\mathbb{R} \times \mathbb{R}$

In this section, we will consider the Cauchy problem for the wave equation in $\mathbb{R} \times \mathbb{R}$. The fact that the wave equation involves second partial derivatives motivates the following definition:

**Definition 2.1.** A function $f$ belongs to the class $C^k$ if it is $k$ times continuously differentiable.

Note that if $u(x,t) \in C^2$, then the partial derivatives $\frac{\partial^2}{\partial x^2} u$ and $\frac{\partial^2}{\partial t^2} u$ are continuously differentiable.

**Definition 2.2.** The Cauchy problem for the wave equation in $\mathbb{R} \times \mathbb{R}$ is to find a function $u \in C^2$, $u(x,t) : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ such that:

(i) $u$ satisfies the wave equation,
(ii) $u(x,0) = f(x)$, where f is the initial form of the wave at time $t = 0$, and
(iii) $\frac{\partial}{\partial t} u(x,0) = g(x)$ where g is the transverse velocity of each point of the wave at time $t = 0$.

We now derive a solution to the Cauchy problem. Note that condition (i) equivalently says that

$$\left(\frac{\partial}{\partial t} + \frac{\partial}{\partial x}\right)\left(\frac{\partial}{\partial t} - \frac{\partial}{\partial x}\right) u(x,t) = 0.$$

Define the function $v(x,t) := \left(\frac{\partial}{\partial t} - \frac{\partial}{\partial x}\right) u(x,t) = \frac{\partial}{\partial t} u(x,t) - \frac{\partial}{\partial x} u(x,t)$. Then

$$\left(\frac{\partial}{\partial t} + \frac{\partial}{\partial x}\right) v(x,t) = \frac{\partial}{\partial t} v(x,t) + \frac{\partial}{\partial x} v(x,t) = 0.$$

**Lemma 2.3.** *If $v(x,t) \in C^1$ satisfies*

$$\begin{cases} \frac{\partial}{\partial t} v(x,t) + \frac{\partial}{\partial x} v(x,t) = 0 \\ v(x,0) = f(x), \end{cases}$$

*then $v(x,t) = f(x-t)$, for some function $f$.*

*Proof.* Consider the function $z : \mathbb{R} \to \mathbb{R}$ defined by $z(s) := v(x+s, t+s)$. Then $z(-t) = v(x-t, 0)$. Since $v(x,0) = f(x)$, it follows that $v(x-t,0) = f(x-t)$. Now observe that the first condition implies that $u$ is constant along the line through some point $(x,t)$ defined parametrically by $(x+s, t+s)$. Hence, it must be true that $v(x,t) = f(x-t)$. □

**Lemma 2.4.** *If $u(x,t) \in C^2$ satisfies*

$$\begin{cases} \frac{\partial}{\partial t}u(x,t) - \frac{\partial}{\partial x}u(x,t) = v(x,t) \\ u(x,0) = f(x), \end{cases}$$

*then $u(x,t) = f(x+t) + \int_0^t v(x+(t-s), s)ds$.*

*Proof.* Consider the function $z(s) := u(x-s, t+s)$. First, note that $z \in C^1$ since $u \in C^1$, and

$$\frac{d}{ds}z(s) = \frac{\partial}{\partial t}u(x-s,t+s) - \frac{\partial}{\partial x}u(x-s,t+s).$$

Observe that

$$z(0) = u(x,t),$$

and

$$z(-t) = u(x+t, 0) = f(x+t).$$

Then

$$u(x,t) - f(x+t) = z(0) - z(-t) = \int_{-t}^{0} \frac{d}{ds}z(s)ds$$

$$= \int_{-t}^{0} \left[ \frac{\partial}{\partial t}u(x-s,t+s) - \frac{\partial}{\partial x}u(x-s,t+s) \right] ds.$$

But, by the initial conditions,

$$\frac{\partial}{\partial t}u(x-s,t+s) - \frac{\partial}{\partial x}u(x-s,t+s) = v(x-s,t+s).$$

Thus,

$$u(x,t) - f(x+t) = \int_{-t}^{0} v(x-s,t+s)ds.$$

Changing variables from $s$ to $s-t$, we have

$$u(x,t) = f(x+t) + \int_0^t v(x+(t-s), s)ds.$$

□

We now return to the equation $\frac{\partial}{\partial t}v(x,t) + \frac{\partial}{\partial x}v(x,t) = 0$. By Lemma 2.3, we have that

$$v(x,t) = a(x-t),$$

where

$$a(x) = v(x,0).$$

And since $v(x,t) = \frac{\partial}{\partial t}u(x,t) - \frac{\partial}{\partial x}u(x,t)$, by Lemma 2.4, we have that

$$u(x,t) = u(x+t, 0) + \int_0^t v(x+(t-s), s)ds.$$

Applying the initial conditions, we have

$$u(x,t) = f(x+t) + \int_0^t a(x+(t-s)-s)ds,$$

and changing variables in the integrand from $x+t-2s$ to $y$ yields

$$u(x,t) = f(x+t) - \frac{1}{2}\int_{x+t}^{x-t} a(y)dy.$$

Now, by the definition of $a(y)$, we have

$$u(x,t) = f(x+t) + \frac{1}{2}\int_{x-t}^{x+t}\left[\frac{\partial}{\partial t}u(y,0) - \frac{\partial}{\partial y}u(y,0)\right]dy.$$

Again applying the initial conditions,

$$u(x,t) = f(x+t) + \frac{1}{2}\left[\int_{x-t}^{x+t} g(y)dy + \int_0^{x-t}\frac{\partial}{\partial y}f(y)dy - \int_0^{x+t}\frac{\partial}{\partial y}f(y)dy\right].$$

Hence,

$$u(x,t) = \frac{1}{2}\left[f(x-t) + f(x+t) + \int_{x-t}^{x+t} g(y)dy\right]. \tag{2.5}$$

(2.5) is known as d'Alembert's formula. This derivation motivates the following theorem.

**Theorem 2.6.** *If $u : \mathbb{R}\times\mathbb{R} \to \mathbb{R}$, $u \in C^2$ is defined by* (2.5)*, then $u$ satisfies:*

$$\begin{cases} \Delta u - \frac{\partial^2}{\partial t^2}u = 0 \\ u(x,0) = f(x) \\ \frac{\partial}{\partial t}u(x,0) = g(x). \end{cases}$$

The proof involves only a calculation to check each condition.

*Remark* 2.7. If $u(x,t)$ is of the form given in (2.5), then

$$u(x,t) = F(x+t) + G(x-t), \tag{2.8}$$

for two functions $F$ and $G$. We can consider the form of solution given in (2.8) as the superposition of a right-traveling and a left-traveling wave, both moving with speed $c = 1$. In fact, we have the following result regarding this form of solution:

**Theorem 2.9.** *All solutions to the wave equation take the form given in* (2.8)*.*

*Proof.* Suppose $u(x,t)$ solves the wave equation. Define $\alpha := x+t$ and $\beta := x-t$, and let $v(\alpha,\beta) = u(x,t)$. Since $u$ solves the wave equation, we have that

$$\frac{\partial^2}{\partial\alpha\partial\beta}v(\alpha,\beta) = 0.$$

Then, by the fundamental theorem of calculus,

$$\left(\frac{\partial}{\partial\beta}\right)v(\alpha,\beta) = \int \frac{\partial^2}{\partial\alpha\partial\beta}v(\alpha,\beta)d\alpha = \int 0\ d\alpha = A(\beta).$$

Integrating again,

$$v(\alpha,\beta) = \int \frac{\partial}{\partial\beta}v(\alpha,\beta)d\beta = \int A(\beta) = F(\alpha) + G(\beta).$$

Therefore,

$$v(\alpha, \beta) = u(x,t) = F(\alpha) + G(\beta) = F(x+t) + G(x-t).$$

□

We may now prove the following uniqueness theorem:

**Theorem 2.10.** *If $u : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ satisfies:*

$$\begin{cases} \frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial t^2} = 0 \\ u(x,0) = f(x) \\ \frac{\partial}{\partial t} u(x,0) = g(x), \end{cases}$$

*then $u$ is of the form given in* (2.5).

*Proof.* We apply the initial conditions of the Cauchy problem and the result of Theorem 2.9 to obtain

$$\begin{cases} F(x) + G(x) = f(x) \\ F'(x) - G'(x) = g(x). \end{cases}$$

From this, we see

$$F'(x) = \frac{f'(x)}{2} + \frac{g(x)}{2} \implies F(x) + C_1 = \frac{1}{2}\left[f(x) + \int g(x)dx\right],$$

and

$$G'(x) = \frac{f'(x)}{2} - \frac{g(x)}{2} \implies G(x) + C_2 = \frac{1}{2}\left[f(x) - \int g(x)dx\right].$$

We see that $C_1 + C_2 = 0$ by checking $u(x,0)$ against the Cauchy conditions. Therefore,

$$v(\alpha, \beta) = u(x,t) = \frac{1}{2}\left[f(x-t) + f(x+t) + \int_{x-t}^{x+t} g(y)dy\right],$$

giving us (2.5), as desired. □

## 3. The Fourier Transform on $\mathbb{R}$

**Definition 3.1.** The Fourier transform of a function $f : \mathbb{R} \to \mathbb{R}$ is a function $\hat{f} : \mathbb{R} \to \mathbb{C}$ defined by

$$\hat{f}(\xi) := \int_{\mathbb{R}} f(x) e^{-2\pi i x \xi} dx. \tag{3.2}$$

Clearly, this integral will not converge for every function $f$. We therefore wish to impose some conditions on $f$ to ensure that the integral given in (3.2) converges. A natural limitation to put on $f$ is to have $|f(x)| \to 0$ sufficiently quickly as $|x| \to \infty$. This consideration motivates the following definition:

**Definition 3.3.** The Schwartz space on $\mathbb{R}$, denoted $\mathcal{S}(\mathbb{R})$, is the space of all infinitely differentiable functions $f : \mathbb{R} \to \mathbb{R}$ such that $f$ and all of its derivatives are rapidly decreasing, which means that for every $k, l \in \mathbb{N} \cup \{0\}$,

$$\sup_{x \in \mathbb{R}} |x|^k \left| \left(\frac{d}{dx}\right)^l f(x) \right| < \infty.$$

Now, if $f \in \mathcal{S}(\mathbb{R})$, then the integral given in (3.2) converges.

**Proposition 3.4.** *If $f$ is a function of the form $f(x) = e^{-ax^2}$, $a \in \mathbb{R}$, then $f \in \mathcal{S}(\mathbb{R})$.*

*Proof.* It is clear that such a function $f$ is infinitely differentiable. To see that $f$ satisfies

$$\sup_{x\in\mathbb{R}} |x|^k \left| \left(\frac{d}{dx}\right)^l f(x) \right| < \infty, \text{ for all } k, l \in \mathbb{R},$$

it suffices to observe that the denominator of the above expression, $|e^{ax^2}|$, will outgrow any polynomial of the form $|(x^k)(-2ax)^l|$ as $|x|$ goes to $\infty$. □

*Remark* 3.5. Functions of the form $e^{-ax^2}$ are known as Gaussians. These functions will play a central role in our study of the Fourier transform on $\mathbb{R}$, and in the generalization of this theory to an arbitrary number of dimensions. In particular, we will consider the Gaussian where $a = \pi$.

**Theorem 3.6.** *If $f(x) = e^{-\pi x^2}$, then $\hat{f}(\xi) = f(\xi)$.*

The proof will require the following two lemmas:

**Lemma 3.7.** *For any function $f \in \mathcal{S}(\mathbb{R})$, the Fourier transform of $f'(x)$ is given by*

$$2\pi i\xi \hat{f}(\xi).$$

*Proof.* Applying the definition of $\hat{f}(\xi)$, we get

$$2\pi i\xi \hat{f}(\xi) = 2\pi i\xi \int_{\mathbb{R}} f(x)e^{-2\pi ix\xi}dx = \int_{\mathbb{R}} f(x)(2\pi i\xi)e^{-2\pi ix\xi}dx.$$

Integrating by parts, we have

$$\int_{\mathbb{R}} f(x)(2\pi i\xi)e^{-2\pi ix\xi}dx = \left[f(x)(-e^{-2\pi ix\xi})\right]_{-\infty}^{\infty} - \int_{\mathbb{R}} f'(x)(-e^{-2\pi ix\xi})dx.$$

Since $f \in \mathcal{S}(\mathbb{R})$, $\left[f(x)(-e^{-2\pi ix\xi})\right]_{-\infty}^{\infty} = 0$, so

$$2\pi i\xi \hat{f}(\xi) = \int_{\mathbb{R}} f'(x)(e^{-2\pi ix\xi})dx,$$

which, by definition, is the Fourier transform of $f'(x)$. □

**Lemma 3.8.** *For any function $f \in \mathcal{S}(\mathbb{R})$, the Fourier transform of $-2\pi ixf(x)$ is given by*

$$\frac{d}{d\xi}\hat{f}(\xi).$$

*Proof.* If we can interchange differentiation and integration, then we have that

$$\frac{d}{d\xi}\hat{f}(\xi) = \int_{\mathbb{R}} (-2\pi ixf(x))e^{-2\pi ix\xi}dx,$$

which, by definition, is the Fourier transform of $-2\pi ixf(x)$. To show that this can be done, we use the fact that $f(x)e^{-2\pi ix\xi} \in \mathcal{S}(\mathbb{R})$ for fixed $\xi$, which follows from the fact that $f \in \mathcal{S}(\mathbb{R})$. Working from the definition of the derivative, we have

$$\frac{d}{d\xi}\hat{f}(\xi) = \lim_{h\to 0} \frac{\int_{\mathbb{R}} f(x)e^{-2\pi ix(\xi+h)} - \int_{\mathbb{R}} f(x)e^{-2\pi ix\xi}dx}{h}$$

$$= \lim_{h\to 0} \frac{\int_{\mathbb{R}} f(x)e^{-2\pi i x\xi}dx + h\int_{\mathbb{R}} \frac{d}{d\xi} f(x)e^{-2\pi i x\xi}dx + h\int_{\mathbb{R}} \mu(x,\xi,h)dx - \int_{\mathbb{R}} f(x)e^{-2\pi i x\xi}dx}{h},$$

where $\mu$ is an error term which goes to 0 pointwise as $h$ goes to 0. Then

$$\frac{d}{d\xi}\hat{f}(\xi) = \int_{\mathbb{R}} \frac{d}{d\xi} f(x)e^{-2\pi i x\xi}dx + \int_{\mathbb{R}} \mu(x,\xi,h)dx.$$

We may now complete the proof if we can show that $\left|\int_{\mathbb{R}} \mu(x,\xi,h)dx\right|$ goes to 0. The Schwartz space is closed under addition, so $\mu \in \mathcal{S}(\mathbb{R})$ for fixed $h, \xi$. For any $\eta > 0$ of our choosing, we have

$$\left|\int_{\mathbb{R}} \mu(x,\xi,h)dx\right| = \left|\int_{|x|<\eta} \mu(x,\xi,h)dx + \int_{|x|\geq\eta} \mu(x,\xi,h)dx\right|.$$

Since $\mu \in \mathcal{S}(\mathbb{R})$, we can choose $k$ and $l$ in the definition of the Schwartz space to obtain

$$[|\mu(x,\xi,h)|] \leq M_1(h) \implies \left|\int_{|x|<\eta} \mu(x,\xi,h)dx\right| \leq 2\eta M_1(h).$$

Moreover, by Taylor's Remainder Theorem, we have that

$$|\mu(x,\xi,h)| \leq |f''(\alpha)|\left(\frac{h}{2}\right), \quad \text{for some } \alpha \in (x, x+h).$$

Applying the fact that $f \in \mathcal{S}(\mathbb{R})$, we have that

$$|\mu(x,\xi,h)| \leq \begin{cases} \left(\frac{M_2}{(x-h)^2}\right)\left(\frac{h}{2}\right), & x > h \\ \left(\frac{M_2}{(x+h)^2}\right)\left(\frac{h}{2}\right), & x < h. \end{cases}$$

Hence,

$$\left|\int_{|x|\geq\eta} \mu(x,\xi,h)dx\right| \leq \frac{M_2 h}{|\eta - h|}.$$

Therefore, we can choose $h$ sufficiently small and $\eta$ sufficiently large so that $\mu(x,\xi,h)$ goes to 0, as desired. □

We are now in a position to prove the theorem.

*Proof.* Suppose $f \in \mathcal{S}(\mathbb{R})$ is given by $f(x) = e^{-\pi x^2}$. We wish to prove $\hat{f}(\xi) = e^{-\pi\xi^2}$. Define a function $G : \mathbb{R} \to \mathbb{R}$ by

$$G(\xi) := \hat{f}(\xi)e^{\pi\xi^2}.$$

Then, by the product rule for differentiation,

$$G'(\xi) = \hat{f}'(\xi)e^{\pi\xi^2} + 2\pi\xi\hat{f}(\xi)e^{\pi\xi^2}.$$

By Lemma 3.8,

$$\frac{d}{d\xi}\hat{f}(\xi) = \int_{\mathbb{R}} -2\pi i x f(x)e^{-2\pi i x\xi}dx.$$

By the definition of $f$,

$$\int_{\mathbb{R}} -2\pi i x f(x)e^{-2\pi i x\xi}dx = i\int_{\mathbb{R}} f'(x)e^{-2\pi i x\xi}dx.$$

By Lemma 3.7,

$$i\int_{\mathbb{R}} f'(x)e^{-2\pi i x\xi}dx = i(2\pi i\xi\hat{f}(\xi)) = -2\pi\xi\hat{f}(\xi).$$

Therefore,

$$G'(\xi) = \hat{f}(\xi)e^{\pi\xi^2}(2\pi\xi - 2\pi\xi) = 0.$$

Thus, $G$ is constant. We claim that $\hat{f}(0) = 1$. Then $G(0) = 1$, and $G = 1$. Then, recalling the definition of $G$, we see that $\hat{f}(\xi) = e^{-\pi\xi^2} = f(\xi)$. All that is left to prove is the claim. Consider

$$(\hat{f}(0))^2 = \int_{\mathbb{R}}\int_{\mathbb{R}} e^{-\pi(x^2+y^2)}dxdy.$$

Converting to polar coordinates, we have

$$(\hat{f}(0))^2 = \int_0^{2\pi}\int_0^{\infty} e^{-\pi r^2} r dr d\theta = \int_0^{\infty} 2\pi r e^{-\pi r^2} dr = \left[-e^{-\pi r^2}\right]_0^{\infty} = 1.$$

The integrand is greater than or equal to 0, so $\hat{f}(0) = 1$, and the claim is proven. □

## 4. The Fourier Transform on $\mathbb{R}^d$

Since we wish to solve the wave equation in $\mathbb{R}^d \times \mathbb{R}$, we must consider Fourier transforms on $\mathbb{R}^d$. We will begin with some preliminary definitions.

**Definition 4.1.** A $d$-tuple $\alpha = (\alpha_1, \ldots, \alpha_d)$ is a multi-index if $\alpha_1, \ldots, \alpha_d \in \mathbb{N} \cup \{0\}$.

**Definition 4.2.** For $x \in \mathbb{R}^d$, $x := (x_1, \ldots, x_d)$, and multi-index $\alpha = (\alpha_1, \ldots, \alpha_d)$,

$$x^\alpha = x_1^{\alpha_1} \cdot \ldots \cdot x_d^{\alpha_d},$$

and

$$\left(\frac{\partial}{\partial x}\right)^\alpha = \left(\frac{\partial}{\partial x_1}\right)^{\alpha_1} \cdot \ldots \cdot \left(\frac{\partial}{\partial x_d}\right)^{\alpha_d}.$$

**Definition 4.3.** The Fourier Transform of a function $f : \mathbb{R}^d \to \mathbb{R}$ is the function $\hat{f} : \mathbb{R}^d \to \mathbb{C}$ given by

$$\hat{f}(\xi) := \int_{\mathbb{R}^d} f(x)e^{-2\pi i(x\cdot\xi)}dx, \tag{4.4}$$

for $\xi \in \mathbb{R}^d$.

Once again, we want to consider functions that decay as $|x| \to \infty$. This motivates the following definition:

**Definition 4.5.** The Schwartz space on $\mathbb{R}^d$, denoted $\mathcal{S}(\mathbb{R}^d)$, is the space of all infinitely differentiable functions $f : \mathbb{R}^d \to \mathbb{R}$ that are rapidly decreasing, which means that they satisfy

$$\sup_{x\in\mathbb{R}^d}\left|x^\alpha\left(\frac{\partial}{\partial x}\right)^\beta f(x)\right| < \infty,$$

for all multi-indexes $\alpha, \beta$.

Now, if $f \in \mathcal{S}(\mathbb{R}^d)$, then the integral given in (4.4) converges.
We will now state two properties of the Fourier transform on $\mathbb{R}^d$, which will be needed for the proof of the main theorem of this section.

**Proposition 4.6.** *If $f \in \mathcal{S}(\mathbb{R}^d)$, then*

$$\hat{f}(\xi + x) = \hat{f}(\xi)e^{2\pi i x\cdot\xi}, \tag{4.7}$$

$$\hat{f}(\delta x) = \frac{f(\frac{x}{\delta})}{\delta^d}. \tag{4.8}$$

The proofs follow from the definition of the Fourier transform on $\mathbb{R}^d$.

We now consider the related concept of good kernels, which have useful implications in Fourier analysis.

**Definition 4.9.** A family of functions $K_\delta : \mathbb{R}^d \to \mathbb{R}$ is a family of good kernels if it satisfies the following properties:

(i)
$$\int_{\mathbb{R}^d} K_\delta(x)dx = 1,$$

(ii) $\exists\ M \in \mathbb{R}$ such that
$$\int_{\mathbb{R}^d} |K_\delta(x)|dx \leq M,$$

(iii) $\forall\ \eta \in \mathbb{R}$ with $\eta > 0$,
$$\lim_{\delta\to 0}\int_{|x|\geq\eta} |K_\delta(x)|dx = 0.$$

**Proposition 4.10.** *The family of functions $K_\delta : \mathbb{R}^d \to \mathbb{R}$ given by*

$$K_\delta(x) := \frac{e^{-\pi|x/\sqrt{\delta}|^2}}{\delta^{d/2}}$$

*is a family of good kernels.*

Each condition can be checked, but we omit the proof (see [1]).

**Lemma 4.11.** *If $f \in \mathcal{S}(\mathbb{R}^d)$ and $K_\delta(x)$ is a family of good kernels, then*

$$\lim_{\delta\to 0}\int_{\mathbb{R}^d} K_\delta(x)f(x)dx = f(0).$$

*Proof.* Let $\epsilon > 0$ be given. By property (ii) of good kernels, $\int_{\mathbb{R}^d} |K_\delta(x)|\, dx \leq M$, for some $M \in \mathbb{R}$. Since $f \in \mathcal{S}(\mathbb{R}^d)$, we have that $|f| \leq L$, for some $L \in \mathbb{R}$. Moreover, $f \in \mathcal{S}(\mathbb{R}^d)$, f is continuous, so there exists some $\eta > 0$ such that if $|x| < \eta$, then $|f(x) - f(0)| < \frac{\epsilon}{2M}$. Then, by property (i) of good kernels,

$$\left|\lim_{\delta\to 0}\int_{\mathbb{R}^d} K_\delta(x)f(x)dx - f(0)\right| = \left|\lim_{\delta\to 0}\int_{\mathbb{R}^d} K_\delta(x)f(x)dx - f(0)\int_{\mathbb{R}^d} K_\delta(x)dx\right|$$
$$= \left|\lim_{\delta\to 0}\left[\int_{|x|\geq\eta} K_\delta(x)(f(x) - f(0))dx + \int_{|x|<\eta} K_\delta(x)(f(x) - f(0))dx\right]\right|.$$

By property (iii) of good kernels and the fact that $f \in \mathcal{S}(\mathbb{R}^d)$,

$$\left|\lim_{\delta\to 0}\int_{|x|\geq\eta} K_\delta(x)(f(x) - f(0))dx\right| \leq \lim_{\delta\to 0}\int_{|x|\geq\eta} |K_\delta(x)|\left[|f(x)| + |f(0)|\right]dx$$
$$\leq 2L\lim_{\delta\to 0}\int_{|x|\geq\eta} |K_\delta(x)|dx \leq \frac{\epsilon}{2}.$$

Moreover, we have that

$$\left|\int_{|x|<\eta} K_\delta(x)(f(x)-f(0))dx\right| \leq \int_{|x|<\eta} |K_\delta(x)|\,|f(x)-f(0)|\,dx$$

$$< \frac{\epsilon}{2M}\int_{|x|<\eta} |K_\delta(x)|dx \leq \frac{\epsilon}{2}.$$

Combining these observations, we see that

$$\left|\lim_{\delta\to 0}\int_{\mathbb{R}^d} K_\delta(x)f(x)dx - f(0)\right| < \epsilon,$$

as desired. □

**Lemma 4.12.** *If $f \in \mathcal{S}(\mathbb{R}^d)$ is given by $f(x) = e^{-\pi|x|^2}$, then*

$$\hat{f}(\xi) = f(\xi).$$

*Proof.* First, observe that

$$\hat{f}(\xi) = \int_{\mathbb{R}^d} e^{-\pi|x|^2} e^{-2\pi i x\cdot\xi} dx$$

$$= \int_{\mathbb{R}} e^{-\pi x_d^2} e^{-2\pi i x_d\cdot\xi_d}\left(\ldots\left(\int_{\mathbb{R}} e^{-\pi x_1^2} e^{-2\pi i x_1\cdot\xi_1} dx_1\right)\ldots\right) dx_d.$$

Applying the result from Theorem 3.6 to the previous expression gives us

$$e^{-\pi\xi_d^2}\cdot\ldots\cdot e^{-\pi\xi_1^2} = e^{-\pi(\xi_d^2+\ldots+\xi_1^2)} = e^{-\pi|\xi|^2}.$$

Combining this with the definition of $f$, we have that

$$\hat{f}(\xi) = e^{-\pi|\xi|^2} = f(\xi).$$

□

Now, if we define $G_\delta(x)$ by

$$G_\delta(x) := e^{-\pi|\sqrt{\delta}x|^2},$$

then by (4.8) and the previous lemma,

$$\hat{G_\delta}(\xi) = \frac{e^{-\pi|\xi/\sqrt{\delta}|^2}}{\delta^{d/2}} =: K_\delta(\xi). \tag{4.13}$$

We need only a few more tools in order to prove the main theorem of this section.

**Proposition 4.14.** *(Multiplication Formula)*
*If $f, g \in \mathcal{S}(\mathbb{R}^d)$, then*

$$\int_{\mathbb{R}^d} \hat{f}(x)g(x)dx = \int_{\mathbb{R}^d} f(\xi)\hat{g}(\xi)d\xi.$$

The proof follows from the definitions and Fubini's theorem.

**Corollary 4.15.** *If $f \in \mathcal{S}(\mathbb{R}^d)$, then*

$$f(0) = \int_{\mathbb{R}^d} \hat{f}(\xi)d\xi.$$

*Proof.* Consider

$$\int_{\mathbb{R}^d} \hat{f}(\xi) G_\delta(\xi) d\xi,$$

with $G_\delta$ and $K_\delta$ defined as above. By the multiplication formula and (4.13), we have that

$$\int_{\mathbb{R}^d} \hat{f}(\xi) G_\delta(\xi) d\xi = \int_{\mathbb{R}^d} f(x) \hat{G}_\delta(x) dx = \int_{\mathbb{R}^d} f(x) K_\delta(x) dx.$$

Taking the limit as $\delta \to 0$ and applying the result of Lemma 4.11, we have that

$$\lim_{\delta \to 0} \int_{\mathbb{R}^d} f(x) K_\delta(x) dx = f(0).$$

I claim that

$$\lim_{\delta \to 0} \int_{\mathbb{R}^d} \hat{f}(\xi) G_\delta(\xi) d\xi = \int_{\mathbb{R}^d} \hat{f}(\xi) d\xi.$$

Indeed, this follows from the fact that $\hat{f} \in \mathcal{S}(\mathbb{R}^d)$, which we use without proof. Let $\epsilon > 0$ be given. We have that

$$\left| \lim_{\delta \to 0} \int_{\mathbb{R}^d} \hat{f}(\xi) G_\delta(\xi) d\xi - \int_{\mathbb{R}^d} \hat{f}(\xi) d\xi \right|$$
$$\leq \lim_{\delta \to 0} \int_{|x| \leq \eta} \left| \hat{f}(\xi) \right| \left| e^{-\pi |\sqrt{\delta} \xi|^2} - 1 \right| d\xi + \lim_{\delta \to 0} \int_{|x| > \eta} \left| \hat{f}(\xi) \right| \left| e^{-\pi |\sqrt{\delta} \xi|^2} - 1 \right| d\xi,$$

for any $\eta > 0$. Since $\hat{f} \in \mathcal{S}(\mathbb{R}^d)$, we can choose $\eta$ sufficiently large so that

$$\lim_{\delta \to 0} \int_{|x| > \eta} \left| \hat{f}(\xi) \right| \left| e^{-\pi |\sqrt{\delta} \xi|^2} - 1 \right| d\xi < \lim_{\delta \to 0} \int_{|x| > \eta} \left| \hat{f}(\xi) \right| d\xi < \frac{\epsilon}{2}.$$

Now observe that $|\hat{f}| \leq M$ for some $M \in \mathbb{R}$ since $\hat{f} \in \mathcal{S}(\mathbb{R}^d)$. Thus,

$$\lim_{\delta \to 0} \int_{|x| \leq \eta} \left| \hat{f}(\xi) \right| \left| e^{-\pi |\sqrt{\delta} \xi|^2} - 1 \right| d\xi < \lim_{\delta \to 0} \left| e^{-\pi |\sqrt{\delta} \eta|^2} - 1 \right| \int_{|x| \leq \eta} \left| \hat{f}(\xi) \right|$$
$$\leq \left( \lim_{\delta \to 0} \left| e^{-\pi |\sqrt{\delta} \eta|^2} - 1 \right| \right) (2\eta M) < \frac{\epsilon}{2},$$

and the claim is proven. Thus,

$$f(0) = \int_{\mathbb{R}^d} \hat{f}(\xi) d\xi.$$

□

We are now in a position to prove the Fourier inversion formula.

**Theorem 4.16.** *If $f \in \mathcal{S}(\mathbb{R}^d)$, then*

$$f(x) = \int_{\mathbb{R}^d} \hat{f}(\xi) e^{2\pi i (x \cdot \xi)} dx. \tag{4.17}$$

*Proof.* Define $F(y) := f(x + y)$. We have that $F \in \mathcal{S}(\mathbb{R}^d)$ since $f \in \mathcal{S}(\mathbb{R}^d)$. Therefore, we can apply the result from Corollary 4.15 and see that

$$F(0) = \int_{\mathbb{R}^d} \hat{F}(\xi) d\xi.$$

From the definition of $F$, we have that

$$f(x) = \int_{\mathbb{R}^d} \hat{f}(x + \xi) d\xi.$$

By (4.7), it follows that

$$f(x) = \int_{\mathbb{R}^d} \hat{f}(\xi) e^{2\pi i x\cdot\xi} d\xi.$$

□

## 5. The wave equation in $\mathbb{R}^d \times \mathbb{R}$

We begin this section by introducing the Cauchy problem for the wave equation in $\mathbb{R}^d \times \mathbb{R}$. This problem asks for a function $u(x,t) : \mathbb{R}^d \times \mathbb{R} \to \mathbb{R}$ which satisfies:

$$\begin{cases} \Delta u - \frac{\partial^2}{\partial t^2} = 0 \\ u(x,0) = f(x) \\ \left(\frac{\partial}{\partial t}\right) u(x,0) = g(x), \end{cases}$$

where $f, g \in \mathcal{S}(\mathbb{R}^d)$.

**Theorem 5.1.** *If $u(x,t) \in C^2$ is defined by*

$$u(x,t) = \int_{\mathbb{R}^d} \left[ \hat{f}(\xi)\cos(2\pi|\xi|t) + \hat{g}(\xi)\frac{\sin(2\pi|\xi|t)}{2\pi|\xi|} \right] e^{2\pi i x\cdot\xi} d\xi, \tag{5.2}$$

*then $u$ is a solution.*

To prove this theorem, we use the following proposition:

**Proposition 5.3.** *If $f \in \mathcal{S}(\mathbb{R}^d)$, then*

$$\Delta \int_{\mathbb{R}^d} f(x)dx = \int_{\mathbb{R}^d} \Delta f(x)dx.$$

The proof is similar to the argument used in the proof of Lemma 3.8. Now Theorem 5.1 can be proved by calculations:

*Proof.* To check that $u$ satisfies the wave equation, note that, by Proposition 5.3,

$$\begin{aligned} \Delta u(x,t) &= \Delta \int_{\mathbb{R}^d} \left[ \hat{f}(\xi)\cos(2\pi|\xi|t) + \hat{g}(\xi)\frac{\sin(2\pi|\xi|t)}{2\pi|\xi|} \right] e^{2\pi i x\cdot\xi} d\xi \\ &= \int_{\mathbb{R}^d} \Delta \left[ (\hat{f}(\xi)\cos(2\pi|\xi|t) + \hat{g}(\xi)\frac{\sin(2\pi|\xi|t)}{2\pi|\xi|}) e^{2\pi i x\cdot\xi} \right] d\xi \\ &= \int_{\mathbb{R}^d} \left[ \hat{f}(\xi)\cos(2\pi|\xi|t) + \hat{g}(\xi)\frac{\sin(2\pi|\xi|t)}{2\pi|\xi|} \right] (-4\pi^2|\xi|^2) e^{2\pi i x\cdot\xi} d\xi. \end{aligned}$$

Similarly,

$$\begin{aligned} \frac{\partial^2}{\partial t^2} u(x,t) &= \frac{\partial^2}{\partial t^2} \int_{\mathbb{R}^d} \left[ \hat{f}(\xi)\cos(2\pi|\xi|t) + \hat{g}(\xi)\frac{\sin(2\pi|\xi|t)}{2\pi|\xi|} \right] e^{2\pi i x\cdot\xi} d\xi \\ &= \int_{\mathbb{R}^d} \frac{\partial^2}{\partial t^2} \left[ (\hat{f}(\xi)\cos(2\pi|\xi|t) + \hat{g}(\xi)\frac{\sin(2\pi|\xi|t)}{2\pi|\xi|}) e^{2\pi i x\cdot\xi} \right] d\xi \\ &= \int_{\mathbb{R}^d} \left[ \hat{f}(\xi)\cos(2\pi|\xi|t) + \hat{g}(\xi)\frac{\sin(2\pi|\xi|t)}{2\pi|\xi|} \right] (-4\pi|\xi|^2) e^{2\pi i x\cdot\xi} d\xi. \end{aligned}$$

Thus,

$$\Delta u(x,t) - \frac{\partial^2}{\partial t^2} = 0,$$

so $u$ satisfies the wave equation.

Now we check the second condition. If $t = 0$, then the sine term in the integrand goes away, and we have that

$$u(x,0) = \int_{\mathbb{R}^d} \hat{f}(\xi) e^{2\pi i x \cdot \xi} d\xi.$$

Then, by the Fourier inversion formula,

$$u(x,0) = f(x).$$

We now check the final condition. Note that

$$\begin{aligned}\frac{\partial}{\partial t} u(x,t) &= \frac{\partial}{\partial t} \int_{\mathbb{R}^d} \left[ \hat{f}(\xi) \cos(2\pi|\xi|t) + \hat{g}(\xi) \frac{\sin(2\pi|\xi|t)}{2\pi|\xi|} \right] e^{2\pi i x \cdot \xi} d\xi \\ &= \int_{\mathbb{R}^d} \frac{\partial}{\partial t} \left[ (\hat{f}(\xi) \cos(2\pi|\xi|t) + \hat{g}(\xi) \frac{\sin(2\pi|\xi|t)}{2\pi|\xi|}) e^{2\pi i x \cdot \xi} \right] d\xi \\ &= \int_{\mathbb{R}^d} \left[ \hat{f}(\xi)(-2\pi|\xi|) \sin(2\pi|\xi|t) + \hat{g}(\xi) \cos(2\pi|\xi|t) \right] e^{2\pi i x \cdot \xi} d\xi.\end{aligned}$$

Again, if $t = 0$, then the sine term disappears, so

$$\frac{\partial}{\partial t} u(x,0) = \int_{\mathbb{R}^d} \hat{g}(\xi) e^{2\pi i x \cdot \xi} d\xi.$$

By the Fourier inversion formula,

$$\frac{\partial}{\partial t} u(x,0) = g(x).$$

□

## 6. Uniqueness

We have thus proven the existence of a solution to the Cauchy problem for the wave equation in $\mathbb{R}^d \times \mathbb{R}$. We will now try to prove that this solution is unique. In order to do so, we will introduce the following definitions:

**Definition 6.1.** For $u : \mathbb{R}^d \times \mathbb{R} \to \mathbb{R}$, $u(x,t) \in C^2$, define $\nabla u$ by

$$\nabla u := \left( \frac{\partial u}{\partial x_1}, \ldots, \frac{\partial u}{\partial x_d} \right).$$

**Definition 6.2.** Define the ball at time $0 \le t \le r_0$ located at $x_0$ of radius $r_0$ by

$$B_t(x_0, r_0) := \left\{ x \in \mathbb{R}^d : |x - x_0| \le r_0 - t \right\}.$$

*Remark* 6.3. Note that a ball in $\mathbb{R}^d$ is usually defined to have a fixed radius. In this case, we define the ball to have a radius that varies with time. This distinction will become important when we consider the time derivative of an integral over this ball.

**Definition 6.4.** The energy $E(t)$ of a solution $u$ is given by

$$E(t) := \frac{1}{2} \int_{B_t(x_0, r_0} |\nabla u|^2 + \left| \frac{\partial}{\partial t} u \right|^2 dx.$$

These definitions allow us to introduce the following theorem.

**Theorem 6.5.** *Suppose that the function $u : \mathbb{R}^d \times \mathbb{R} \to \mathbb{R}$, $u \in C^2$, satisfies:*

$$\begin{cases} \Delta u = \frac{\partial^2 u}{\partial t^2} \\ u(x,0) = 0 \\ \frac{\partial}{\partial t} u(x,0) = 0. \end{cases}$$

*Then $u = 0$.*

In order to prove this theorem, we will use the following identity for $f, g \in C^2$:

$$\nabla \cdot (f \nabla g) = (\nabla f) \cdot (\nabla g) + f \Delta g. \tag{6.6}$$

We will also use the Divergence Theorem where $f \in C^2$ is a vector-valued function and where $v$ is the outward-pointing normal vector:

$$\int\int\int_V (\nabla \cdot f)\, dV = \int\int_S (f \cdot v) dS, \tag{6.7}$$

*Proof.* (of Theorem 6.5)
First, note that the initial conditions imply that $E(0) = 0$, and the definition of energy implies that $E(t) \geq 0$ for all $t \geq 0$. Therefore, if we can prove that $\frac{dE}{dt} \leq 0$, then we will have shown that $E(t) = 0$ for all $t \geq 0$, which will imply that $u = 0$. We now compute

$$\frac{dE}{dt} = \frac{1}{2} \frac{d}{dt} \int_{B_t(x_0,r_0)} |\nabla u|^2 + \left| \frac{\partial}{\partial t} u \right|^2 dx$$

$$= \frac{1}{2} \int_{B_t(x_0,r_0)} \frac{d}{dt} \left[ |\nabla u|^2 + \left| \frac{\partial}{\partial t} u \right|^2 \right] dx - \frac{1}{2} \int_{\partial B_t(x_0,r_0)} \left[ |\nabla u|^2 + \left| \frac{\partial}{\partial t} u \right|^2 \right] d\sigma(\gamma),$$

where $d\sigma(\gamma)$ is the infinitesimal surface element of the ball $B_t(x_0, r_0)$. Then

$$\frac{1}{2} \int_{B_t(x_0,r_0)} \frac{d}{dt} \left[ |\nabla u|^2 + \left| \frac{\partial}{\partial t} u \right|^2 \right] dx$$

$$= \frac{1}{2} \int_{B_t(x_0,r_0)} \left[ 2 (\nabla u) \cdot \left( \nabla \frac{\partial u}{\partial t} \right) + 2 \frac{\partial u}{\partial t} \frac{\partial^2 u}{\partial t^2} \right] dx$$

$$= \int_{B_t(x_0,r_0)} (\nabla u) \cdot \left( \nabla \frac{\partial u}{\partial t} \right) dx + \int_{B_t(x_0,r_0)} \frac{\partial u}{\partial t} \frac{\partial^2 u}{\partial t^2} dx.$$

By (6.6), we have

$$\frac{dE}{dt} = \int_{B_t(x_0,r_0)} \nabla \cdot \left( u \nabla \frac{\partial u}{\partial t} \right) dx - \int_{B_t(x_0,r_0)} \Delta u \frac{\partial u}{\partial t} dx$$

$$+ \int_{B_t(x_0,r_0)} \frac{\partial^2 u}{\partial t^2} \frac{\partial u}{\partial t} dx - \frac{1}{2} \int_{\partial B_t(x_0,r_0)} \left[ \left| \nabla u^2 \right| + \left| \frac{\partial}{\partial t} u^2 \right| \right] d\sigma(\gamma).$$

Since $u$ satisfies the wave equation, we have

$$\frac{dE}{dt} = \int_{B_t(x_0,r_0)} \nabla \cdot \left( u \nabla \frac{\partial u}{\partial t} \right) dx - \int_{B_t(x_0,r_0)} \frac{\partial^2 u}{\partial t^2} \frac{\partial u}{\partial t} dx$$

$$+ \int_{B_t(x_0,r_0)} \frac{\partial^2 u}{\partial t^2} \frac{\partial u}{\partial t} dx - \frac{1}{2} \int_{\partial B_t(x_0,r_0)} \left[ \nabla u^2 + \frac{\partial}{\partial t} u^2 \right] d\sigma(\gamma)$$

$$= \int_{B_t(x_0,r_0)} \nabla \cdot \left( u \nabla \frac{\partial u}{\partial t} \right) dx - \frac{1}{2} \int_{\partial B_t(x_0,r_0)} \left[ \left| \nabla u^2 \right| + \left| \frac{\partial}{\partial t} u^2 \right| \right] d\sigma(\gamma).$$

Applying (6.7), we have that

$$\int_{B_t(x_0,r_0)} \nabla \cdot \left( u \nabla \frac{\partial u}{\partial t} \right) dx = \int_{\partial B_t(x_0,r_0)} u \nabla \frac{\partial u}{\partial t} \cdot v d\sigma(\gamma),$$

where $v$ denotes the outward pointing normal vector with respect to the surface $\partial B_t(x_0, r_0)$.

Thus, we have

$$\frac{dE}{dt} = \int_{\partial B_t(x_0,r_0)} u \nabla \frac{\partial u}{\partial t} \cdot v d\sigma(\gamma) - \frac{1}{2} \int_{\partial B_t(x_0,r_0)} \left[ |\nabla u|^2 + \left| \frac{\partial}{\partial t} u \right|^2 \right] d\sigma(\gamma).$$

Therefore, in order to prove that $\frac{dE}{dt} \leq 0$, it suffices to show

$$\nabla u \frac{\partial u}{\partial t} \cdot v \leq \frac{1}{2} \left[ |\nabla u|^2 + \left| \frac{\partial}{\partial t} u \right|^2 \right].$$

Observe that

$$0 \leq \frac{1}{2} \left| \nabla u - \frac{\partial u}{\partial t} v \right|^2 = \frac{1}{2} \left[ |\nabla u|^2 + \left| \frac{\partial u}{\partial t} (v \cdot v) \right|^2 \right] - \left| \nabla u \frac{\partial u}{\partial t} \cdot v \right|.$$

Thus,

$$\nabla u \frac{\partial u}{\partial t} \cdot v \leq \left| \nabla u \frac{\partial u}{\partial t} \cdot v \right| \leq \frac{1}{2} \left[ |\nabla u|^2 + \left| \frac{\partial u}{\partial t} (v \cdot v) \right|^2 \right].$$

Therefore, $\frac{dE}{dt} \leq 0$, and $E(0) = 0$, and $E(t) \geq 0$ for all $t \geq 0$. Hence, $E = 0$, and it follows that $u = 0$. □

We can now use this result to prove the uniqueness of the solution to the Cauchy problem for the wave equation.

**Theorem 6.8.** *If $u, v$ satisfy:*

$$\begin{cases} \Delta^2 u - \frac{\partial^2 u}{\partial t^2} = 0 = \Delta^2 v - \frac{\partial^2 v}{\partial t^2} \\ u(x,0) = f(x) = v(x,0) \\ \frac{\partial u}{\partial t} u(x,0) = g(x) = \frac{\partial u}{\partial t} v(x,0), \end{cases}$$

*then $u = v$.*

*Proof.* We apply the result of Theorem 6.5 to the function $w := u - v$. Then $w = 0$, so $u = v$. □

## 7. Closed Form Solution for $\mathbb{R}^3$

We will now consider the Cauchy problem for the wave equation in $\mathbb{R}^3$ and attempt to derive a closed form solution analagous to d'Alembert's Formula for the 1-dimensional case. We begin with the following definition:

**Definition 7.1.** The spherical mean of a function $f : \mathbb{R}^3 \to \mathbb{R}$ over a sphere of radius $t$ centered at $x$ is given by

$$M_t(f)(x) := \frac{1}{4\pi} \int_{\mathrm{S}^2} f(x - t\gamma) d\sigma(\gamma),$$

where $d\sigma(\gamma)$ is the unit surface element.

We now state the following lemma without proof (See [1]).

**Lemma 7.2.** $\hat{f}(\xi)\frac{\sin(2\pi|\xi|t)}{2\pi|\xi|t} = \widehat{M_t(f)}(\xi)$.

We can now prove the following corollary:

**Corollary 7.3.** $M_t(f)(x) = \int_{\mathbb{R}^3} \hat{f}(\xi)\frac{\sin(2\pi|\xi|t)}{2\pi|\xi|t}e^{2\pi i x\cdot\xi}d\xi$.

*Proof.* By Lemma 7.2, we have that

$$\int_{\mathbb{R}^3} \hat{f}(\xi)\frac{\sin(2\pi|\xi|t)}{2\pi|\xi|t}e^{2\pi i x\cdot\xi}d\xi = \int_{\mathbb{R}^3} \widehat{M_t(f)}(x)e^{2\pi i x\cdot\xi}d\xi.$$

Then, by the Fourier Inversion Formula, we have

$$\int_{\mathbb{R}^3} \widehat{M_t(f)}(x)e^{2\pi i x\cdot\xi}d\xi = M_t(f)(x).$$

□

We are now in a position to prove the main theorem of this section:

**Theorem 7.4.** *If* $u \in C^2$, $u : \mathbb{R}^3 \times \mathbb{R} \to \mathbb{R}$ *satisfies*

$$\begin{cases} \Delta u - \frac{\partial^2 u}{\partial t^2} = 0 \\ u(x,0) = f(x) \\ \frac{\partial}{\partial t}u(x,0) = g(x), \end{cases}$$

*then*

$$u(x,t) = \frac{\partial}{\partial t}\left[tM_t(f)(x)\right] + tM_t(g)(x).$$

*Proof.* We begin by applying the result of Theorem 5.1. We see that

$$u(x,t) = \int_{\mathbb{R}^3} \left[\hat{f}(\xi)\cos(2\pi|\xi|t) + \hat{g}(\xi)\frac{\sin(2\pi|\xi|t)}{2\pi|\xi|}\right]e^{2\pi i x\cdot\xi}d\xi$$

$$= \int_{\mathbb{R}^3} \hat{f}(\xi)\cos(2\pi|\xi|t)e^{2\pi i x\cdot\xi}d\xi + \int_{\mathbb{R}^3} \hat{g}(\xi)\frac{\sin(2\pi|\xi|t)}{2\pi|\xi|}e^{2\pi i x\cdot\xi}d\xi$$

$$= \int_{\mathbb{R}^3} \hat{f}(\xi)\frac{\partial}{\partial t}\left[\frac{\sin(2\pi|\xi|t)}{2\pi|\xi|}\right]e^{2\pi i x\cdot\xi}d\xi + \int_{\mathbb{R}^3} \hat{g}(\xi)\frac{\sin(2\pi|\xi|t)}{2\pi|\xi|}e^{2\pi i x\cdot\xi}d\xi$$

$$= \frac{\partial}{\partial t}\int_{\mathbb{R}^3} \hat{f}(\xi)\frac{\sin(2\pi|\xi|t)}{2\pi|\xi|}e^{2\pi i x\cdot\xi}d\xi + \int_{\mathbb{R}^3} \hat{g}(\xi)\frac{\sin(2\pi|\xi|t)}{2\pi|\xi|}e^{2\pi i x\cdot\xi}d\xi$$

$$= \frac{\partial}{\partial t}t\int_{\mathbb{R}^3} \hat{f}(\xi)\frac{\sin(2\pi|\xi|t)}{2\pi|\xi|t}e^{2\pi i x\cdot\xi}d\xi + t\int_{\mathbb{R}^3} \hat{g}(\xi)\frac{\sin(2\pi|\xi|t)}{2\pi|\xi|t}e^{2\pi i x\cdot\xi}d\xi.$$

Applying the result of Corollary 7.3 to both terms, we see that

$$u(x,t) = \frac{\partial}{\partial t}\left[tM_t(f)(x)\right] + tM_t(g)(x),$$

as desired. □

## 8. Closed Form Solution for $\mathbb{R}^2$

Solving the wave equation in 2 dimensions is not as straightforward as the 3-dimensional case. In this section, we will see that the closed form solution for the wave equation in 2 dimensions can be derived from the solution in three dimensions. This method of finding a solution in a higher dimension and using it to derive a solution in a lower dimension is called the method of descent. We begin this section with the following definition:

**Definition 8.1.** The spherical mean of a function $f$ over a disc of radius $t$ centered at $x$ is given by

$$N_t(f)(x) := \frac{1}{2\pi}\int_{|y|\leq 1}\frac{f(x-ty)}{\sqrt{1-|y|^2}}dy.$$

The following proposition motivates our study of the method of descent.

**Proposition 8.2.** *Given $\beta > 0$, define $\eta : \mathbb{R} \to \mathbb{R}$ such that*

$$\eta(s) := \begin{cases} 1, & s < \beta \in \mathbb{R} \\ 0, & \text{otherwise.} \end{cases}$$

*Given $f_0 \in \mathcal{S}(\mathbb{R}^2)$, define $f(x_1, x_2, x_3) := f_0(x_1, x_2)\eta(x_3) = f_0(x_1, x_2)$ for all $x_3 < \beta$, so that $f \in \mathcal{S}(\mathbb{R}^3)$. Then*

$$M_t(f)(x) = N_t(f_0)(x).$$

*Remark* 8.3. Here, $\beta$ is arbitrary, so we can choose $\beta >> 1$.

*Proof.* Beginning with the definitions and working in spherical coordinates (for $x_3 < \beta$),

$$\begin{aligned} M_t(f)(x) &= \frac{1}{4\pi}\int_0^{2\pi}\int_0^{\pi} f(x_1 - t\sin(\theta)\cos(\phi), x_2 - t\sin(\theta)\sin(\phi), x_3 - t\cos(\theta))\sin(\theta)d\theta d\phi \\ &= \frac{1}{4\pi}\int_0^{2\pi}\int_0^{\pi} f_0(x_1 - t\sin(\theta)\cos(\phi), x_2 - t\sin(\theta)\sin(\phi))\sin(\theta)d\theta d\phi \\ &= \frac{1}{2\pi}\int_0^{2\pi}\int_0^{\frac{\pi}{2}} f_0(x_1 - t\sin(\theta)\cos(\phi), x_2 - t\sin(\theta)\sin(\phi))\sin(\theta)d\theta d\phi. \end{aligned}$$

Making the substitution $r = \sin(\theta)$, the previous line becomes

$$\frac{1}{2\pi}\int_0^{2\pi}\int_0^{1} f_0(x_1 - rt\cos(\phi), x_2 - rt\sin(\phi))\frac{r}{\sqrt{1-r^2}}drd\phi.$$

Converting to polar coordinates, we have

$$M_t(f)(x) = \frac{1}{2\pi}\int_{|y|\leq 1}\frac{f_0(x-ty)}{\sqrt{1-|y|^2}}dy = N_t(f_0)(x).$$

□

Proposition 8.2 establishes the connection between the spherical means in 2 and 3 dimensions, showing why we can use the method of descent. The next theorem puts this method into practice.

**Theorem 8.4.** *Suppose $f, g \in \mathcal{S}(\mathbb{R}^2)$, and suppose $u \in C^2$, $u : \mathbb{R}^2 \times \mathbb{R} \to \mathbb{R}$ satisfies*

$$\begin{cases} \Delta u(x,t) - \frac{\partial^2}{\partial t^2} u(x,t) = 0 \\ u(x,0) = f(x) \\ \frac{\partial}{\partial t} u(x,0) = g(x). \end{cases}$$

*Then a closed form solution for $u$ is given by*

$$u(x,t) = \frac{\partial}{\partial t}\left[tN_t(f)(x)\right] + tN_t(g)(x).$$

*Proof.* Let $\beta >> 1$ be given. As above, we define $\eta : \mathbb{R} \to \mathbb{R}$ by

$$\eta(s) := \begin{cases} 1, \ s < \beta \in \mathbb{R} \\ 0, \ \text{otherwise.} \end{cases}$$

Define $f^*, g^* : \mathbb{R}^3 \to \mathbb{R}$ by

$$f^*(x_1, x_2, x_3) := f(x_1, x_2)\eta(x_3) = f(x_1, x_2), \ \forall \ x_3 < \beta,$$

and

$$g^*(x_1, x_2, x_3) := g(x_1, x_2)\eta(x_3) = g(x_1, x_2), \ \forall \ x_3 < \beta.$$

Then $f^*, g^* \in \mathcal{S}(\mathbb{R}^3)$, and it suffices to consider $f^*(x_1, x_2, 0)$ and $g^*(x_1, x_2, 0)$. Define $u^*(x_1, x_2, x_3, t)$ by

$$u^*(x_1, x_2, x_3, t) := u(x_1, x_2, t)\eta(x_3).$$

Then we have that

$$\begin{cases} u^*(x_1, x_2, 0, t) := u(x_1, x_2, t)\eta(x_3) \\ u^*(x_1, x_2, 0, 0) = f^*(x_1, x_2, 0) \\ \frac{\partial}{\partial t} u^*(x_1, x_2, 0, 0) = g^*(x_1, x_2, 0). \end{cases}$$

Then $u^*$ satisfies the wave equation since $u$ does. Thus, we can apply the result of Theorem 7.4 to see

$$u^*(x_1, x_2, 0, t) = \frac{\partial}{\partial t}\left[tM_t(f^*)(x_1, x_2, 0)\right] + tM_t(g^*)(x_1, x_2, 0).$$

Now, by Proposition 8.2, we have

$$u(x,t) = \frac{\partial}{\partial t}\left[tN_t(f)(x_1, x_2)\right] + tN_t(g)(x_1, x_2),$$

as desired. □

The notable fact that the closed form solution in 2 dimensions is derived from the closed form solution in 3 dimensions leads us to conclude with the following remark.

*Remark* 8.5. Let $B(x,r) := \{x \in \mathbb{R}^d : |x| < r\}$ be the closed ball defined in a plane at $t = 0$ with radius $r$ and centered at $x$. Consider the backward light cone originating from $x$, given by

$$\mathcal{L}_{B(x,r)} := \{(x,t) \in \mathbb{R}^d \times \mathbb{R} : |x - x_0| < r - t, t \in [0,r]\}.$$

For odd $d \geq 3$, $u(x,t)$ depends only on the values of $f$ and $g$ at the boundary of the base of the backward light cone, a phenomenon known as Huygen's principle. On the other hand, for even $d \geq 2$, $u(x,t)$ depends on the values of $f$ and $g$ over the entire base of the cone. This qualitative difference between the solutions is easily

observed in the cases of the closed form solutions to the Cauchy problem for the wave equation in 2 and 3 dimensions derived above.

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## 10. bibliography

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