HOMOTOPY THEORY FROM SUBDIVISION AND A-SPACE MODELS

ELEANOR MCSPIRIT

Abstract. This paper takes an essentially combinatorial approach to homotopy theory by using the tools available in working with Alexandroff T_0 spaces to model homotopy invariant information of topological spaces and continuous maps. Not only do we conclude with a natural bijection between the colimit of a system built from the subdivision of posets, but we produce a method to model co-H-spaces using subdivision that allows for the construction of a group isomorphism in particular cases.

Contents

1. Introduction

A-spaces have proven themselves to be unexpectedly interesting spaces that possess a wealth of topological information. While they are fruitful objects of study in their own right, the relationship between the categories of A-spaces and topological spaces given by weak homotopy equivalence has allowed for much of the structure and several of the problems of algebraic topology to be effectively transferred to the combinatorial domain of posets. For example, weak homotopy equivalences between A-spaces induce homotopy equivalences between their respective geometric realizations. Thus, it is a natural to ask if we can encode the topological information of a space that is invariant up to homotopy by turning our focus to weakly equivalent A-spaces.

The respective works of Clader and Thibault allow us to consider arbitrary topological spaces as represented by a system of subdivided models. Formally, our first

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result says that given a locally finite A-space X, its realization $|\mathscr{K}(X)|$ is homotopy equivalent to the inverse limit of the directed system:

$$\cdots \xrightarrow{\inf} (\mathrm{Sd}^2 X)^{op} \xrightarrow{\inf} (\mathrm{Sd} X)^{op} \xrightarrow{\inf} X^{op}\ .$$

After establishing this result, we turn our attention away from topological spaces and instead focus on modeling continuous functions. While we can associate to every CW complex a partially ordered set, the category of posets suffers from a "deficiency of morphisms" that limits our ability to model functions in the category of topological spaces. To remedy this issue, we follow the work of Hardie and Vermeulen as well as Thibault to formulate an elegant relationship between A-spaces and their geometric realizations using the subdivision functor.

The main result establishes that, given a finite A-space X and an arbitrary A-space Y, there is a natural bijection between the colimit of the system:

$$[X, Y] \xrightarrow{\sup^*} [\mathrm{Sd} X, Y] \xrightarrow{\sup^*} [\mathrm{Sd}^2 X, Y] \xrightarrow{\sup^*} \cdots$$

and $[|\mathscr{K}(X)|, |\mathscr{K}(Y)|]$. In plain terms, this result allows us to model continuous maps between topological spaces with order-preserving maps between posets, up to homotopy. These two results in conjunction effectively justify that the category of A-spaces is an appropriate place in which to study algebraic topology.

To add algebraic importance to these results, we develop the final piece of this paper. In general, an isomorphism of sets is the strongest relationship we can hope for between this colimit and the homotopy classes of maps between the realizations of two A-spaces, since for arbitrary topological spaces X and Y, $[X, Y]$ need not be a group. However, if we restrict our attention to basepoint-preserving homotopy classes of the form $\langle \Sigma X, Y \rangle$, where ΣX is the reduced suspension of a space X, we have an important group structure. Here, we recover a *group isomorphism* between

$$\operatorname*{colim}_n \langle \mathrm{Sd}^n(\mathbb{S}^{op} X), Y \rangle \quad \text{and} \quad \langle \Sigma |\mathscr{K}(X)|, |\mathscr{K}(Y)| \rangle$$

by demonstrating an A-space model of the co-H-space structure on ΣX. We conclude by focusing on spheres as a particular class of suspensions that allow our system to model the group structure of homotopy groups, as well as return some results regarding the contractibility of particular spaces.

We will use $\mathbb{S}^n$ to denote the minimal finite model of the n-sphere, consisting of $2n+2$ points, which we construct inductively by taking the non-Hausdorff suspension of $\mathbb{S}^{n-1}$.

When describing n-simplices of $\mathscr{K}(X)$, we will write $\{x_0, x_1, ..., x_n\}$, where each x_i and element of the underlying poset X and we assume $x_0 < x_1 < ... < x_n$. Likewise, when referring to the subdivision of a partially ordered set, we will abuse the language of n-simplices to refer to the elements of the subdivision that correspond to totally ordered subsets with $n+1$ elements.

2. A-spaces and Ordered Simplicial Complexes

Definition 2.1. An *A-space* is a space coupled with an Alexandroff T_0 topology. As with an Alexandroff topology, the arbitrary intersection of open sets is decidedly open, and given two points x and y, there exists an open set U such that x is in U but y is not in U.

Given the natural equivalence of the category of A-spaces and that of posets constructed by regarding $x \leq y$ if and only if every open set containing y also contains x, this paper will often switch between the language of the two as is useful for the problem at hand. We begin by first developing a toolbox of functors and other results that link A-spaces and ordered simplicial complexes. As we will see, there is an intuitive way to associate an A-space to an ordered simplicial complex, and vice versa. These functors generate a convenient notion of subdivision in both categories that are intimately related and will be the foundational notion that will allow for the constructions in this paper.

Definition 2.2. The *order complex functor* $\mathscr{K} : \mathscr{P}os \to \mathscr{OSC}$ maps a poset X to the ordered simplicial complex whose poset of vertices $V(\mathscr{K}(X))$ is X itself and whose simplices are determined by the totally ordered finite subsets of X. Given a map of posets $f : X \to Y$, $\mathscr{K}(f)$ maps vertices to vertices, and for $\sigma = \{x_0, x_1, ..., x_n\} \in \mathscr{K}(X)$, we define $\mathscr{K}(f)(\sigma) := \{f(x_0), f(x_1), ..., f(x_n)\}$.

Definition 2.3. The *face poset functor* $\mathscr{X} : \mathscr{SC} \to \mathscr{P}os$ maps a simplicial complex K to the poset $\mathscr{X}(K)$ whose elements are the simplices of K, where if σ and σ' are simplices of K and σ is a face of σ', then $\mathscr{X}(\sigma) \leq \mathscr{X}(\sigma')$.

Note that the face poset functor may act on ordered simplicial complexes if we consider precomposition with the forgetful inclusion of the category of ordered simplicial complexes into the category of simplicial complexes $\iota : \mathscr{OSC} \to \mathscr{SC}$; we will abuse notation and still write $\mathscr{X}(K)$ to mean $\mathscr{X}(\iota(K))$, keeping in mind that $\mathscr{X}$ does not see the ordering of an ordered simplicial complex K.

One should verify that composing $\mathscr{K}$ and $\mathscr{X}$ in either order does *not* necessarily return the original A-space or simplicial complex; instead, the elements of the resultant space are chains of elements from the starting space in both cases. The next two definitions formalize this notion.

Definition 2.4. If K is an (ordered) simplicial complex, then the *barycentric subdivision* of K is $\mathscr{K}\mathscr{X}(K)$, which we denote $\mathrm{Sd}K$.

In a parallel fashion, for a an A-space X, we define its subdivision $\mathrm{Sd}X$ to be the composition $\mathscr{X}\mathscr{K}(X)$.

Then by definition, $\mathscr{K}(\mathrm{Sd}^n X) = \mathrm{Sd}^n(\mathscr{K}(X))$, a relationship that will allow us to fluidly move between discussing the subdivisions of A-spaces and those of ordered simplicial complexes, which will prove useful for defining and utilizing further notions in both areas. As one should expect, these definitions coincide both with the classical barycentric subdivision of a simplicial complex and the subdivision of a poset when regarded as a category.

In order to establish a relationship between A-spaces and more general topological spaces, we remind the reader of the geometric realization functor $|\cdot| : s\mathscr{S}et \to \mathscr{T}op$ which we extend to ordered simplicial complexes by their inclusion as simplicial sets satisfying Properties A, B, and C [7].

Now that we have constructed the composite functor $|\mathscr{K}(\cdot)| : \mathscr{P}os \to \mathscr{T}op$ (which we will often call the realization of an A-space for simplicity), we have a way to assign a topological space to an arbitrary A-space. However, the motivation for our results will come from understanding relationships in the opposite direction; topological spaces are often the objects we would like to study by looking at corresponding A-spaces. For this reason, we introduce the following:

Definition 2.5. A continuous map $f: X \to Y$ is a *weak homotopy equivalence* if the induced homomorphisms

$$f_* : \pi_n(X) \to \pi_n(Y)$$

are bijections for all $n \geq 0$. In this case, we say X and Y are *weakly equivalent*.

Note that weak equivalence of topological spaces is *not* transitive, so we must refrain from a notion of weak equivalence classes unless we undergo a process to formally invert these induced maps, which will not be necessary for our purposes.

Definition 2.6. Given a topological space X, an A-space Y is said to be an *A-space model of* X if there exists a chain of weak equivalences from X to Y. That is, there is a finite collection consisting of both topological spaces and A-spaces X_i along with weak equivalences $f_i : X_i \to X_{i+1}$ that give the chain

$$X = X_0 \xrightarrow{f_0} X_1 \xrightarrow{f_1} \cdots \xrightarrow{f_{n-2}} X_{n-1} \xrightarrow{f_{n-1}} X_n = Y$$

which in turn induces a bijection on all homotopy groups of X and Y.

Proposition 2.7. *If* Y *is an A-space model of a CW-complex* X*, then* $|\mathscr{K}(Y)| \simeq X$.

This is a direct application of Whitehead's Theorem. In light of this result, we see that once we have found a model or, looking ahead, a collection of related models for a space, we may study them instead of our space directly, as they encode the same homotopy-invariant information of the original space after realization. Particularly, since every CW complex is homotopy equivalent to the geometric realization of some poset, these relationships are sufficient to study the homotopical properties of CW complexes. While many of the general theoretic results of this paper exist independent of these facts, the results presented in this paper would be of little use for studying recognizable topological spaces.

3. Modeling Topological Spaces

While the functors $\mathscr{K}$ and $\mathscr{X}$ give us a way to build subdivisions of posets and simplicial complexes, it will also be useful for us to define a way to go from the subdivision of an A-space back to the underlying space. With this in mind, we define the following map.

Definition 3.1. Given an A-space X, let $\sup : \mathrm{Sd}X \to X$ be the map such that if $\sigma = \{x_0, x_1, ..., x_k\}$ is an element of $\mathrm{Sd}X$, then $\sup(\sigma) = x_k$.

Proposition 3.2. *The map* sup *is continuous.*

Proof. Suppose $\sigma \leq \sigma'$. Since the ordering on $\mathrm{Sd}X$ is induced by simplicial inclusion in $\mathscr{K}(X)$, we have $\sigma \subseteq \sigma'$ when regarded as simplices of $\mathscr{K}(X)$. Inclusion requires $x_i \in \sigma \implies x_i \in \sigma'$, so in particular $x_k \leq x'_{k'}$, and thus sup is continuous. $\square$

Proposition 3.3. *The map* sup *is is a weak homotopy equivalence.*

It is a straightforward check to see that $|\mathscr{K}(f)| : |\mathscr{K}(\mathrm{Sd}X)| \to |\mathscr{K}(X)|$ is a homotopy equivalence. This implies f is a weak homotopy equivalence of A-spaces.

Definition 3.4. Let X be an A-space. Define $\inf : (\mathrm{Sd}X)^{op} \to X^{op}$ to be the map that coincides pointwise with sup on $\mathrm{Sd}X$ to X.

One should note that sup returns the largest element of a given chain, and inf, the smallest. We will use this descriptive language in proofs, as it illuminates the continuity of some maps and the commutativity of some diagrams that we will introduce in the coming pages.

To establish a concrete relationship between an A-space and its realization that will allow us to regard X as an A-space model of $|\mathscr{K}(X)|$, we present the following theorem.

Definition 3.5. For an A-space X, let u be a point in $|\mathscr{K}(X)|$. Then u is in the interior of a unique simplex $\{x_0, x_1, ..., x_k\}$ of $\mathscr{K}(X)$. Define the map $p : |\mathscr{K}(X)| \to X$ by $p(u) = x_0$.

Theorem 3.6. [8] *For an A-space X, $p : |\mathscr{K}(X)| \to X$ is a natural weak homotopy equivalence.*

We can in fact extend this result to see that there is a weak homotopy equivalence $p_n : |\mathscr{K}(X)| \to (\mathrm{Sd}^n X)^{op}$. While it may seem odd for us to pick the opposite topology, one should note that each p_n as defined above returns the maximal element of the unique simplex in $\mathrm{Sd}^n X$ containing a point u. When composed with the inf map as above, we generate the following commutative diagram. [1]

$$\begin{tikzcd} & {|\mathscr{K}(X)|} \arrow[dl, "p_0"'] \arrow[d, "p_1"] \arrow[dr, "p_2"] & & \\ X^{op} & (\mathrm{Sd}X)^{op} \arrow[l, "\mathrm{inf}"] & (\mathrm{Sd}^2X)^{op} \arrow[l, "\mathrm{inf}"] & \cdots \arrow[l, "\mathrm{inf}"] \end{tikzcd}$$

Given this diagram, we take the inverse limit of the system of the bottom row, given by

$$\tilde{X} = \prod_n (\mathrm{Sd}^n X)^{op}/(\sim)$$

where equivalence is generated by the inf map. We would like to imagine that the increasingly fine subdivisions of a simplex "converge" to a point in $|\mathscr{K}(X)|$. To formalize this intuition, we offer the following setup:

We define the map $\tilde{p} : |\mathscr{K}(X)| \to \tilde{X}$ by $\tilde{p}(a) = (p_0(a), p_1(a), ...)$, which associates to each point a in the realization of X the corresponding sequence of images of a under each p_n. One may note that this is a sequence of nested simplices all containing a. Since the maps p_i give a cone to our inverse system, so continuity is clear. In tandem, we offer the following map that will act as an inverse up to homotopy:

Definition 3.7. Let $x = (x_0, x_1, ...) \in \tilde{X}$, with each $x_i \in (\mathrm{Sd}^i X)^{op}$. Pick some $a_i \in \tilde{p_i}^{-1}(x_i)$ for each $x_i \in x$. Then $\{a_n\}$ converges to a point $a \in |\mathscr{K}(X)|$. Let $G : \tilde{X} \to |\mathscr{K}(X)|$ denote this map.

The restriction in the following theorem that X be a *locally finite* A-space, i.e. an A-space where each element has finite closure and a finite neighborhood, is required

[1] Note the importance of the opposite topology and the substitution of the inf map to make this a commutative diagram of continuous maps. For a more explicit construction of each p_n, which Thibault denotes $\tilde{p}_n$, his thesis gives full details that make the checks on this diagram and the inverse limit much more concrete. We elect to omit these here, and refer the skeptical and intrigued reader to [9].

to ensure G is continuous. These complete checks of continuity and well-definedness can be found in [9], which additionally correct a detail on the inherited topology of $\tilde{X}$ in Clader's original proof.

Theorem 3.8. *Let X be a locally finite A-space. Then $\tilde{X}$ and $|\mathscr{K}(X)|$ are homotopy equivalent and $|\mathscr{K}(X)|$ is a deformation retract of $\tilde{X}$.*

Proof. First, observe that $G \circ \tilde{p}$ is in fact the identity map on $|\mathscr{K}(X)|$, so we need only show $\tilde{p} \circ G \simeq id_{\tilde{X}}$. Suppose $x, x' \in \tilde{X}$ and say $x \sim x'$ if $G(x) = G(x')$. If E denotes a subset of $\tilde{X}$ corresponding to one equivalence class partitioned by $(\sim)$, define a homotopy $h_E : E \times [0,1] \to E$ by

$$h_E(x,t) = \begin{cases} x & t < 1 \\ \tilde{p}(G(x)) & t = 1 \end{cases}.$$

Thibault proves that this homotopy is continuous in [9], so we have an explicit deformation retract of each equivalence class under $(\sim)$. These maps allow us to define globally $H : \tilde{X} \times [0,1] \to \tilde{X}$.

To show H itself is continuous, consider $U \subset \tilde{X}$ open. Suppose $x \in E$, and let $y = G(x) \in |\mathscr{K}(X)|$. Because of our partition, y is the same for all $x \in E$. Observe that if $\tilde{p}(y) \notin (U \cap E)$, then $h_E^{-1}(U \cap E) = (U \cap E) \times [0,1)$. Likewise, if $(U \cap E)$ does contain $\tilde{p}(y)$, then $h_E^{-1}(U \cap E) = (U \cap E) \times [0,1]$.

Define $U' = \{x \in U \mid \tilde{p}(G(x)) \notin U\}$. Then $H^{-1}(U) = (U \times [0,1]) \setminus (U' \times \{1\})$. Since $\tilde{p}$ and G are continuous, U' is closed in U. So H is continuous on all of $\tilde{X}$, and gives a homotopy from the identity to $\tilde{p} \circ G(\tilde{X})$. Therefore, $\tilde{X}$ deformation retracts onto $\tilde{p} \circ G(\tilde{X})$, and since G is surjective, $\tilde{p} \circ G(\tilde{X}) = \tilde{p}(|\mathscr{K}(X)|)$. □

It should be surprising that the inverse limit and geometric realization act equivalently up to homotopy on a locally finite A-space X. This should not be a readily accepted result, as these functors typically do not behave nicely with one another. After all, the geometric realization functor is left adjoint, which in general does not preserve limits. While the intuition laid out in this proof makes the conclusion feel motivated, one should keep in mind the interesting mathematical relationships at play.

4. Motivation and Colimit Setup

Now, we turn our attention to another system that, while similar in style to the former, marks a departure from modeling topological spaces to instead modeling continuous functions. The benefit of a model built similarly to our inverse limit system is that with increasingly finer subdivisions of X, we are able to "fill in the gaps" between morphisms, so to speak.

Given the way we will construct our directed system for a finite A-space X and an arbitrary A-space Y, its colimit is given by

$$\coprod_n [\mathrm{Sd}^n X, Y]/(\sim)$$

subject to the equivalence relation on homotopy classes generated by precomposition with the sup map. That is, elements, which for us are equivalence classes of maps, $[f] \in [\mathrm{Sd}^i X, Y]$ and $[g] \in [\mathrm{Sd}^j X, Y]$ are identified in the colimit if there exists an $N \geq 0$ such that $[f \circ \sup^{N-i}] = [f \circ \sup^{N-j}] \in [\mathrm{Sd}^N X, Y]$. This relationship based on the sup map simply consolidates extraneous information; the crux of the result will come from the Simplicial Approximation Theorem.

To prove that this equivalence relation coincides precisely with those maps whose realizations are homotopic, we introduce a basic vocabulary to understand homotopy classes in the setting of A-spaces, and turn to the simplicial notion of contiguity that provides us with precisely the tools necessary to prove that the above colimit is a useful model.

5. Function Spaces and Homotopies

For arbitrary A-spaces X and Y, $Y^X := \mathrm{Hom}(X, Y)$ need not also be an A-space. However, under the restriction that X is a finite A-space, then Y^X is an A-space whose order is determined by that of Y: for maps $f, g : X \to Y$, we say $f \leq g$ in Y^X if $f(x) \leq g(x)$ for all $x \in X$.

This ordering on Y^X gives us a natural way to extend the notion of (path) connectivity of A-spaces to homotopy equivalence of maps between A-spaces.

Proposition 5.1. *Let $f \leq g$, where f and g are maps from a finite A-space X to an A-space Y. Then $f \simeq g$.*

Proof. f and g are connected by the path $H : I \to Y^X$ such that

$$H(t) = \begin{cases} f & t < 1 \\ g & t = 1 \end{cases}$$

We may reformulate this result to resemble the standard formulation of a homotopy by noting that the existence and continuity of H implies that of $H^* : X \times I \to Y$ where $H(x, 0) = f(x)$ and $H(x, 1) = g(x)$. Thus, f and g are homotopic. □

Conversely, we may say that if $f \simeq g$, there exists a finite chain of comparable maps (i.e. maps $\{f_i\}$ for $1 \leq i \leq n$ such that either $f_i \leq f_{i+1}$ or $f_i \geq f_{i+1}$ for $1 \leq i \leq n-1$) connecting f and g. This is proven in Section 2.2 of [7] by noting that the image of I in Y^X is compact, and therefore contains a finite number of elements given Y^X is an A-space.

With this understanding, we turn to a notion that is different from but related to homotopy that allows us to compare maps residing in different components of our colimit, as well as characterize the behavior of related maps under realization.

6. Contiguity

In order to create a formal relationship between maps with different subdivisions as their domains, we utilize the concept of contiguity of simplicial maps to generate a notion of equivalence of maps in distinct components in our colimit. We conclude many of our intermediate results in the setting of simplicial complexes, as the notion of contiguity does not extend perfectly to A-spaces. This will be sufficient for our purposes, but for a more thorough treatment of contiguity of A-space maps, see [7] and [9].

Definition 6.1. Let K and L be simplicial complexes. Two simplicial maps $u : K \to L$ and $v : \mathrm{Sd}^m K \to L$ are *contiguous* if for each simplex $\sigma \in \mathrm{Sd}^m K$, there is a simplex $\tau \in L$ such that $u(\sup^{(m)}(\sigma)) \subset \tau$ and $v(\sigma) \subset \tau$.

If we have an A-space X such that $\mathscr{K}(X) = K$, we can somewhat extend this definition to maps between A-spaces. However, it should be clear by this definition that we are limited by the existence of simplices, which requires at least one subdivision of X to be sensible. Thus, we say that two maps f and g between A-spaces are contiguous if $\mathscr{K}(f)$ and $\mathscr{K}(g)$ are contiguous as simplicial maps.

Now that we have presented the notion contiguity as it pertains to simplicial and A-space maps, we offer the following relationships between contiguity and homotopy that link equivalence in the colimit to equivalence after realization.

Proposition 6.2. *Suppose $\mathscr{K}(f) : \mathscr{K}(X) \to \mathscr{K}(Y)$ and $\mathscr{K}(g) : \mathscr{K}(\mathrm{Sd}^n X) \to \mathscr{K}(Y)$ are contiguous. Then $|\mathscr{K}(f)| \simeq |\mathscr{K}(g)|$.*

Since $\mathscr{K}(f)$ and $\mathscr{K}(g)$ are contiguous, they satisfy the definition of simplicial closeness given in [7]. Thus, there is a linear homotopy connecting their realizations.

Theorem 6.3. [4] *For a finite A-space X and arbitrary A-space Y, if $f : |\mathscr{K}(X)| \to |\mathscr{K}(Y)|$ is continuous, then there exists an $n \geq 0$ and a map $g : \mathrm{Sd}^n X \to Y$ with $|\mathscr{K}(g)| \sim f$.*

Proof. By the classical Simplicial Approximation Theorem, there exists a simplicial map $u : \mathrm{Sd}^{n-1}\mathscr{K}(X) \to \mathscr{K}(Y)$ such that $|u| \simeq f$. Noting $\mathrm{Sd}^{n-1}\mathscr{K}(X) = \mathscr{K}(\mathrm{Sd}^{n-1}X)$, define $g = \sup \circ \mathscr{X}(u)$. Then g is a map from $\mathrm{Sd}^n X$ to Y, and given $\sigma = \{S_1, S_2, ..., S_k\}$, an arbitrary simplex of $\mathscr{K}(\mathrm{Sd}^n(X))$, we have

$$(\mathscr{K}(\sup \circ \mathscr{X})(u))(\sigma) = \sup\{u(S_1), u(S_2), ..., u(S_k)\} = u(\sup(\sigma)),$$

so by definition u and $\mathscr{K}(g)$ are contiguous. Following Proposition 6.2, this implies $|\mathscr{K}(g)| \simeq |u|$. Thus, $|\mathscr{K}(g)| \simeq f$. □

We say any map g satisfying this condition is an *A-space approximation* of f.

Proposition 6.4. [7] *Let X and Y be A-spaces. If $g : \mathrm{Sd}^i X \to Y$ and $g' : \mathrm{Sd}^j X \to Y$ are both simplicial approximations of $f : |\mathscr{K}(X)| \to |\mathscr{K}(Y)|$, then g and g' are contiguous.*

Proposition 6.5. *Suppose f and g are maps from X to Y, where X is a finite A-space and Y is an arbitrary A-space. If $f \simeq g$, then $|\mathscr{K}(f)| \simeq |\mathscr{K}(g)|$.*

Proof. Since $f \simeq g$ implies that there is a sequence of comparable maps connecting the two, and since both $\mathrm{Im}(f)$ and $\mathrm{Im}(g)$ are finite, it suffices to show the desired result for f and g such that $f(x) = g(x)$ for all but one x', where $f(x') \leq g(x')$. For a simplex σ of X that does not contain x', we have $f(\sigma) = g(\sigma)$, which is clearly contained in a simplex of Y. If $x' \in \sigma$, then $x' = x_i$ for some i and both $f(\sigma)$ and $g(\sigma)$ are contained in the simplex given by

$$\{f(x_0), f(x_1), ..., f(x'), g(x'), g(x_{i+1}), ..., g(x_k)\}$$

after removing repetitions. By Proposition 6.2, $|\mathscr{K}(f)| \simeq |\mathscr{K}(g)|$. □

7. Modeling Homotopy Classes of Maps

We are ready to consider a method to model the homotopy classes of maps between topological spaces based on the framework we have laid.

Theorem 7.1. *Let X be a finite A-space and let Y be an arbitrary A-space. Then there is a natural bijection between $[|\mathscr{K}(X)|, |\mathscr{K}(Y)|]$ and the colimit of the system*

$$[X, Y] \xrightarrow{\sup^*} [\mathrm{Sd}X, Y] \xrightarrow{\sup^*} [\mathrm{Sd}^2X, Y] \xrightarrow{\sup^*} \cdots$$

This bijection, which we denote $K : \operatorname*{colim}_n [\mathrm{Sd}^nX, Y] \to [|\mathscr{K}(X)|, |\mathscr{K}(Y)|]$, maps $[f] \in [\mathrm{Sd}^iX, Y]$ to $[|\mathscr{K}(f)|] \in [|\mathscr{K}(X)|, |\mathscr{K}(Y)|]$.

Proof. From Proposition 6.5, homotopies are preserved by each component map from $[\mathrm{Sd}^nX, Y]$ to $[|\mathscr{K}(X)|, |\mathscr{K}(Y)|]$ so K is well-defined. Suppose $[f] \in [\mathrm{Sd}^iX, Y]$ and $[g] \in [\mathrm{Sd}^jX, Y]$. If $[f]$ and $[g]$ are identified in the colimit, then there exists an N such that $f \circ \sup^{(N-i)} \simeq g \circ \sup^{(N-j)}$. Again by Proposition 6.5, $|\mathscr{K}(f \circ \sup^{(N-i)})| \simeq |\mathscr{K}(g \circ \sup^{(N-j)})|$. Thus, we need only check that $|\mathscr{K}(f)| \simeq |\mathscr{K}(f \circ \sup^{(N-i)})|$ to ensure $|\mathscr{K}(f)| \simeq |\mathscr{K}(g)|$. Noting that f and $f \circ \sup^{(N-i)}$ are tautologically contiguous, we have by Proposition 6.2 that $|\mathscr{K}(f)| \simeq |\mathscr{K}(f \circ \sup^{(N-i)})|$. Thus, $|\mathscr{K}(f)| \simeq |\mathscr{K}(g)|$.

By Theorem 6.3, each homotopy class of maps has associated to it at least one A-space approximation, so K is surjective.

To prove injectivity, suppose $|\mathscr{K}(f)| \simeq |\mathscr{K}(g)|$. Then f and g are by definition A-space approximations for, say, $|\mathscr{K}(f)|$. Suppose $[f] \in [\mathrm{Sd}^iX, Y]$ and $[g] \in [\mathrm{Sd}^jX, Y]$. Then by Proposition 6.4, f and g are contiguous. This is sufficient co conclude $[f] = [g]$ in $\operatorname*{colim}_n [\mathrm{Sd}^nX, Y]$. Thus, we have shown all necessary criteria for the existence of this canonical bijection. □

While a bijection certainly creates a beautiful relationship here, the surjectivity of this map should be regarded as the most interesting component of this result; It means that given an arbitrary continuous map between $|\mathscr{K}(X)|$ and $|\mathscr{K}(Y)|$, only a finite number of subdivisions on the level of A-spaces are necessary to model this map up to homotopy. The fact that we can transfer the study of continuous maps between topological spaces to that of order preserving maps between posets should be startling. The rest of the framework and proof simply collapses down maps redundant up to homotopy.

8. Suspensions and the Group Structure on $\langle X, Y\rangle$

In this section, we narrow our focus to a particular class of topological spaces that induce additional structure on homotopy classes of maps. For arbitrary topological spaces X and Y, $[X, Y]$ is in general simply a set, so a bijection is the strongest relationship we can hope for. However, the fact that under certain conditions based homotopy classes admit a group structure may raise the questions as to whether our bijection can also be extended to a group isomorphism in these cases. Keeping the motivating example of homotopy groups in mind, we widen our attention to suspensions, of which spheres are an example, to demonstrate a way to define a group structure on our colimit using subdivisions.

First, we show the bijection outlined in Section 7 can be modified to a correspondence, to borrow Hatcher's notation for pointed homotopy classes in [3], $\langle |\mathscr{K}(X)|, |\mathscr{K}(Y)|\rangle$ and the colimit of

$$\langle X, Y\rangle \xrightarrow{\sup^*} \langle \mathrm{Sd}X, Y\rangle \xrightarrow{\sup^*} \langle \mathrm{Sd}^2X, Y\rangle \xrightarrow{\sup^*} \cdots$$

by noticing that fixing basepoints $x_0 \in X$ and $y_0 \in Y$ in partially ordered set allows for a notion of based maps where we require $f(x_0) = y_0$. The corresponded totally ordered subsets $\{x_0\}$ and $\{y_0\}$ indeed realize to points in $|\mathscr{K}(X)|$ and $|\mathscr{K}(Y)|$ respectively, since they denote 0-simplices and thus realize to 0-subcomplexes. Our choice of 0-cell for the basepoint is preserved as expected under subdivisions of X, since there is a canonical inclusion map $i : X \to \mathrm{Sd}X$ that takes an element x to the corresponding single element subset of $\{x\}$ of X, which denotes an element of $\mathrm{Sd}X$. Since under the sup map, $\{x_0\} \mapsto x_0$, taking pointed spaces is compatible with our colimit and our definition of contiguity, which importantly extends the Simplicial Approximation Theorem to this new result.

However, to make this variation play well with the sup map and contiguity, we must require that x_0 is a maximal point of X, which is no trouble since we have already assumed X is finite. This will agree with our methods in the following section.

Now that we have sufficiently restricted our homotopy class bijection, we may proceed with our group structure construction.

Recall that given a topological space X, we define the *suspension of* X, denoted SX, to be the quotient space $(X \times I)/(\sim)$, where $(x,t) \sim (y,s)$ if and only if $(x,t) = (y,s)$ or $s = t = 0$ or $s = t = 1$. We say the suspension is *reduced* if we additionally identify all points of the form (x_0, t) for $t \in [0,1]$. For this, we write ΣX. Clearly, this collapsing map gives a homotopy equivalence from SX to ΣX.

Definition 8.1. We define the finite analog of these constructions to be the *non-Hausdorff suspension* $\mathbb{S}X$, which is the resulting space after adding two points $+$ and $-$ to X such that the only open sets containing them are $\mathbb{S}X$ itself, $\{X \cup +\}$, and $\{X \cup -\}$.

In terms of posets, where we will set our following constructions, this amounts to adding two new maximal points to the poset of X.

Theorem 8.2. [7] *For any space X, the map $\gamma : SX \to \mathbb{S}X$ is a weak homotopy equivalence. For any weak homotopy equivalence $f : X \to Y$, the maps $Sf : SX \to SY$ and $\mathbb{S}f : \mathbb{S}X \to \mathbb{S}Y$ are weak homotopy equivalences. Therefore $\gamma^n : S^n X \to \mathbb{S}^n X$ is a weak homotopy equivalence for any space X.*

Further, this implies that there is a basepoint-preserving weak homotopy equivalence $\gamma^n : (\Sigma^n X, x_0) \to (\mathbb{S}^n X, x_0)$.

Unfortunately, the non-Hausdorff suspension of a space will not give quite the structure we need to make our desired map continuous. Thus, we define an alternative configuration:

Definition 8.3. For a topological space X, the *non-Hausdorff opposite suspension* of X is given formally by $(\mathbb{S}(X^{op}))^{op}$, which we interpret as adding two new minimal points to X, and denote $\mathbb{S}^{op}X$.

From this will arise a nice space intimately related to the classical suspension that will allow us to model a defining property of suspensions of topological spaces.

There is a natural map from $\mathscr{K}(\mathbb{S}X)$ to $\mathscr{K}(\mathbb{S}^{op}X)$ which, for example, takes a simplex $\sigma = \{x_0, x_1, ..., x_k, +\}$ to the simplex $\{+, x_0, x_1, ..., x_k\}$ in $\mathscr{K}(\mathbb{S}^{op}X)$ and preserves ordering. From this, it follows that $|\mathscr{K}(\mathbb{S}X)| = |\mathscr{K}(\mathbb{S}^{op}X)|$, as the geometric realization functor acts independently of the order on the underlying poset.

The map γ in Theorem 8.2 and the map p defined by McCord together generate a weak equivalence $\tilde{p} : \Sigma|\mathscr{K}(X)| \to \mathbb{S}X$. This, coupled with the above equivalence, gives that $\Sigma|\mathscr{K}(X)| \simeq |\mathscr{K}(\mathbb{S}^{op}X)|$. Keeping this relationship in mind, we digress to lay some mathematical framework that will help us make use of this homotopy equivalence.

Definition 8.4. [1] A pointed topological space (X, x_0) is a *co-H-space* if there exists a map $\psi : X \to X \vee X$ such that $p_1\psi \simeq id_X$ rel x_0 and $p_2\psi \simeq id_X$ rel x_0, where $p_1, p_2 : X \vee X \to X$ are the two projection maps. (X, x_0) is a *co-group* if it is co-associative and there is a co-inverse map $\xi : X \to X$ up to based homotopy such that the co-group axioms are satisfied.

Above, we require that $X \vee X$ be wedged at x_0 and call x_0 the identity of the co-H-space.

Example 8.5. $(\Sigma X, x_0)$ is a co-group (and thus a co-H-space) for all pointed topological spaces (X, x_0).

We are particularly interested in suspensions and their properties because of their presence in many important constructions in algebraic topology. For example, the n-sphere is homeomorphic to n iterative suspensions of the 0-sphere, generating a class of topological spaces central to algebraic topology. Additionally, the co-group structure of a suspension plays an important role in the Eckmann-Hilton duality, as understanding the loop spaces of certain topological spaces often gives a wealth of information.

Given that our motivation for looking at A-spaces is to encode information about topological spaces more generally, one may ask if we are able to model this co-H-space structure using finite A-spaces. Unfortunately, following a dual result for H-spaces from Stong, we have the following:

Theorem 8.6. [5] *Suppose X is a finite connected space. Then X admits a co-H-space structure if and only if it is contractible.*

Thus, any finite model that is also a co-H-space is a model of a space with only trivial homotopy groups, which is both terribly narrow and uninteresting. This seems to imply that we have no way to model such a structure using finite spaces. For this reason, we turn to the colimit we have constructed for inspiration on how to model the co-H-space structure of a suspension that makes use of the subdivision functor in a way that strengthens the main result of Section 8.

Theorem 8.7. *Suppose that X is a finite A-space. Then there exists an order-preserving map $\phi : \mathrm{Sd}(\mathbb{S}^{op}X) \to \mathbb{S}^{op}X \vee \mathbb{S}^{op}X$ such that $p_1|\mathscr{K}(\phi)| \simeq id_{\Sigma|\mathscr{K}(X)|}$ and $p_2|\mathscr{K}(\phi)| \simeq id_{\Sigma|\mathscr{K}(X)|}$, where p_1 and p_2 are the respective projection maps.*

Construction 8.8. We assume that the wedge of two copies of $\mathbb{S}X$ identifies different minimal points (i.e. $+ \sim -$) in each copy of X to make the description of our map cleaner, but the reader should check that a similar map exists regardless of choice of minimal basepoint on each copy.

Let $\sigma = \{x_0, x_1, ..., x_n\}$ be an element of $\mathrm{Sd}(\mathbb{S}^{op}X)$. Define an equivalence relation on $\mathrm{Sd}(\mathbb{S}^{op}X)$ by the following two generating conditions:

(i) Identify all such σ where x_0 was an original element of X (in other words, $\inf(\sigma) \neq +$ or $-$).

(ii) $\sigma \sim \sigma'$ if $\sup(\sigma) = \sup(\sigma')$ and $\inf(\sigma) = \inf(\sigma')$.

Call the identification space generated by this partition X', equipped with the quotient topology (which corresponds to inheriting an order from $\mathbb{S}^{op}X$). Note that if X is a finite A-space, X' is a finite A-space. Let ρ denote the identification map from $\mathrm{Sd}(\mathbb{S}^{op}X)$ to X'.

Proposition 8.9. *$X' \cong \mathbb{S}^{op}X \vee \mathbb{S}^{op}X$.*

Proof. Define a map $f : X' \to \mathbb{S}^{op}X \vee \mathbb{S}^{op}X$ as follows. Given an element σ of $\mathrm{Sd}(\mathbb{S}^{op}X)$ such that $\inf(\sigma)$ is not minimal in $\mathbb{S}^{op}X$, let $f(\sigma) = *$, the basepoint of the wedge. Otherwise, $\inf(\sigma)$ is one of two minimal points of $\mathbb{S}^{op}X$. Let which minimal point σ contains be regarded as a coordinate to denote one of the copies of $\mathbb{S}^{op}X$. Then, restricted to that copy, let $f(\sigma) = \sup(\sigma)$.

That this map is surjective follows from the fact that sup is surjective. To see that it is injective, consider σ and σ' such that $f(\sigma) = f(\sigma')$. Then either $f(\sigma) = f(\sigma') = *$, in which case neither contained a minimal element of $\mathbb{S}^{op}X$ so they were identified together in X', or both the inf and sup map agree, which implies $\sigma \sim \sigma'$ in $\mathrm{Sd}\,\mathbb{S}^n$, so $\sigma = \sigma'$ in X.

To prove f is continuous, suppose $\sigma \le \sigma'$. Then $\rho(\sigma) \le \rho(\sigma')$ by the quotient topology, and there are four cases to consider: first, that neither σ nor σ' have a minimal element. Then $f\rho(\sigma) = * = f\rho(\sigma')$. Suppose σ does not have a minimal element but σ' does. Then $f\rho(\sigma) = *$ which is tautologically less than or equal to $f\rho(\sigma')$. Then, suppose both contain a minimal element. If they do not agree on the minimal element, they were not comparable subsets to begin with. Then supposing they do, this means that ρ is determined by sup, which is continuous. So f preserves order and is therefore continuous.

Let $g : \mathbb{S}^{\mathrm{op}} \vee \mathbb{S}^{\mathrm{op}} \to X'$ be the map that sends the wedge point to the equivalence class of all chains not containing a minimal element. Suppose $x \in \mathbb{S}^{\mathrm{op}} \vee \mathbb{S}^{\mathrm{op}}$ is not the wedge point. Then define where x is sent by first noting which copy of $\mathbb{S}^{\mathrm{op}}$ x belongs to (the copies correspond to the minimal points $+$ and $-$ respectively) and which point in $\mathbb{S}^{\mathrm{op}}$ maps to it under the inclusion $i : \mathbb{S}^{\mathrm{op}} \to \mathbb{S}^{\mathrm{op}} \vee \mathbb{S}^{\mathrm{op}}$ into that copy. Let the resulting $+$ or $-$ association determine the minimal point of the chain and the preimage under inclusion the maximal point, completely determining an element of X'. This construction makes it a clear inverse, and checking continuity is simply another process of checking assignment cases while recognizing the ordering of X' is fully determined by that of $\mathrm{Sd}(\mathbb{S}^{op}X)$. $\square$

We end by renaming the composition $f \circ \rho$ as ϕ for simplicity. For the next checks, we will use the construction above with instead the assumption that the points "+" were identified for the basepoint of the wedge. Again, one should check that this choice is arbitrary and that the inconsistency in choice in this paper is instead to give the reader the simplest and most elegant constructions, as should be the point.

Proposition 8.10. *Let*

$$\phi_* : (|\mathscr{K}(\mathrm{Sd}(\mathbb{S}^{op}X)|, |\mathscr{K}(+)|) \to (|\mathscr{K}(\mathbb{S}^{op}X)| \vee |\mathscr{K}(\mathbb{S}^{op}X)|, |\mathscr{K}(*)|)$$

be the map induced by ϕ. Then $p_1\phi_* \simeq id_{\Sigma|\mathscr{K}(X)|}$ rel $|\mathscr{K}(+)|$ and $p_2\phi_* \simeq id_{\Sigma|\mathscr{K}(X)|}$ rel $|\mathscr{K}(+)|$, i.e. ϕ induces a co-H-space structure on $(\Sigma|\mathscr{K}(X)|, |\mathscr{K}(+)|)$.

Proof. To show that ϕ induces a co-H-space structure on $\Sigma|\mathscr{K}(X)|$, we must show that the induced map $\phi_* = |\mathscr{K}(\phi)|$ satisfies the co-H-structure properties of ψ as above. Basepoint preservation comes from keeping track of its point-wise assignment, so we omit that tracking. Since our map is symmetric, we demonstrate the proof explicitly for only p_1.

Since there is a canonical homeomorphism from $|\mathscr{K}(\mathrm{Sd}(\mathbb{S}^{op}X))|$ to $|\mathscr{K}(\mathbb{S}^{op}X)|$ [10], the homotopy equivalence $|\mathscr{K}(\sup)| : |\mathscr{K}(\mathrm{Sd}(\mathbb{S}^{op}X))| \to |\mathscr{K}(\mathbb{S}^{op}X)|$ is in fact a map homotopic to the identity. This simplifies our problem to showing that the following diagram commutes up to homotopy:

$$\begin{tikzcd} \mathrm{Sd}^{k+1}(\mathbb{S}^{op}X) \arrow[r, "\phi"] \arrow[rr, bend right, "\sup"'] & \mathrm{Sd}^k(\mathbb{S}^{op}X) \vee \mathrm{Sd}^k(\mathbb{S}^{op}X) \arrow[r, "p_1"] & \mathrm{Sd}^k(\mathbb{S}^{op}X) \end{tikzcd}$$

We begin by demonstrating that $p_1\phi \simeq \sup$. Let $\sigma = \{x_0, x_2, ..., x_k\}$ be an element of $\mathrm{Sd}(\mathbb{S}^{op}X)$. We consider three cases. First, suppose x_0 is not a minimal element of $\mathbb{S}^{op}X$. Then $\phi(\sigma)$ is minimal in $\mathbb{S}^{op}X \vee \mathbb{S}^{op}X$, which is preserved under projection. If x_0 is a minimal element, but is the minimal element associated with the second copy of $\mathbb{S}^{op}X$, the projection map will ultimately crush σ to $*$, which is minimal. In both of these cases, it trivially follows that $p_1\phi(\sigma) \leq \sup(\sigma)$. If σ contains the minimal point sent to "first" sphere, then $p_1\phi$ sends σ to $\sup(\sigma)$, so we tautologically have that $p_1\phi(\sigma) \leq \sup(\sigma)$.

It follows from Proposition 5.1 that if $p_1\phi(\sigma) \leq \sup(\sigma)$ is true for all σ, then $p_1\phi \simeq \sup$. By Proposition 6.5 we have that $p_1\phi \simeq \sup$ implies $|\mathscr{K}(p_1\phi)| \simeq |\mathscr{K}(\sup)|$. Thus, $p_1\phi_* \simeq id$ rel $|\mathscr{K}(+)|$. $\square$

Proposition 8.11. [1] *Let (X, x_0) and (Y, y_0) be arbitrary topological spaces. Then*

(i) There is a unital binary operation on $\langle X, Y\rangle$ if and only if X is a co-H-space.

(ii) $\langle X, Y\rangle$ is a group if and only if X is a co-group.

Given this result, we should expect $\langle|\mathscr{K}(\mathbb{S}^{op}X)|, |\mathscr{K}(Y)|\rangle$ to be a group since $|\mathscr{K}(\mathbb{S}^{op}X)|$ is homotopic to the reduced suspension of the geometric realization of X, a CW complex, and thus has a co-group structure. Below we will demonstrate how our finite model of the co-H-space structure of $\mathbb{S}^{op}X$ captures enough information to model the group structure of these homotopy classes of maps.

Suppose $[f] \in \langle\mathrm{Sd}^k(\mathbb{S}^{op}X), Y\rangle$ and $[g] \in \langle\mathrm{Sd}^m(\mathbb{S}^{op}X), Y\rangle$ with $k \leq m$. Then we may construct the following chain of maps:

$$\begin{tikzcd} \mathrm{Sd}^{m+1}(\mathbb{S}^{op}X) \arrow[r, "\phi"] & \mathrm{Sd}^m(\mathbb{S}^{op}X) \vee \mathrm{Sd}^m(\mathbb{S}^{op}X) \arrow[d, "\sup\, \vee\, id"'] & & \\ & \mathrm{Sd}^k(\mathbb{S}^{op}X) \vee \mathrm{Sd}^m(\mathbb{S}^{op}X) \arrow[r, "f\vee g"] & Y \vee Y \arrow[r, "\nabla"] & Y \end{tikzcd},$$

where $\nabla : Y \vee Y \to Y$ denotes the codiagonal map. Define the homotopy class of the composition to be $[f] + [g]$. Thus, this co-H-space model structure allows us to define a unital binary operation in the colimit.

We would like our induced binary operation to be associative and have inverses, as to model the full group structure on these homotopy classes. For this, we use the construction of the bijection in the previous section to induce this additional structure from the modeled homotopy class of maps.

Given $[[f] + [g]] + [h]$ in the colimit, we have $[[f] + [g]] + [h] \mapsto K([[f] + [g]]) + K([h])$, which we know equals $K([f]) + K([[g] + [h]])$ by the associativity of the group structure on $[S^n, X]$. Then via the existence of K^{-1}, we get $[[f] + [g]] +$

$[h] = [f] + [[g] + [h]]$, showing the operation is associative. Likewise, suppose we have $[f]$ in the colimit. Define $[f]^{-1}$ as the unique element $K^{-1}((K([f])^{-1})$, whose existence and uniqueness is guaranteed by the existence and uniqueness of inverses in $\langle \Sigma|\mathscr{K}(X)|, |\mathscr{K}(Y)|\rangle$ and our one-to-one correspondence. Therefore we have constructed a model for the full group structure of $\langle \Sigma|\mathscr{K}(X)|, |\mathscr{K}(Y)|\rangle$ using the result of Theorem 7.1.

9. Applications to Homotopy Groups

One very important place we can use the group structure induced from suspensions is in the homotopy groups of a topological space X. Homotopy groups are an essential element of algebraic topology, and the ability to model them using a system built from A-spaces should be incredibly motivating. Since for $n > 1$, the n-sphere is nothing more than the suspension of S^{n-1}, it should be unsurprising that we will conclude this paper by discussing applications to homotopy groups.

For this section, we will restrict our attention to connected (and hence path-connected) CW complexes X, and must pay some mind of based homotopy classes, since we have the relation $\pi_k(X) \cong \langle S^k, X\rangle$. Thus, applying the above sections to conclude homotopy-theoretic results is a natural extension.

Here, we omit the opposite suspension construction for the sphere, as $\mathbb{S}^{op}\mathbb{S}^n \cong \mathbb{S}\mathbb{S}^n$ because of the symmetric nature of the minimal finite model for S^n. Thus, we simplify our notation by dropping the opposites, and leave the reader to check that the same map ϕ suffices to give a map from Sd $\mathbb{S}^n$ to $\mathbb{S}^n \vee \mathbb{S}^n$ which induces a co-H-space structure on S^n. Given that we demonstrated both a bijection and group homomorphism in the previous sections, and under the assumption that X is a connected A-space, we may conclude $\operatorname*{colim}_n \langle (\mathrm{Sd}^k\ \mathbb{S}^n, *), (X, x_0)\rangle \cong \pi_n(|\mathscr{K}(X)|, |\mathscr{K}(x_0)|)$ as groups for $n \geq 2$.

We can say more about spheres, as we know that for $n \geq 2$, we have a cocommutative co-group that induces an abelian group structure on $\pi_n(X)$. We demonstrate this below:

Proposition 9.1. [1] *Let (X, x_0) be a co-H-space. Then $\psi : X \to X \vee X$ is cocommutative if and only if $\langle X, Y\rangle$ is a cocommutative cogroup for all spaces Y.*

Given $[f] + [g]$ in the colimit, we have $[f] + [g] \mapsto K([f]) + K([g])$, which we know is equal to $K([g]) + K([f])$ in $\langle S^n, X\rangle$. Then $[f] + [g] = [g] + [f]$.

Applying the results of this paper to homotopy-theoretic results, we begin by showing that if a connected CW complex X admits an A-space model A with a sole maximal (or minimal) point, then X is contractible.

Let $*$ denote this maximal (or minimal) point. Then any based map $f : \mathrm{Sd}^k(\mathbb{S}^n) \to A$ for some values k and n, the image consists of finitely many points, and the constant map given by $g(x) \equiv *$ is always greater than or equal to f since $* \geq \mathrm{Im}(f)$ (or less than or equal to f since $* \leq \mathrm{Im}(f)$). Thus, $f \simeq g$ for all maps f based at $*$. This gives that $\langle S^n, X\rangle$ is trivial in every grading, giving trivial homotopy groups by inclusion of π_n. Since trivial homotopy groups imply contractibility for CW complexes by Whitehead's theorem, we conclude X is contractible.

Diverging to poset-theoretic results, this also implies that the geometric realization of a lattice is contractible. For a similar but perhaps more intriguing application, we set up the following definition:

Definition 9.2. The *infinite sphere* S^∞ is defined to be the directed colimit of S^n, where the map from S^{n-1} to S^n is given by inclusion.

For our purposes, we would like to generalize the A-space model we have for S^n to an A-space model for S^∞. As we understand $\mathbb{S}^n$ as being n successive suspensions of S^0, the same construction applies here. In particular, we typically represent the finite model of S^2 by the Hasse diagram below:

$$\begin{tikzcd} \bullet & \bullet \\ \bullet \arrow[u] \arrow[ur] & \bullet \arrow[u] \arrow[ul] \\ \bullet \arrow[u] \arrow[ur] & \bullet \arrow[u] \arrow[ul] \end{tikzcd}$$

For S^∞, we get much the same picture, except the tower stretches infinitely upwards. The A-space corresponding to this Hasse diagram we will denote $\mathbb{S}^\infty$.

Proposition 9.3. *S^∞ is contractible.*

Proof. Given our A-space model for S^∞, we can consider $\operatorname*{colim}_n \langle \mathrm{Sd}^n\mathbb{S}^k, \mathbb{S}^\infty\rangle$ to model the kth homotopy group of S^∞. While working with a colimit does not seem necessarily advantageous compared to working with a topological space itself, it does allow us to conclude some important results here.

It is clear to see that for any n the image of a map $f : \mathrm{Sd}^n\mathbb{S}^k \to \mathbb{S}^\infty$ will be finite since $\mathrm{Sd}^n\mathbb{S}^k$ has only finitely many points. Since $\mathbb{S}^\infty$ is infinite, and in particular never attains maximal points in its poset representation, there will always exist an $x_K \in \mathbb{S}^\infty$ such that $f(x) < x_K$ for all $x \in \mathrm{Sd}^n\mathbb{S}^k$. Then the constant map that sends every element of $\mathrm{Sd}^n\mathbb{S}^k$ to x_K is homotopic to f by Proposition 5.1. Further, given maps f and g, we are ensured by the nonexistence of a maximal element that we can always find some x_K s.t. f and g are both dominated by the constant map to x_K. Thus, any two choices of f and g will be homotopic regardless of the number of subdivisions we take, and therefore by the same argument involving basepoints as above, $\pi_n(S^\infty)$ is trivial for all n. S^∞ is also a CW complex, so this is sufficient to conclude contractibility. $\square$

While the applications of this section involve little calculation, and do not require the results of Section 8, the potential applications of framing some homotopic information within the scope of combinatorial computation could be used more broadly. While the computation of these homotopy classes of maps between subdivided A-spaces remain unreasonably computable at present, there is no *theoretical* barrier to this method of finding the the homotopy classes of maps between spaces, particularly for finite homotopy groups. Following this paper, I hope to determine if some sort of upper bound on subdivisions for the bijection to hold can be obtained in the case where $[X, Y]$ is a finite set, and explore whether the Freudenthal Suspension Theorem can be combinatorially concluded.

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