GEOMETRIC GROUP THEORY

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ABSTRACT. This paper is an introduction to certain topics in geometric group theory. We begin with an introduction to Cayley graphs and the word metric. We then move to the notion of quasiisometry, and how this interacts with group actions, making a connection with Riemannian geometry. Following this connection, we examine the growth of balls around the identity, and classify groups by their asymptotic growth type. We focus particularly on groups of exponential growth.

Contents

1. Introduction

Geometric group theory aims to study finitely-generated groups by their geometric properties. Given a generating set of a group, we will define a metric on the group with respect to this generating set by letting the distance between two group elements be the length of the shortest sequence of group elements which begins at one element and ends at the other, and such that each subsequent element is obtained from the previous element by multiplication by a generator (or the inverse of a generator). Metrics given by different generating sets turn out to be equivalent.

There is a natural equivalence relation on metric spaces, which we will call *quasiisometry*, which allows us to build a connection between groups and spaces on which they act. We will show, for example, that the fundamental group of a compact Riemannian manifold is quasiisometric to its universal cover. We will also see that quasiisometry preserves the number of ends of spaces.

The primary analogy we will draw between groups and Riemannian manifolds is the growth of balls. It is a well-known theorem of Riemannian geometry that

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any simply connected Riemannian manifold with constant curvature 0 is isometric to $\mathbb{R}^n$ (with the standard Euclidean metric) and the volume of a ball of radius r is $C_n r^n$ for some constant C_n, and so grows polynomially in r. Also, any simply connected Riemannian manifold with constant curvature -1 is isometric to $\mathbb{H}^n$ (with the standard hyperbolic metric) and the volume of a ball of radius r grows exponentially in r. We will similarly study and classify groups by the asymptotic rate of growth of balls around the identity, with respect to a suitable equivalence relation.

2. Cayley Graphs and Word Metrics

A common way of representing a group is by giving one of its presentations. A presentation of a group consists of a pair of sets, written $\langle S|R\rangle$ where S is a generating set of the group and R is a set of "relations" on these generators, i.e. words in the generators which are equal to the identity in the group, which is maximal in the sense that for any word in the generators which is equal to the identity in the group, the fact that it is equal to identity is derivable from the relations in R. This can be formalized in the following way:

Definition 2.1. Let F_S be the free group on the set S, $R \subseteq F_S$, and $N = \left\langle R^{F_S} \right\rangle \trianglelefteq F_S$ the normal closure of R in F_S. Then $\langle S|R\rangle := F_S/N$.

Remarks 2.2. We will frequently identify elements of S with their images in F_S/N, so that S is a generating set for $\langle S|R\rangle$.

It is easy to see that any group G is isomorphic to some $\langle S|R\rangle$ by taking $S = G$ and $R = \{c^{-1}ab \in F_G : ab = c \text{ in } G\}$, since this forces the multiplication table of $\langle S|R\rangle$ to be the same as that of G.

Moreover, for any generating set S of G, we have a presentation $G = \langle S|R\rangle$ for some R. To see this, simply take R to be the set of all words in the elements of S which are the identity in G.

In general, there will be many different presentations for the same group.

Definition 2.3. A group G is said to be *finitely generated* if it has a presentation with a finite generating set.

For example, finite groups, $\mathbb{Z}^n$, and F_n are finitely generated, while $\mathbb{R}^n$ and S^1 are not. Geometric group theory is primarily concerned with finitely generated groups, for reasons which will become clear in Prop. 2.8 below.

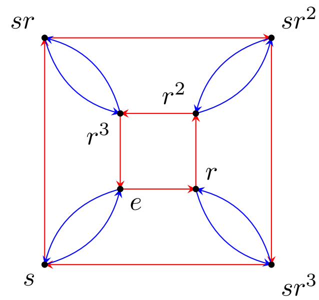
Given a finite set of generators S of a group G, we can define a graph which captures the information of the group as follows:

Definition 2.4. The *(decorated) Cayley graph* $\Gamma(G,S)$ of a group G with generating set S is the directed graph with edges colored by elements of S and vertex set G where the edge relation is given by $g \xrightarrow{s} gs$.

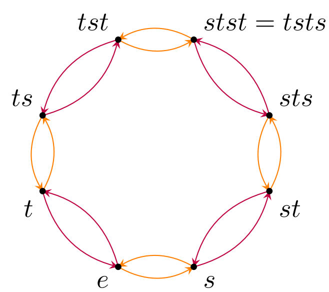
The *(undecorated) Cayley graph*, which we will also denote by $\Gamma(G,S)$ is obtained from the decorated Cayley Graph by forgetting the directions, colors, and multiedges.

Unless otherwise specified, we will usually refer to the undecorated Cayley graph as simply the *Cayley graph*.

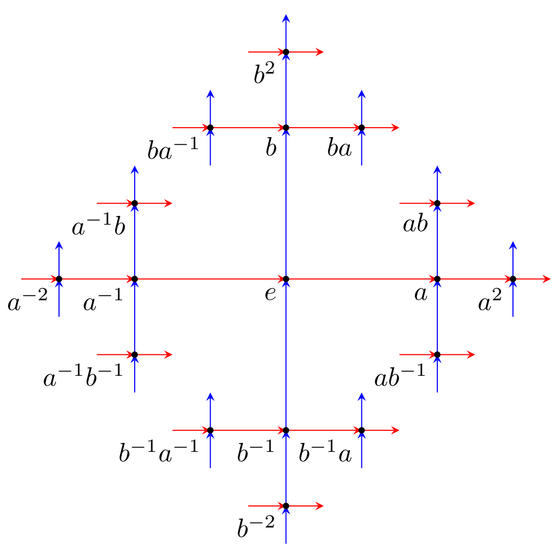
Examples 2.5. We will show the decorated Cayley graphs of D_4 for two different presentations. The first is $D_4 = \left\langle r, s | r^4, s^2, rsrs \right\rangle$.



The second is $D_4 = \left\langle s,t | s^2, t^2, s^{-1}t^{-1}s^{-1}t^{-1}stst \right\rangle$ (note that an isomorphism between these two presentations of D_4 is given by $s \mapsto s$ and $t \mapsto rs$).



We also draw the decorated Cayley graph for $F_2 = \langle a, b | \emptyset \rangle$ up to three steps away from the identity.



And the undecorated Cayley Graphs for the above groups and generating sets are obtained by simply forgetting the colors, orientations, and multiedges.

Remark 2.6. It is possible to see a graph both as a combinatorial object (a set of vertices together with a set of unordered pairs of vertices) or as a geometrical object (with edges glued together as in the corresponding diagram, and where each edge is isometric to $[0, 1]$). We will use these two interpretations interchangeably.

Similar to the fact that every group G can be represented as a quotient $\langle S|R\rangle$ of a free group F_S, we also clearly have that the Cayley graph $\Gamma(G,S)$ is naturally a quotient of the $\#S$-ary tree.

We can now define a notion of distance in groups by way of the distance in the graph (note that the the combinatorial distance and geometrical distance in a graph agree on the vertices).

Definition 2.7. Given a group G and a generating set S, define $d_S(g,h)$ to be the distance in $\Gamma = \Gamma(G,S)$ between g and h. In other words,

$$\begin{aligned} d_S(g,h) &= \min\{\ell_S(\gamma) : \gamma \text{ is a path in } \Gamma \text{ from } g \text{ to } h\} \\ &= \min\{\ell_S(\sigma) : \sigma \in F_S \text{ and } g\sigma = h \text{ in } G\} \end{aligned}$$

where $\ell_S(\gamma)$ denotes the (metric) length of γ and $\ell_S(\sigma)$ denotes the (combinatorial) length of σ. It is straightforward to check that d_S is a metric.

Now, the metric d_S depends on the choice of generating set S, but from the next easy proposition we see that all metrics of this type are equivalent when S is finite. (Recall that two metrics d and $\tilde{d}$ on a space X are equivalent if there is a constant C such that for all $x,y \in X$, we have $\frac{1}{C}\tilde{d}(x,y) \le d(x,y) \le C\tilde{d}(x,y)$, or equivalently, $\frac{1}{C}d(x,y) \le \tilde{d}(x,y) \le Cd(x,y)$.) This shows why geometric group theory is primarily suited to finitely generated groups.

Proposition 2.8. *For any two finite generating sets S and T of a finitely generated group G, the metrics d_S and d_T are equivalent.*

Proof. Let $C = \max(\{\ell_S(t) : t \in T\} \cup \{\ell_T(s) : s \in S\})$. Then, if $\sigma \in F_S$ is such that $\ell_S(\sigma) = d_S(g,h)$ for some $g,h \in G$ and $g\sigma = h$ in G, we can replace each entry s in σ with a string of length $\ell_T(s) \le C$ in T, and call this new string τ. Then $\ell_T(\tau) \le C\ell_S(\sigma)$ and $g\tau = h$ in G, so that $d_T(g,h) \le Cd_S(g,h)$. And similarly $d_S(g,h) \le Cd_T(g,h)$, and so d_S and d_T are equivalent. $\square$

3. Quasiisometries

Although the word metrics on a finitely generated group are all equivalent, as we have seen above the Cayley graphs of a such group need not be isomorphic as graphs (in other words, isometric when seen as metric spaces as in in Remark 2.5). However, there is a notion of equivalence weaker than graph isomorphism under which all Cayley graphs of the same group are equivalent. We give a definition of this notion which applies not only to graphs, but to metric spaces in general. This notion of equivalence will not require there to be a bijection (continuous or otherwise) between equivalent metric spaces. Intuitively, it sees only the features of the space which are visible from infinitely far away.

Definition 3.1. For metric spaces X and Y, a *quasiisometry* $f : X \to Y$ is a (not necessarily continuous) map such that for some constants $L \ge 1$ and $C, R \ge 0$, for all $w, x \in X$ we have

$$\frac{1}{L}d_X(w,x) - C \le d_Y(f(w), f(x)) \le Ld_X(w,x) + C,$$

and for all $y \in Y$, there is an $x \in X$ such that

$$d_Y(f(x), y) < R.$$

If there is a quasiisometry $f : X \to Y$, we say that X and Y are *quasiisometric*.

The following two propositions show why we do not require quasiisometries to be continuous, and why it is a useful relation when talking about word metrics.

Proposition 3.2. *Quasiisometry is an equivalence relation on metric spaces.*

Proof. It is clearly reflexive, since the identity map $\mathrm{id} : X \to X$ is a quasiisometry with constants $L = 1$, $C = 0$ and $R = \varepsilon > 0$. Also it is straightforward to check that if $f : X \to Y$ and $g : Y \to Z$ are quasiisometries, then so is $g \circ f : X \to Z$ (where the constants of $g \circ f$ depend on the constants for f and g), and so quasiisometry is transitive.

The only interesting condition is symmetry. To see that it is symmetric, suppose there is a quasiisometry $f : X \to Y$ with constants L, C, R as above. Define a map $g : Y \to X$ as follows: for any $y \in Y$, take some point $z \in f(X) \cap B_R(y)$ (which exists by the second condition of a quasiisometry). Then there is some $x \in X$ such that $f(x) = z$. Define $g(y) = x$. If we happen to have $y \in f(X)$, we should choose $z = y$, so that $f(x) = y$.

Now, we need to check that g is a quasiisometry. Since f is a quasiisometry, we have by definition that

$$\frac{1}{L} d_X(w, x) - C \leq d_Y(f(w), f(x)) \leq L d_X(w, x) + C$$

for any $w, x \in X$ and therefore, isolating $d_X(w, x)$ on each side,

$$\frac{1}{L} d_Y(f(w), f(x)) - \frac{C}{L} \leq d_X(w, x) \leq L d_Y(f(w), f(x)) + LC$$

Therefore, for any $y, z \in Y$, we have

$$(*) \quad \frac{1}{L} d_Y(f \circ g(y), f \circ g(z)) - \frac{C}{L} \leq d_X(g(y), g(z)) \leq L d_Y(f \circ g(y), f \circ g(z)) + LC$$

By our construction of g, we also have for any $y, z \in Y$ that,

$$d_Y(y, z) - 2R \leq d_Y(f \circ g(y), f \circ g(z)) \leq d_Y(y, z) + 2R$$

Therefore,

$$\frac{1}{L} d_Y(y, z) - \frac{1}{L}(C + 2R) \leq d_X(g(y), g(z)) \leq L d_Y(y, z) + L(C + 2R)$$

And, since $L > 1$, we can weaken the lower bound to

$$\frac{1}{L} d_Y(y, z) - L(C + 2R) \leq d_X(g(y), g(z)) \leq L d_Y(y, z) + L(C + 2R)$$

This shows that g satisfies the first condition for quasiisometry.

Now, for any $x \in X$, we have by the upper bound in $(*)$ that

$$\begin{aligned} d_X(x, g \circ f(x)) &\leq L d_Y(f(x), f \circ g \circ f(x)) + LC \\ &\leq L d_Y(f(x), f(x)) + LC \\ &\leq LC \end{aligned}$$

and so for any $x \in X$ there is a $y \in Y$ (for example $y = f(x)$) such that

$$d_X(g(y), x) < LC + \varepsilon$$

for any $\varepsilon > 0$.

Therefore, we see that g is a quasiisometry with constants L, $L(C + 2R)$, and $LC + \varepsilon$. Frequently, g is called the *quasiisometric inverse* or *coarse inverse* of f.

This shows that quasiisometry is symmetric, and therefore an equivalence relation. □

Proposition 3.3. *(i) For any finite generating set S of a fintely generated group G with the word metric d_S, G is quasiisometric to the Cayley graph $\Gamma(G, S)$.*

(ii) If S and T are two finite sets of generators for a finitely generated group G, then (G, d_S) and (G, d_T) are quasiisometric.

(iii) If S and T are two finite sets of generators for a finitely generated group G, then their corresponding Cayley graphs $\Gamma(G, S)$ and $\Gamma(G, T)$ are quasiisometric.

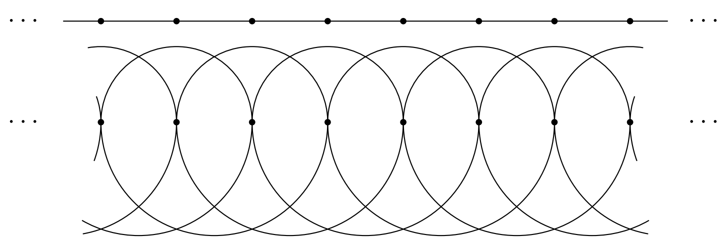
Proof. Since quasiisometry is an equivalence relation, (iii) follows from (i) and (ii).

(i) Let $\Gamma = \Gamma(G, S)$ and define a map $f : G \to \Gamma$ which sends each $g \in G$ to the corresponding vertex $g \in \Gamma$. Since the combinatorial distance between vertices in Γ agrees with word distance in G, we see that $d_S(g, h) = d_\Gamma(f(g), f(h))$ where d_Γ is the metric on Γ. Furthermore, any point on an edge in Γ is at most distance $\frac{1}{2}$ from a vertex. Therefore, $B_1(f(G)) = \Gamma$. So f is a quasiisometry (with constants 1,0,1) between G and Γ.

(ii) Let $\mathrm{id} : G \to G$ be the identity map on G. We have seen that all word metrics on G are equivalent in Proposition 2.8, so id must satisfy the first quasiisometric condition. And since id is surjective, it must also satisfy the second quasiisometric condition. Thus id is a quasiisometry (with constants $1, 0, \varepsilon$ for $\varepsilon > 0$) between (G, d_S) and (G, d_T). □

We now give straightforward examples of quasiisometric metric spaces.

Example 3.4. Consider the generating sets $\{1\}$ and $\{2, 3\}$ of the infinite cyclic group $\mathbb{Z}$. The Cayley graphs of $\mathbb{Z}$ with respect to these generators are shown below:



These graphs must be quasiisometric.

Example 3.5. The inclusion map $\mathbb{Z}^n \hookrightarrow \mathbb{R}^n$ is a quasiisometry.

Example 3.6. All bounded metric spaces (in particular all finite groups) are quasiisometric.

Given the previous three examples, one might think that the notion of quasiisometry is too weak to be of any use. After all, we have shown several spaces which look fairly different and yet are quasiisometric. However, we will see their usefulness in the remainder of this paper. In the next section, we will show connections between coarse geometry and differential geometry. First, we will discuss a relationship with group actions, which shows even more quasiisometry equivalences. We will then discuss a relationship with ends of metric spaces, which will at last give us a method of proving that two spaces are *not* quasiisometric. The following section will discuss growth functions, which provide another powerful way of showing that two spaces are not quasiisometric.

4. Group Actions and Quasiisometry

First we will recall some basic definitions.

Definitions 4.1. A metric space is *geodesic* if any two points have a path between them which minimizes the length between any two points on the path.

A metric space is *proper* if every closed ball is compact.

A group action $G \curvearrowright X$ on a metric space is *proper* if the map $G \times X \to X \times X$ defined by $(g, x) \mapsto (g \cdot x, x)$ is proper.

A group action $G \curvearrowright X$ on a metric space is *cocompact* if the quotient $G \backslash X$ is compact.

Theorem 4.2. *Let X be a metric space which is geodesic and proper. Let G be a group with an action $G \curvearrowright X$ by isometries which is proper and cocompact.*

Then G is finitely generated and quasiisometric to X. In particular, for any fixed $x \in X$, the map $g \mapsto g \cdot x$ is a quasiisometry.

Before proving this theorem, we will prove an easy corollary.

Corollary 4.3. *Let M be a compact Riemannian manifold. Then $\pi_1(M)$ is finitely generated and is quasiisometric to the universal cover $\tilde{M}$ of M.*

Proof. This follows immediately from the theorem using the usual action $\pi_1(M) \curvearrowright \tilde{M}$. □

Proof of Theorem 4.2. Let $\pi : X \to G \backslash X$ be the canonical projection, and define the metric $d_{G \backslash X}$ on $G \backslash X$ by

$$d_{G \backslash X}(p, q) = \inf\{d_X(x, y) : x \in \pi^{-1}(p), y \in \pi^{-1}(q)\}$$

It is straightforward to check that $d_{G \backslash X}$ is a metric (and this is where the requirement that X be proper is used).

Now, $G \backslash X$ is compact, and so has diameter $R < \infty$. Fix a point $x \in X$. Let $D \subseteq X$ be the closed ball of radius R centered at x. Then $\{g \cdot D\}_{g \in G}$ covers X. Let

$$S = \{g \in G \setminus \{1\} : D \cap g \cdot D \neq \emptyset\}.$$

Note that S is finite since the action is proper. We will show that S is a generating set for G, so that G is finitely generated. Assume that S is not all of $G \setminus \{1\}$, since otherwise there is nothing to show.

Because the action $G \curvearrowright X$ is proper, the set

$$\{g \in G \setminus (S \cup \{1\}) : d_X(D, g \cdot D) \leq C\}$$

is finite for any $C > 0$. There is some element h of $G \setminus (S \cup \{1\})$, and so for $C > d_X(D, h \cdot D)$, this set must be nonempty. Therefore, there is some $g \in G \setminus (S \cup \{1\})$ which minimizes $d_X(D, g \cdot D)$. Let μ be this minimum.

Let $g \in G \setminus (S \cup \{1\})$. Since X is geodesic, we can choose $y_0 = x, y_1, ..., y_n, y_{n+1} = g \cdot x$ in X with $d_X(y_k, y_{k+1}) < \mu$. Since $\{g \cdot D\}_{g \in G}$ covers X, we can choose $h_0 = 1, h_1, ..., h_n, h_{n+1} = g$ in G such that $y_k \in h_k \cdot D$ for each k. Set $s_k = h_k^{-1} h_{k+1}$ so that $g = s_0 \cdots s_n$. We have that

$$\begin{aligned} d_X(D, s_k \cdot D) &\leq d_X(h_k^{-1} y_k, s_k h_{k+1}^{-1} y_{k+1}) \\ &\leq d_X(y_k, y_{k+1}) \\ &< \mu \end{aligned}$$

Therefore $s_k \in S \cup \{1\}$. Thus, S generates G, so that G is finitely generated.

Now, we will show that the map $f : G \to X$ given by $f(g) = g \cdot x$ is a quasiisometry by slightly modifying the method above. When choosing $y_0, ..., y_{n+1}$, let n be the integer defined by

$$n\mu \leq d_X(x, g \cdot x) < (n+1)\mu$$

so that $y_0, ..., y_{n+1}$ uses as few steps as possible to get from x to $g \cdot x$. We then have

$$d_S(1, g) \leq n + 1 = \left\lceil \frac{1}{\mu} d_X(x, g \cdot x) \right\rceil \leq \frac{1}{\mu} d_X(x, g \cdot x) + 1$$

Also, if $M = \max_{s \in S} d_X(x, s \cdot x)$, then we have

$$\frac{1}{M} d_X(x, s \cdot x) \leq d_S(1, g)$$

From these two inequalities, for any $g, h \in G$ we have

$$\frac{1}{M} d_X(f(g), g(h)) \leq d_S(g, h) \leq \frac{1}{\mu} d_X(f(g), f(h)) + 1$$

by the left-invariance of the metrics d_X and d_S with respect to the actions $G \curvearrowright X$ and $G \curvearrowright G$ (by multiplication).

And by the definition of R, we have $B_{R+\varepsilon}(f(G)) = X$ for $\varepsilon > 0$.

Therefore, f is a quasiisometry (with constants $\max\left\{\frac{1}{\mu}, M\right\}, 1, R + \varepsilon$). □

An important group-theoretical fact about quasiisometries is that they cannot distinguish between a group, its finite index subgroups, and its quotients by finite groups. This is a generalization of the fact that all finite groups are quasiisometric. Moreover, this reflects our intuition that the quasiisometry relation only sees things which are visible from infinitely far away, since it it cannot see finite quotients and finite extensions.

Corollary 4.4. *(i) Let G be a finitely generated group and $H \leq G$ a finite index subgroup. Then H is finitely generated and quasiisometric to G.*

(ii) Given a short exact sequence of groups

$$1 \to N \to G \to H \to 1$$

(for example if $G = N \rtimes H$) with H finitely generated and N finite, then G is finitely generated and quasiisometric to H.

(iii) Let G be finitely generated and $N \trianglelefteq G$ a finite normal subgroup. Then G/N is finitely generated and quasiisometric to G.

Proof. Let Γ denote the Cayley graph of G and Δ denote the Cayley graph of H.

(i) This follows from Theorem 4.2 and the natural action $H \curvearrowright \Gamma$

(ii) This follows from Theorem 4.2 and the natural action $G \curvearrowright \Delta$

(iii) If G is finitely generated, then so is G/N, so we have a short exact sequence

$$1 \to N \to G \to G/N \to 1$$

and so (iii) follows from (ii). □

Example 4.5. It is a fact from group theory that F_2 contains a copy of F_n as a subgroup of finite index for any $2 \leq n < \infty$. Therefore all the F_n for $2 \leq n < \infty$ are quasiisometric (as are all the $2n$-ary trees).

Now we move to the number of ends of a space, which we will see is a quasiisometry invariant.

Definition 4.6. Let X be a proper geodesic metric space. Then the *number of ends of* X is the supremum over all compact subspaces $K \subseteq X$ of unbounded connected components of $X \setminus K$.

Remarks 4.7. This definition makes sense for arbitrary metric spaces, but we will only consider proper geodesic metric spaces.

Also, if $K \subseteq L \subseteq X$ with both K and L compact, then the number of unbounded connected components of $X \setminus L$ is at least the number of unbounded connected components of $X \setminus K$. In particular, the supremum may be taken over closed balls around some base point.

Definition 4.8. The *set of ends* $\epsilon(X)$ of a proper geodesic metric space X is the set of sequences $U_1 \supseteq U_2 \supseteq \cdots$ of unbounded connected components of complements $X \setminus K_k$ for some fixed exhaustion $K_1 \subseteq K_2 \subseteq \cdots$ of X by compact sets, where U_k is an unbounded connected component of $X \setminus K_k$.

Remark 4.9. $\#\epsilon(X)$ does not depend on the choice of exhaustion, and is equal to the number of ends of X, but we will not prove this here.

Lemma 4.10. *The number of ends is quasiisometry-invariant.*

Proof. Let X and Y be proper geodesic metric spaces, and let $f : X \to Y$ be a quasiisometry with constants L, C, R. Fix basepoints $x \in X$ and $y = f(x) \in Y$. Note that in a geodesic space connectedness is equivalent to path-connectedness.

Let $D = \overline{B}_r(x) \subseteq X$ be the closed ball of radius r around x in X, and suppose $X \setminus D$ has at least n connected components for some integer n. Now, we would like to have (analogous to the case where f is an isometry) that f takes the unbounded connected components of $X \setminus D$ to the unbounded connected components of $Y \setminus f(D)$, but this is not quite true. Instead, we will show that $f : X \to Y$ induces an bijective map $f^* : \epsilon(X) \to \epsilon(Y)$

We cannot hope that f takes connected sets to connected sets because it is not necessarily continuous. However, if $d_X(v, w) < 1$, then $d_Y(f(v), f(w)) < L + C$. So $B_{L+C}(f(A))$ will be connected when A is connected, since for points $a, b \in A$ there is a sequence $c_0 = a, c_1, ..., c_{k-1}, c_k = b$ with $d_X(c_i, c_{i+1}) < 1$ and thus $d_Y(f(c_i), f(c_{i+1})) < L+C$, so we can connect c_i to c_{i+1} in $B_{L+C}(f(A))$ by geodesics.

Fix an exhaustion $\{K_k\}$ of X by compact sets such that $B_1(K_k) \subseteq K_{k+1}$. Now, $\overline{f(K_k)}$ is compact because proper metric spaces have the Heine-Borel property. And since f is a quasiisometry, $\{\overline{f(K_k)}\}$ is an exhaustion of Y. Moreover $B_{L+C}(\overline{f(K_k)}) \subseteq K_{k+1}$. Let $\epsilon(X)$ and $\epsilon(Y)$ be the sets of ends of X and Y with respect to these exhaustions.

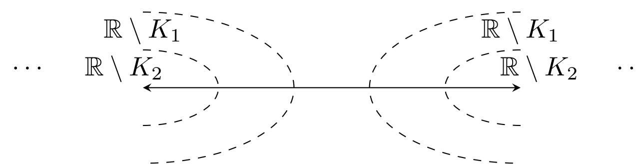
Now, define a map $f^* : \epsilon(X) \to \epsilon(Y)$ as follows. Then for any $\eta = \{U_1 \subseteq U_2 \subseteq \cdots\} \in \epsilon(X)$, we know that $B_{L+C}(f(U_{k+1}))$ is unbounded, connected, and contained in some unbounded connected component V_k of $Y \setminus \overline{f(K_k)}$. So define $f^*(\eta) = \{V_1 \subseteq V_2 \subseteq \cdots\} \in \epsilon(Y)$.

It is straightforward to check that if g is the coarse inverse of f, then $g^* \circ f^*(\eta) = \eta$. Therefore $f^* : \epsilon(X) \to \epsilon(Y)$ is a bijection with inverse g^*. So X and Y have the same number of ends. $\square$

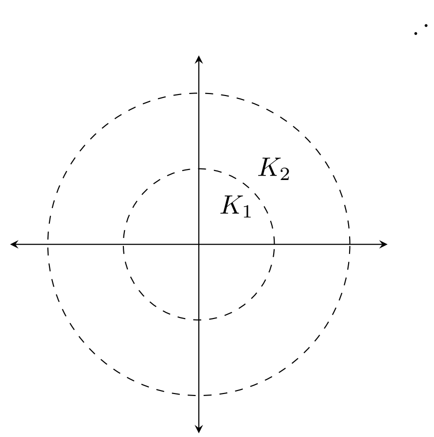
This lemma gives a reason for the intiution that the quasiisometry relation sees the features of a metric space which are visible from infinitely far away. In view of this lemma, we can define the number of ends of a finitely generated group by declaring it to be the number of ends of its Cayley graph.

We can now show that some groups are *not* quasiisometric!

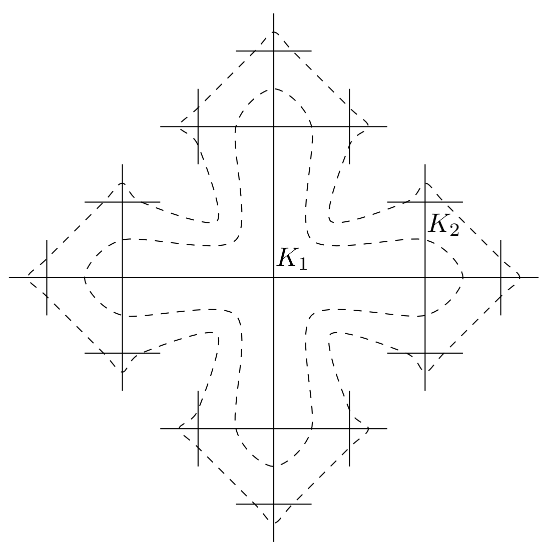
Example 4.11. $\mathbb{Z}$, $\mathbb{Z}^2$, and F_2 are pairwise *not* quasiisometric (we have already seen that F_2 is quasiisometric to every F_n for $2 \leq n < \infty$). This is because they are quasiisometric to $\mathbb{R}$, $\mathbb{R}^2$ and T_4 (the 4-ary tree), respectively, and these spaces have two, one, and infinitely many ends, respectively. We show pictures of exhaustions of these spaces by closed balls.



Every complement $\mathbb{R} \setminus K_k$ of this exhaustion has 2 unbounded connected components, so $\#\epsilon(\mathbb{R}) = 2$.



Every complement $\mathbb{R}^2 \setminus K_k$ of this exhaustion has 1 unbounded connected component, so $\#\epsilon(\mathbb{R}^2) = 1$.



Every complement $T_4 \setminus K_k$ of this exhaustion has $4 \cdot 3^k$ unbounded connected components, so $\#\epsilon(T_4) = \sup_{k\in\mathbb{N}}\{4 \cdot 3^k\} = \infty$.

Now, it happens that $\mathbb{Z}^n$ and $\mathbb{Z}^m$ are *not* quasiisometric when $n \neq m$. But they both have one end, and so we cannot distinguish them by Lemma 4.10. However, in the next section we will see how they can be distingushed.

5. Growth Functions

It is a fact from Riemannian geometry that in Euclidean space, the volume of balls $\mathrm{vol}B_r(x)$ grows polynomially in r, while in hyperbolic space, the volume of balls $\mathrm{vol}B_r(x)$ grows exponentially in r. In this section we will analyze finitely generated groups in a similar fashion.

Definitions 5.1. Let G be a finitely generated group with finite generating set S. We define the *length* $\ell_S(g)$ of an element $g \in G$ (with respect to S) to be $d_S(1, g)$.

We also define the *growth function of* G by $\beta(k) = \#\{g \in G : \ell_S(g) \leq k\} = \#\overline{B}_k(1)$.

Now, we might hope that if groups G and H are quasiisometric, then their respective growth functions would be asymptotically equivalent. However, F_2 and F_3 are quasiisometric, but the growth function for F_2 is $4 \cdot 3^k$ while the growth function for F_3 is $6 \cdot 5^k$, which are not asymptotically equivalent. However, there is a weaker equivalence relation on functions which is quasiisometry-invariant.

Definition 5.2. Let X be an ordered set (e.g. $\mathbb{N}$ or $\mathbb{R}$). Given functions $f, g : X \to \mathbb{R}$, we say that $f \preceq g$ if there are $\lambda, \mu > 0$ such that for all $x \in X$, we have $\mu x \in X$ and

$$f(x) \leq \lambda g(\mu x).$$

If $f \preceq g$ and $g \preceq f$, we write $f \asymp g$.

Remarks 5.3. It is easy to check that $\preceq$ is transitive, and then that $\asymp$ is an equivalence relation.

Also note that usual asymptotic equivalence $f \sim g$ entails $f \asymp g$.

Proposition 5.4. *If G and H are quasiisometric finitely generated groups, then their respective growth functions β_G and β_H satisfy $\beta_G \asymp \beta_H$.*

Proof. Let $f : G \to H$ be a quasiisometry with constants L, C, R and let $g : H \to G$ be its coarse inverse. Let $\ell = \max\{\ell_H(f(1_G)), \ell_G(g(1_H))\}$. Then we have

$$f(\overline{B}_k(1_G)) \subseteq \overline{B}_{Lk+\ell}(1_H) \quad \text{and} \quad g(\overline{B}_k(1_H)) \subseteq \overline{B}_{Lk+\ell}(1_G)$$

Also, if $f(a) = f(b)$, then $d_G(a, b) \leq LC$, so at most $\beta_G(LC)$ elements of G can map under f to a given element of H. And similarly at most $\beta_H(LC)$ elements of H can map under g to a given element of G. Let $M = \max\{\beta_G(LC), \beta_H(LC)\}$. Therefore, we have

$$\beta_G(k) = \#\overline{B}_k(1_G) \leq M\#\overline{B}_k(1_H) = M\beta_H(k)$$

and

$$\beta_H(k) = \#\overline{B}_k(1_H) \leq M\#\overline{B}_k(1_G) = M\beta_G(k)$$

So $\beta_G \asymp \beta_H$. ∎

There are many equivalent definitions of the relation $\preceq$ when the functions involved are growth functions of infinite finitely generated groups. These are given by the following proposition, which we will not prove here.

Proposition 5.5. *Let G and H be infinite finitely-generated groups with growth functions β and γ, respectively. Then the following are equivalent:*
(i) $\beta \preceq \gamma$
(ii) There is a constant λ such that $\beta(k) \leq \lambda\gamma(\lambda k)$ for all k.
(iii) There are constants λ and C such that $\beta(k) \leq \lambda\gamma(\lambda k + C) + C$ for all k.
(iv) There is a constant λ such that $\beta(k) \leq \beta(\lambda k)$ for all k.

Remark 5.6. Note that (i) and (ii) are equivalent for any nonnegative functions β and γ.

Moreover, (i), (ii), and (iii) are all equivalent as long as β is nonnegative and γ is at least 1 for k sufficiently large.

Only the equivalence of (iv) relies on the fact that G and H are infinite finitely generated groups.

Now, we state some immediate properties of growth functions of finitely generated groups.

Proposition 5.7. *Let G be a finitely generated group, let ℓ be the length function and β be the growth function.*
(i) ℓ is symmetric and subadditive, i.e.

$$\ell(g^{-1}) = \ell(g)$$
$$\ell(gh) \leq \ell(g) + \ell(h)$$

(ii) β is nondecreasing. And, if G is infinite, then it is strictly increasing.
(iii) β is submultiplicative, i.e.

$$\beta(k + j) \leq \beta(k)\beta(j)$$

(iv) If G has an n-element generating set (i.e. is a quotient of F_n), then

$$\beta(k) \leq 2n(2n-1)^{k-1}$$

since $2n(2n-1)^{k-1}$ is the growth function of F_n.

The connection with Riemannian geometry which we saw before extends naturally to growth functions.

Theorem 5.8 (Schwarzc-Milnor)**.** *Let X be a complete Riemannian manifold with fixed basepoint x, and let G be a group which acts properly and cocompactly by isometries on X. By Theorem 4.2, G is finitely generated and quasiisometric to X. Let β be a growth function for G. Then $\beta(k) \asymp \mathrm{vol}B_k(x)$.*

Remark 5.9. By Remark 5.6, $\beta(k)$ and $\mathrm{vol}B_k(p)$ will satisfy all of (i), (ii), and (iii) from Proposition 5.5, but they may not satisfy (iv).

Proof. We use notation similar to that in Theorem 4.2:
$R < \infty$ is the diameter of $G\backslash X$.
$D \subseteq X$ is the closed ball of radius R centered at x, and $\{g \cdot D\}_{g\in G}$ covers X.
$S = \{g \in G \setminus \{1\} : D \cap g \cdot D \neq \emptyset\}$ is a finite set of generators of G.
$\mu = \min_{g\in G\setminus(S\cup\{1\})} d_X(D, g \cdot D)$
$M = \max_{s\in S} d_X(x, s \cdot x)$

Additionally, let a be the order of $G_x = \{g \in G : g \cdot x = x\}$ in G. Note that for any $g \in G_x$, we have $x \in D \cap g \cdot D$, so $g \in S$. Therefore G_x is finite because it is contained in the finite set S.

Note that for $y \in G \cdot x$, the balls $\overline{B}_{\frac{\mu}{3}}(y)$ are pairwise disjoint. Therefore, we have

$$\frac{1}{a}\beta(k)\mathrm{vol}\overline{B}_{\frac{\mu}{3}}(x) \leq \mathrm{vol}\overline{B}_{kM+\frac{\mu}{3}}(x)$$

Thus $\beta(k) \preceq \mathrm{vol}\overline{B}_k(x) = \mathrm{vol}B_k(x)$.

Let $y \in B_k(x)$. There is a $g \in G$ such that $y \in \overline{B}_R(g \cdot x)$. Recall from the proof of Theorem 4.2 that

$$d_S(1, g) \leq \frac{1}{\mu} d_X(x, g \cdot x) + 1$$

which implies that

$$d_S(1, g) \leq \frac{1}{\mu} d_X(x, y) + C \leq \frac{k}{\mu} + C$$

for some C Therefore $\overline{B}_k(x)$ is covered by $h \cdot D$ where h satisfies $d_S(1, h) \leq \frac{1}{\mu} d_X(x, y) + C$. So,

$$\mathrm{vol}\overline{B}_k(x) \leq \beta\left(\frac{k}{\mu} + C\right)\mathrm{vol}D$$

So $\mathrm{vol}\overline{B}_k(x) = \mathrm{vol}B_k(x) \preceq \beta(k)$. And thus, we have

$$\beta(k) \asymp \mathrm{vol}B_k(x)$$

□

Proposition 5.10. *$\mathbb{Z}^n$ and $\mathbb{Z}^m$ are not quasiisometric when $n \neq m$.*

Proof. The growth function of $\mathbb{Z}^n$ for the usual generating set

$$\{(1, 0, ..., 0), ..., (0, ..., 0, 1)\}$$

is asymptotically equivalent $C_n k^n$ for some C_n. It is easy to see that $C_n k^n \not\asymp C_m k^m$ when $n \neq m$. Therefore $\mathbb{Z}^n$ and $\mathbb{Z}^m$ are not quasiisometric when $n \neq m$. □

Corollary 5.11. *We thus have a complete classification of F_n and $\mathbb{Z}^n$ up to quasi-isometry into classes:*

$$\begin{array}{ccc} \{F_1, \mathbb{Z}\} & \{F_2, F_3, ...\} & \\ \{\mathbb{Z}^2\} & \{\mathbb{Z}^3\} & \cdots \end{array}$$

6. Growth Types

Definitions 6.1. The *exponential growth rate* of a finitely generated group is

$$\omega = \omega(G, S) = \limsup_{k \to \infty} \sqrt[k]{\beta(k)}$$

We say that G is of

polynomial growth if $\beta(k) \preceq k^d$ for some $d \in \mathbb{N}$
subexponential growth if $\omega = 1$
intermediate growth if it is of subexponential but not polynomial growth
exponential growth if $\omega > 1$

Remark 6.2. Let $(a_k)_{k\in\mathbb{N}}, (b_k)_{k\in\mathbb{N}}$ be nondecreasing sequences of positive real numbers and $\omega = \limsup_{k\to\infty} \sqrt[k]{a_k}$ and $\psi = \limsup_{k\to\infty} \sqrt[k]{b_k}$. If $a_k \preceq b_k$, then $\omega > 1 \Rightarrow \psi > 1$.

Therefore the properties of being of exponential, subexponential, polynomial, and intermediate growth are invariant under quasiisometry.

Remark 6.3. If $H = \langle T\rangle \leq G = \langle S\rangle$ with $T = S \cap H$, then $\omega(H, T) \leq \omega(G, S)$. So, if G has a subgroup of exponential growth, then it is of exponential growth.

If $H \trianglelefteq G = \langle S\rangle$ and $\pi : G \to G/H$ is the canonical projection, then $G/H = \langle \pi(S)\rangle$ and $\omega(G/H, \pi(S)) \leq \omega(G, S)$. So, if G has a quotient of exponential growth, then it is of exponential growth.

Remark 6.4. By Proposition 5.7(iv), we always have

$$1 \leq \omega \leq 2n - 1$$

if G is a finitely generated group with a generating set of size n.

Examples 6.5. F_n is of exponential growth.

$\mathbb{Z}^n$ is of polynomial growth.

Groups of intermediate growth are more difficult to come by. The first was found by Grigorchuk. We will describe this group, but we will not prove that it is of intermediate growth.

Let T be the binary tree, whose elements we think of as finite strings with entries in $\{\pm 1\}$. The Grigorchuk group is a subgroup of the automorphism group of T. Define $a(i_0, i_1, ..., i_k) = (-i_0, i_1, ..., i_k)$. Also define recursively

$$b(i_0, i_1, i_2, ..., i_k) = \begin{cases} (0, -i_1, i_2, ..., i_k) & i_0 = 0 \\ (1, c(i_1, i_2, ..., i_k)) & i_0 = 1 \end{cases}$$

$$c(i_0, i_1, i_2, ..., i_k) = \begin{cases} (0, -i_1, i_2, ..., i_k) & i_0 = 0 \\ (1, d(i_1, i_2, ..., i_k)) & i_0 = 1 \end{cases}$$

$$d(i_0, i_1, i_2, ..., i_k) = \begin{cases} (0, i_1, i_2, ..., i_k) & i_0 = 0 \\ (1, b(i_1, i_2, ..., i_k)) & i_1 = 1 \end{cases}$$

The Grigorchuk group is then $G = \langle a, b, c, d\rangle$. A proof that this group is of intermediate growth, along with some additional exposition about this group can be found in [2], Ch. VIII.

7. Exponential Growth

We can use the growth of subgroups and subsemigroups to obtain an easy lower bound on the growth of a group. For example, we have the following proposition:

Proposition 7.1. *A finitely generated group which has a free subsemigroup on two generators is of exponential growth.*

Proof. The growth function of the free semigroup on two generators is given by $2^{k+1} - 1$ (with respect to the generating set consisting of the two generators). We can then choose a finite set of generators for the original group which contains a free set of generators for the free subsemigroup. So, the growth function of the original group must be $\geq 2^{k+1} - 1$. Thus the original group must be of exponential growth. $\square$

Proposition 7.2. *Let X be a set and $G \curvearrowright X$. If there are $g, h \in G$ and disjoint $A, B \subseteq X$ such that*

$$g \cdot (A \cup B) \subseteq A \quad h \cdot (A \cup B) \subseteq B$$

then the semigroup generated by g and h is free. In particular, G is of exponential growth.

Proof. Let SF_2 be the free semigroup on two generators a, b, and define a semigroup homomorphism $\varphi : \mathrm{SF}_2 \to X$ which sends

$$a \overset{\varphi}{\mapsto} g \quad b \overset{\varphi}{\mapsto} h$$

Let $\sigma, \tau \in \mathrm{SF}_2$ be such that $\varphi(\sigma) = \varphi(\tau)$.

If $\sigma = 1_{\mathrm{SF}_2}$, then τ cannot begin with a since this would imply $\varphi(\tau)(B) \subseteq A$, contradicting $\varphi(\sigma)(B) = B$. Similarly τ cannot begine with b, so we must also have $\tau = 1_{\mathrm{SF}_2}$.

Now assume neither σ nor τ is 1_{SF_2}. They are thus of the form

$$\sigma = s \star \rho \quad \tau = t \star \upsilon$$

for $s, t \in \{a, b\}$ and $\rho, \upsilon \in \mathrm{SF}_2$. The actions of $\varphi(\sigma)$ and $\varphi(\tau)$ on A and B show that $s = t$. And since φ is a semigroup homomorphism, we must then have $\varphi(\rho) = \varphi(\upsilon)$.

Then, by induction, $\sigma = \tau$. So φ is injective and $\varphi(\mathrm{SF}_2)$ is a free subsemigroup of G with two generators. And by the previous proposition G must have exponential growth. $\square$

Example 7.3.

$$\left\langle \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \right\rangle$$

is of exponential growth.

Proof. We can easily see that

$$G = \left\langle \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \right\rangle = \left\{ \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} : a = 2^n,\ n \in \mathbb{Z},\ b \in \mathbb{Z}\left[\frac{1}{2}\right] \right\}$$

We can define an action $G \curvearrowright \mathbb{R}$ by

$$\begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} \cdot x = ax + b$$

Note that if $a \neq 1$, the matrix $\begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix}$ has a unique fixed point $\frac{b}{1-a}$. Choose a, b, c, d such that $a, c < 1$ and $\frac{b}{1-a} \neq \frac{d}{1-c}$ and let

$$A = \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} \quad B = \begin{pmatrix} c & d \\ 0 & 1 \end{pmatrix}$$

Choose disjoint intervals $I, J \subseteq \mathbb{R}$ such that $\frac{b}{1-a} \in I$ and $\frac{d}{1-c} \in J$. Let $H \subseteq \mathbb{R}$ be an interval containing $I \cup J$.

For large enough k, ℓ, we have that $A^k(H) \subseteq I$ and $B^\ell(H) \subseteq J$ since $a, c < 1$. Therefore, by the previous proposition G is of exponential growth. $\square$

In the previous section, we showed that being of exponential growth is invariant under quasiisometry. However, if G is a finitely generated group of exponential growth, we can define $\omega_{\inf}(G) = \inf \omega(G, S)$ where the inf is taken over all finite generating sets. We may then wonder whether $\omega_{\inf} > 1$.

Definition 7.4. G is of *uniformly exponential growth* if $\omega_{\text{inf}} > 1$.

Example 7.5. F_n for $n \geq 2$ is of uniformly exponential growth, and in fact $\omega_{\text{inf}} = 2k - 1$

Proof. Let S be a finite generating set for F_n. And let $\pi : F_n \to F_n^{\text{ab}} = \mathbb{Z}^n$ be the abelianization projection. Note that $\mathbb{Z}^n = \langle \pi(S) \rangle$. There is then a subset $R \subseteq \pi(S)$ of size n which generates a subgroup of finite index in $\mathbb{Z}^n$. Let $T \subseteq S$ be a subset which projects bijectively onto R. The subgroup $\langle T \rangle \leq F_n$ is free (since it is a subgroup of a free group), of rank at most n (since $\#T = n$), and of rank at least n (since $\langle T \rangle^{\text{ab}} = \pi(\langle T \rangle) = \langle \pi(T) \rangle$ is isomorphic to $\mathbb{Z}^n$ and so has rank n). Therefore $\langle T \rangle$ is isomorphic to F_n, and consequently $\omega(F_n, S) \geq \omega(\langle T \rangle, T) = 2k - 1$. □

Uniform exponential growth behaves similarly to other kinds of growth with respect to subgroups and quotients, which we will state but not prove.

Proposition 7.6. *Let G be a finitely generated group.*

(i) If G has a subgroup of finite index with uniformly exponential growth, then G is of uniformly exponential growth.

(ii) If G has a quotient of uniformly exponential growth, then G is of uniformly exponential growth.

(iii) If G has a subgroup of finite index which has a nonabelian free quotient, then G has uniformly exponential growth.

In [3], Wilson gave the first example of a group of exponential but not uniformly exponential growth. Still open is the question of whether uniformly exponential growth is invariant under quasiisometry.

Question 7.7. *If G, H are quasiisometric finitely generated groups of exponential growth, and G is of uniformly exponential growth, is H also of uniformly exponential growth? In other words,*

$$\omega_{\text{inf}}(G) > 1 \Rightarrow \omega_{\text{inf}}(H) > 1?$$

8. Additional Topics: Growth Series and Følner Sequences

In this section, we will present some additional variants and generalizations of the growth of groups. The first are given by the following definitions:

Definition 8.1. Given a finitely generated group G with finite generating set S, the *spherical growth function* $\sigma(k)$ is defined to be

$$\sigma(k) = \#\{g \in G : d_S(1, g) = k\}$$

Unlike the usual growth function, the spherical growth function is not necessarily monotone.

Example 8.2. Let $G = \langle s, t | s^3, t^3, (st)^3 \rangle$. This group can be viewed as the orientation preserving isometries in the group generated by the reflections defined by the sides of an equilateral triangle in the plane. The spherical growth function for G is given by

$$\sigma(k) = \begin{cases} 1 & k = 0 \\ 4 & k = 1 \\ 10k - 2 & k \text{ even and } \geq 2 \\ 8k - 2 & k \text{ odd and } \geq 3 \end{cases}$$

which is not monotone. For example, $\sigma(10) = 48$ which $\sigma(11) = 46$.

Definition 8.3. The growth series of G is given by

$$\mathrm{B}(z) = \sum_{k=0}^{\infty} \beta(k) z^k \in \mathbb{Z}[[z]]$$

and the spherical growth series of G is given by

$$\Sigma(z) = \sum_{k=0}^{\infty} \sigma(k) z^k \in \mathbb{Z}[[z]]$$

Note that ω is the radius of convergence of B. It is clear that in general $\Sigma(z) = (1-z)\mathrm{B}(z)$. It is not well understood which properties of a group are captured by its growth series.

Now, we move on to Følner sequences.

Definition 8.4. Given a subset $A \subseteq G$, the *S-boundary* of A is

$$\partial_S A = \{sa : s \in S, a \in A\} \setminus A$$

Definition 8.5. A *Følner sequence* in G is a sequence $\{A_k\}_{k\in\mathbb{N}}$ of finite subsets such that

$$\lim_{k\to\infty} \frac{\#(A_k \cup \partial_S A_k)}{\# A_k} = 1$$

It is easy to check that whether a sequence $\{A_k\}_{k\in\mathbb{N}}$ is Følner does not depend on the finite generating set. Moreover, we have the following proposition, which we will not prove.

Proposition 8.6. *(i) If G is of polynomial growth, then $\{B_k(1)\}$ is a Følner sequence.*

(ii) If G is of intermediate growth, then some subsequence of $\{B_k(1)\}$ is a Følner sequence.

(iii) If G is of exponential growth, then no subsequence of $\{B_k(1)\}$ is a Følner sequence.

9. Conclusion

We began by introducing natural metrics on finitely generated groups and their Cayley graphs, and defining the notion of quasiisometry, which allows us to consider all the Cayley graphs for different finitely generated presentations of a group as the same. We then saw that a finitely generated group is quasiisometric to a (geodesic, proper) metric space on which it acts (properly and cocompactly) by isometries. This allowed us to draw a connection with Riemannian geometry, via the action of the fundamental group on the universal cover of a Riemannian manifold. Furthermore, we showed that quasiisometry does not distinguish between a group and its finite index subgroups or its quotients by finite normal subgroups. We then arrived at the notion of a number of ends of a metric space, saw that it was a quasiisometry invariant and used it to distinguish certain groups and spaces under quasiisometry.

Motivated by the connection with Riemannian geometry, we investigated the growth of balls around the identity. We saw that the usual asymptotic equivalence is not quasiisometry invariant, but the weaker relation $\asymp$ is. We saw that we can classify the growth rate of groups into polynomial, intermediate, and exponential growth, and that the growth rate can be used to distinguish between groups under quasiisometry. We focused in particular on exponential growth, and saw ways to identify groups of exponential growth by examining their subgroups and actions. Finally, we introduced the notion of uniformly exponential growth, and presented the open question of whether this property is quasiisometry invariant.

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