THE WEIERSTRASS PREPARATION THEOREM AND SOME APPLICATIONS

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Abstract. In this paper we revisit the Weierstrass preparation theorem, which describes how to represent a holomorphic function of several complex variables that vanishes at some point as the product of a nonvanishing analytic function and a polynomial in a neighborhood of the given point. After giving the statement and proof of the theorem, we show how it is related to the implicit function theorem.

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1. Introduction

The Weierstrass preparation theorem is an important theorem regarding the local form of a holomorphic function of several complex variables at a given point. In the ring of germs of holomorphic functions at a point, the theorem states that such a function is equivalent, up to a unit, to a Weierstrass polynomial. A related and equivalent theorem is the Weierstrass division theorem, which defines the division of germs by Weierstrass polynomial.

The Weierstrass preparation theorem has many applications. For instance, it can prove that the ring of germs of analytic functions in n variables is a Noetherian ring. For simplicity, this paper applies the idea of proving Weierstrass preparation theorem to the proof of the implicit function theorem.

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2. Holomorphic functions

2.1. **Holomorphic functions of one variable.**

Consider a function $f : D \mapsto \mathbb{C}$ defined on an open set $D \subset \mathbb{C}$.

Definition 2.1. A function f is said to be holomorphic in D if f is differentiable at every point in D. That is, for all $z \in D$,

$$f'(z) := \lim_{h \to 0} \frac{f(z+h) - f(z)}{h} \tag{2.2}$$

exists.

Definition 2.3. A function f is said to be analytic if for all z_0 in D, one can write

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n \tag{2.4}$$

and the series converges to $f(z)$ in a neighborhood of z_0.

Remark 2.5. A standard result is that a function is complex holomorphic if and only if f is complex analytic.
see Lang [7]

2.2. **Holomorphic functions of several variables.**

Definition 2.6. A polydisc is a Cartesian product of open discs. Let $D(a, r)$ denote an open disc centered at a with radius r in the complex plane, an open polydisc is a set of the form

$$P := D(a_1, r_1) \times \cdots \times D(a_m, r_m), \tag{2.7}$$

or equivalently,

$$P := \{z = (z_1, \cdots, z_m) \in \mathbb{C}^m : |z_k - a_k| < r_k, k = 1, \cdots m\}. \tag{2.8}$$

Definition 2.9. A function f defined on a domain $U \subset \mathbb{C}^m$ is said to be holomorphic on U if for every point $a \in U$, there exists an open polydisc $P \subset U$ centered at a, and

$$f(z) = \sum_{j_1, \cdots, j_m} a_{j_1, \cdots, j_m} (z_1 - a_1)^{j_1} \cdots (z_m - a_m)^{j_m}, \tag{2.10}$$

for all z in P.

3. Germs of holomorphic functions

Definition 3.1. Consider a point x_0 of a topological space X, any set Y, and two maps $f : X \mapsto Y$, $g : X \mapsto Y$. Then f and g define the same germ at x_0 if $f(x) = g(x)$ for all x in a neighborhood U of x_0. If $S \subset X$ and $T \subset X$, then the same germ at x is defined if there is a neighborhood U of x such that $S \cap U = T \cap U$.

Let S be a subset of $\mathbb{C}^m$, and f, g be two holomorphic functions defined on open sets U and V in $\mathbb{C}^m$ respectively. Consider the set pairs (U, f) and (V, g). We define the equivalence relation $\sim$ as $(U, f) \sim (V, g)$ if and only if $f = g$ on an open set $W, W \subset U \cap V$, and W containing S.

Suppose S contains a single point z. Two holomorphic functions are equal in a neighborhood of z if and only if they have the same series expansion at z. We denote the germs of holomorphic functions at z by $\mathcal{H}_z^m$.

4. The Weierstrass preparation theorem

An analytic function f with one complex variable can be represented locally as a convergent power series. This was proved by Cauchy in 1830-1840. Suppose f has a zero of order m at z_0, and its series representation is

$$\sum_{k=1}^{\infty} \frac{f^k(z_0)}{k!}(z-z_0)^k. \tag{4.1}$$

Since the first m terms in the series vanish and thus

$$f = \sum_{k=m}^{\infty} \frac{f^k(z_0)}{k!}(z-z_0)^k = (z-z_0)^m g(z) \tag{4.2}$$

where $g(z)$ is analytic and nonzero at z_0. Weierstrass preparation theorem generalizes this result in higher dimensional space.

For the rest of the paper, we denote the origin in $\mathbb{C}^m$ by 0, i.e. $0 = (0_1, 0_2, \cdots, 0_m)$.

Definition 4.3. A Weierstrass polynomial near 0 is a function P of m complex variables z_1, $\cdots$, z_{m-1}, w, defined in a neighborhood of 0, with the form

$$P(z', w) = w^d + a_{d-1}(z')w^{d-1} + \cdots + a_0(z') \tag{4.4}$$

where $z' = (z_1, \cdots, z_{m-1})$ and d is a non-negative integer. Each $a_i(z')$ is a holomorphic function defined on a neighborhood of $0'$ such that $a_i(0') = 0$ where $0' = (0_1, \cdots, 0_{m-1})$.

4.1. **The Weierstrass preparation theorem.**

Lemma 4.5. *(Argument principal) Let γ be an orientated simple closed curve. Let $U \subset \mathbb{C}^m$ be an open set containing γ, such that f is analytic on U except for finitely many points. Then,*

$$\int_\gamma \frac{f'(z)}{f(z)} dz = 2\pi i(N-P) \tag{4.6}$$

where N and P denote respectively the number of zeros and poles of f(z) inside the contour γ and both counted with multiplicity. (This is a consequence of the residue theorem)

Theorem 4.7. *Given an element $f \in \mathcal{H}_z^m$, suppose $f \not\equiv 0$ in a neighborhood of 0. Then:*

1. *We can choose a basis of $\mathbb{C}^m$ such that $f(0', z_m)$ does not vanish identically in any neighborhood of $z_m = 0$.*
2. *If the basis of $\mathbb{C}^m$ satisfy $f(0', z_m) \not\equiv 0$ in any neighborhood of $z_m = 0$ in $\mathbb{C}$, then there exist an unit $u \in \mathcal{H}_z^m$ and a Weierstrass polynomial g of degree p such that*

$$f \sim ug, \quad g = (z_m)^p + \sum_{k=1}^{p} c_k(z')(z_m)^{p-k}. \tag{4.8}$$

(3) *And if, with respect to the same basis, there exist h such that $f \sim vh$,*

$$h = (z_m)^q + \sum_{k=1}^{m} d_k(z')(z_m)^{q-k}, \tag{4.9}$$

then $p = q$, and c_k and d_k induce the same element of $\mathcal{H}_0^m$. In other words, an representation of f in this form is unique.

Proof.

(1) Let U be an open convex neighborhood of 0 and f be a holomorphic function defined on U. Since $f \not\equiv 0$, there is a point $a \in U(a \neq 0)$, such that $f(a) \neq 0$. If we choose any basis of $\mathbb{C}^m$ such that the m^{th} element is a, then $f(0', z_m)$ does not vanish identically on the connected open set $z_m \in \mathbb{C} : (0, z_m) \in U$ (f is holomorphic on that set).

(2) As a consequence of (1), now we may suppose that f does not vanish identically to 0 on any neighborhood of 0. Let P be a polydisc with center at 0 and radius r such that f is defined and holomorphic on P. Then there exists a number r_1, $0 < r_1 < r$, such that $f(0', z_m) \neq 0$ if $0 < |z_m| < r_1$. Since f is holomorphic on P, we can define an open polydisc P' in $\mathbb{C}^{m-1}$ with center $0'$ and radius $< r$, such that $f(z', z_m) \neq 0$ for $z' \in P'$ and $|z_m| = r_1$. Hence, $f(z', z_m)$ is nonzero and holomorphic in a neighborhood of the tube $V = (z', z_m) : z' \in P', |z| = r_1$. Notice that the r_1 and the radius of P' can be chosen arbitrarily small. Let γ be a circular curve $|w| = r_1$ oriented in positive direction in $\mathbb{C}$. For z' in P',

$$c_0(z') = \frac{1}{2\pi i} \int_\gamma \frac{\partial f(z', w)}{\partial w} \frac{dw}{f(z', w)}, \tag{4.10}$$

which counts the zeros of $f(z', w)$ in $(z', w) : |w| < r_1$. Since $f(z', w)$ is holomorphic in $|w| < r_1$ and does not vanish on the curve γ, c_0 is integer valued and continuous on P'. Suppose $c_0(0') = n$ say, then p is the number of zeros counted with multiplicity for $f(z', w)$ in $|w| < r_1$. For z' in P', let $b_1(z'), \cdots, b_p(z')$ be the zeros of $f(z', w)$ counted with multiplicity. For $k \geq 1$, let

$$c_k(z') = b_1^k(z') + \cdots + b_p^k(z'), z' \in \mathcal{P}'. \tag{4.11}$$

And

$$c_k(z') = \frac{1}{2\pi i} \int_\gamma b^k \frac{\partial f(z', w)}{\partial w} \frac{dw}{f(z', w)} \tag{4.12}$$

shows that all the c_k's are holomorphic functions on P'.

$$|w| < r_1, \; f(0', w) = 0 \quad \Rightarrow \quad w = 0 \quad \Rightarrow \quad b_i(0') = 0, \tag{4.13}$$

for $i = (1, \cdots, p)$. Let $a_i(z')$ be the symmetric polynomial in $b_i(z'), a_i(0') = 0$. Set

$$g(z', w) = \prod_{i=1}^{p} (w - b_i(z')) = w^p + a_{p-1}(z')w^{p-1} + \cdots + (-1)^p a_0(z'). \tag{4.14}$$

Each $a_i(z')$ can be expressed in terms of $\sum_{i=1}^{p} b_i^k(z')$ which is shown before as an integral. And therefore each $a_i(z')$ is holomorphic. Hence, $g(z', w)$ defines a Weierstrass polynomial.
Now we need to show that $f(z', w) = u(z', w)g(z', w)$ for u holomorphic on some

neighborhood of zero. For fixed z', f and g are functions in variable w. And they have the same zeros. Thus u is and holomorphic in w. It can be expressed as

$$u(z', w) = \frac{f(z', w)}{g(z', w)} = \frac{1}{2\pi i}\int_\gamma b^k \frac{u(z', t)}{t - w} dt \tag{4.15}$$

by Cauchy's integral formula. Since f is non-zero and holomorphic in a neighborhood of $\mathcal{V}$ and, f and g have the same zeros, g is also non-zero and holomorphic in a neighborhood of $\mathcal{V}$.
Notice that the integrand is continuous, thus u is holomorphic in a neighborhood of $\mathcal{V}$ and implies that f and ug induce equivalent germs and thus $f \sim ug$.

(3) Let

$$h = (z_m)^q + \sum_{k=1}^{m} d_k(z')(z_m)^{q-k} \tag{4.16}$$

such that $f \sim h$.
Then

$$ug = (z_m)^p + \sum_{k=1}^{m} c_k(z')(z_m)^{p-k} \tag{4.17}$$

and $ug \sim h$ (since $\sim$ is a equivalence relation). Notice that r_1 can be chosen arbitrarily small. Under the assumptions that each d_k are holomorphic on V, and $h(z', w) = v(z', w)g(z', w)$ for some holomorphic and non-zero function v on some neighborhood of 0, we have $h(0', w) = v(0', w)g(0', w)$. Since v does not vanish in $|w| < r_1$, we must have $q = p$. And f and g must have the same data of roots in a neighborhood of zero since they have the same zeros. Thus g can be understood as the unique (germ of) Weierstrass polynomial and h is also uniquely determined $\in \mathcal{H}_0^m$ i.e. c_k and d_k induce the same element of $\mathcal{H}_0^m$.

□

4.2. **The Weierstrass division theorem.**

Theorem 4.18. *Given an element $f \in \mathcal{H}_z^m$, and a Weierstrass polynomial g of degree k.*
(1) Then there exists a unit $h \in \mathcal{H}_z^m$ and polynomial $r \in \mathcal{H}_z^{m-1}$ with degree less than k, such that $f = gh + r$.
(2) If, with respect to the same basis, $f = gq + d$ for some $q \in \mathcal{H}_z^m$ and polynomial $d \in \mathcal{H}_z^{m-1}$ with degree less than k. Then $h = q$ and $r = d$. In other words, an representation of f in this form is unique.

Proof. (1) Let P be a polydisc with center at 0 and radius r such that g is holomorphic on P. Then there exists a number r_1, $0 < r_1 < r$, such that $g(0', z_m) \neq 0$ if $0 < |z_m| < r_1$. Similarly, we can define an open polydisc P' in $\mathbb{C}^{m-1}$ with center $0'$ and radius $< r$, such that $g(z', z_m) \neq 0$ if $z' \in P'$ and $|z_m| = r_1$. Hence, $f(z', z_m)$ is nonzero and holomorphic in a neighborhood of the tube $V = \{(z', z_m) : z' \in P', |z| = r_1\}$.

Set

$$h(z', w) = \frac{1}{2\pi i} \int_{\gamma = r_1} \frac{f(z', t)}{g(z', t)(t - w)} dt, \tag{4.19}$$

h is holomorphic in a neighborhood around zero.
From straight computation,

$$r = f - gh = \frac{1}{2\pi i} \int_{\gamma = r_1} \frac{f(z', t)}{g(z', t)} \left(\frac{g(z', t) - g(z', w)}{t - w}\right) dt. \tag{4.20}$$

And

$$g(z', w) = w^k + a_{k-1}(z') w^{k-1} + \cdots + a_0(z') \tag{4.21}$$

by definition of Weierstrass polynomial. Thus, $\frac{g(z',t) - g(z',w)}{t-w}$ is a polynomial with degree less than k, and r is in $\mathcal{H}_z^{m-1}$ with degree less than k.

(2) Taking z' sufficiently close to zero, $g(z', w)$ acts as a function of the variable w with k zeros (since it is a polynomial with degree k). Let $gh - r = 0$. Since g is non-zero in some neighborhood of zero and r has degree less than g does, h must be zero in order to satisfy the equality. And so does r need to be zero. This shows the uniqueness of h and r.

□

5. Application to the implicit function theorem

When considering a differentiable function with real variables, the inverse function theorem states that the existence of an inverse function is guaranteed locally near a base point. Moreover, the inverse function is differentiable at the image of the base point. When considering an analytic function of two variables $F(x, y)$, the implicit function theorem states that the equation $F(x, y) = 0$ can be solved for y in terms of x provided $\frac{\partial F}{\partial x} \neq 0$. It turns out, the inverse function theorem can be deduced from the implicit function theorem. In this sense, these two theorems are equivalent.

5.1. The implicit function theorem for complex variable.

Theorem 5.1. *Suppose f is a analytic function on a neighborhood V of the point (z_0, w_0) in $\mathbb{C}^2$. If $f(z_0, w_0) = 0$ and $\frac{\partial f}{\partial w}(z_0, w_0) \neq 0$.*
Then there exists, an open ball $B(z_0, r_1)$ centered at z_0 with radius r_1, an open ball $B(w_0, r_2)$ centered at w_0 with radius r_2 such that $B(z_0, r_1) \times B(w_0, r_2) \subset V$, and an analytic function $g : B(w_0, r_2) \mapsto B(z_0, r_1)$ such that for $(z, w) \in B(z_0, r_1) \times B(w_0, r_2)$, we have exactly one solution $w = g(z)$ to the equation $f(z, w) = 0$ in $B(w_0, r_2)$.

The proof of implicit function theorem shares the same idea with the Weierstrass preparation theorem as we will see.

Proof. $f(z_0, w_0) = 0$ and $\frac{\partial f}{\partial w}(z_0, w_0) \neq 0 \implies W \mapsto f(z_0, w_0)$ has a single root at w_0. And since the roots are isolated, there exists a $r_2 > 0$ such that $f(z_0, w) \neq 0$, for $0 < |w - w_0| < r_2$. Therefore, we have $f(z, w) \neq 0$, for all (z, w) in $\{z_0\} \times \partial B(w_0, r_2)$.

Moreover, since f is continuous and $\partial B(w_0, r_2)$ is a compact set, there exists a $r_1 > 0$ such that $f(z, w) \neq 0$, for all (z, w) in $B(z_0, r_1) \times \partial B(w_0, r_2)$.
let γ be the circle with center at w_0 and radius r_2 oriented in the positive direction, and

$$g(z) = \frac{1}{2\pi i} \int_\gamma \frac{\frac{\partial f(z,w)}{\partial w}}{f(z, w)}. \tag{5.2}$$

Since $g(z)$ counts the number of roots for each z of $w \mapsto f(z, w)$ on $B(z_0, r_1)$ by lemma 3.3, it is integer valued. And since $g(z)$ is analytic (therefore continuous) on $B(z_0, r_2)$, it is a constant.
Suppose $g(z) = 1$, it follows that $g(z) = 1$ for all $z \in B(z_0, r_1)$. For each $z \in B(z_0, r_1), f(z, h(z)) = 0, h(z) = w$ has exactly one solution. We define

$$h(z) = \frac{1}{2\pi i} \int_\gamma w \frac{\frac{\partial f(z,w)}{\partial w}}{f(z, w)} \tag{5.3}$$

by lemma 3.3. Thus, h(z) is analytic for each $z \in B(z_0, r_1)$.

By Taylor's theorem, $f(z, w) = (w - h(z))j(z, w)$ for $z \in B(z_0, r_1)$, and $j(z, w) \neq 0$ for $w \in B(w_0, r_2)$. If we replace $(w - h(z))$ by a monic polynomial, we shall see the Weierstrass preparation theorem in two variables case.

□

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