BEILINSON-BERNSTEIN LOCALIZATION

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Abstract. The purpose of this paper is to give an overview of the Beilinson-Bernstein localization theorem. It is a major result which introduced new geometric methods into representation theory. The theorem led to advances such as the resolution of the Kazhdan-Lusztig conjectures and is an important starting point of modern geometric representation theory. The fundamental example of $\mathfrak{sl}_2$ will be developed through the paper in the hopes of giving the reader a concrete idea of what is happening behind the theory.

Contents

1. Introduction

In this paper we hope to give a relatively accessible introduction to the basics of the geometric representation theory of semisimple lie algebras, with a focus on the Beilinson-Bernstein localization theorem. This theorem is a foundational starting point of geometric representation theory which today is a varied and active field of mathematical research. The ideas of geometric representation theory are being applied in fields as diverse as the physics of supersymmetric gauge theories by people like Edward Witten [2], and the arithmetic Langlands program in the work of people like Ngô Bảo Châu [3] and Xinwen Zhu [4].

These applications are modern and will not be the topic of this paper. Instead, we will focus more on the roots of the subject. Geometric representation theory really started with the Borel-Weil (later Borel-Weil-Bott) theorem. This is a fascinating theorem from the 50's [5] which realizes all the irreducible representations of a semi-simple Lie algebra through line bundles on a corresponding flag variety. The theorem by Borel-Weil constructed the irreducible representations as the global

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sections of certain line bundles, whereas Bott calculated the sheaf cohomology of all relevant line bundles. As we will see later this cohomology calculation will be a key ingredient in the proof of the Beilinson-Bernstein theorem.

The Beilinson-Bernstein theorem is in a way the most general form of Borel-Weil-Bott, in that it sets up a correspondence between all representations of a semisimple Lie algebra (not just finite dimensional irreducible representations) and certain geometric objects called D-modules. In their original paper, Beilinson and Bernstein used the theorem to resolve the Kazhdan-Lusztig conjectures [6], which had been attacked for years using purely algebraic methods ([7], p. 295). Another striking application of the theorem is that it provides a way to prove the Langlands classification, which describes the irreducible representations of a reductive real Lie group [8]. These applications of the theorem have cemented it as a central tool in modern representation theory.

This paper is intended for someone with a background in algebraic geometry, most importantly in sheaves and their cohomology. We also assume some knowledge of Lie groups, Lie algebras, their universal enveloping algebras, and their representations.

2. Representations of semi-simple Lie algebras

In this section we will give a quick overview of the representation theory of semi-simple Lie algebras. For proofs and more details, a standard reference is Humphreys [9]. We always assume our Lie algebras are finite dimensional complex vector spaces, and some definitions and results should be modified in more general settings.

Definition 2.1. A Lie algebra $\mathfrak{g}$ is semi-simple if all its solvable ideals are 0.

We will never actually need this definition of semi-simple; instead, we mainly use the following two facts.

Theorem 2.2. *(Cartan criterion) A Lie algebra $\mathfrak{g}$ is semi-simple if and only if the Killing form $Tr(ad_x ad_y)$ is non degenerate. Here, for each $x \in \mathfrak{g}$, $ad_x : \mathfrak{g} \to \mathfrak{g}$ is defined as $ad_x(y) = [x, y]$.*

Theorem 2.3. *(Weyl's complete reducibility theorem) Every finite dimensional representation of a semi-simple Lie algebra splits as the direct sum of irreducible representations.*

The reducibility theorem is crucial to the study of semisimple Lie algebras. It implies that to understand a finite dimensional representation, one only needs to understand the irreducible representations. The way to understand irreducible representations is to consider them as representations of easier to understand subalgebras called Cartan subalgebras.

Definition 2.4. A subalgebra $\mathfrak{h}$ of a semisimple Lie algebra $\mathfrak{g}$ is called a Cartan subalgebra if it is a maximal abelian subalgebra having the property that ad_h is diagonalizable for all $h \in \mathfrak{h}$.

The general theory then shows that because $\mathfrak{h}$ is abelian, the operators ad_h for $h \in \mathfrak{h}$ are simultaneously diagonalizable. In other words, $\mathfrak{g}$ splits as the direct sum of one-dimensional subspaces $\mathfrak{g}_\lambda$ on each of which $\mathfrak{h}$ acts by scalars. That is, for $x \in \mathfrak{g}_\lambda$ and $h \in \mathfrak{h}$ there is a $\lambda(h) \in \mathbb{C}$ such that $ad_h x = \lambda(h)x$. The assignments $h \mapsto \lambda(h)$

are characters $\lambda : \mathfrak{h} \to \mathbb{C}$, and the characters showing up in the decomposition $\mathfrak{g} = \bigoplus \mathfrak{g}_\lambda$ are called the *roots* of the Lie algebra $\mathfrak{g}$.

The roots of $\mathfrak{g}$ constitute what is called a root system in the dual vector space $\mathfrak{h}^*$. We will not need most of the theory of root systems, but will recall a few things.

Definition 2.5. Let V be a vector space and $\Delta \subseteq V$ a root system. A positive root system Δ^+ in Δ is a subset such that

- $\Delta^+ \cup -\Delta^+ = \Delta$
- $\Delta^- \cap -\Delta^+ = \varnothing$
- For $\alpha, \beta \in \Delta^+$, if $\alpha + \beta$ is in Δ then $\alpha + \beta$ is in Δ^+.

If a positive root α can not be written as the sum of two other positive roots, it is called a simple root. We write the collection of simple roots as $\Pi = \{\alpha_1, \dots, \alpha_n\}$ and this is always a basis of V.

Remark 2.6. There is no natural choice of positive root system, and later there will be theorems stated that seem to be wrong if one chooses another positive root system. We will see that a fundamental object, the Borel subgroup, will depend on the choice of positive roots, and this will make everything consistent.

If we choose a positive root system Δ^+ for $\mathfrak{g}$ then we know that the set of roots of $\mathfrak{g}$ is $\Delta^+ \cup \{0\} \cup \Delta^-$. Therefore we get a decomposition

$$\mathfrak{g} = \bigoplus_{\alpha \in \Delta^+} \mathfrak{g}_\alpha \oplus \mathfrak{h} \oplus \bigoplus_{\alpha \in \Delta^-} \mathfrak{g}_\alpha.$$

Definition 2.7. This decomposition determines the following important subspaces of $\mathfrak{g}$ which in fact are subalgebras:

- $\mathfrak{n} = \bigoplus_{\alpha \in \Delta^+} \mathfrak{g}_\alpha$
- $\mathfrak{n}^- = \bigoplus_{\alpha \in \Delta^-} \mathfrak{g}_\alpha$
- The Borel subalgebra is $\mathfrak{b} = \mathfrak{h} \oplus \mathfrak{n}$.

The symmetries of the root system are especially important to representation theory.

Let α be an element of $\mathfrak{h}^*$. If we have an element $\alpha^\vee \in \mathfrak{h}$ ($\mathfrak{h}$ being thought of as $\mathfrak{h}^{**}$) such that $\alpha^\vee(\alpha) = 2$ then we can define a reflection $s_{\alpha^\vee,\alpha}$ by $s_{\alpha^\vee,\alpha}(v) = v - \alpha^\vee(v)\alpha$. It is part of the definition of a root system that for each $\alpha_i \in \Pi$ we can define a dual element $\alpha_i^\vee \in \mathfrak{h}$ such that $\alpha_i^\vee(\alpha_i) = 2$ and $s_{\alpha_i^\vee,\alpha_i}(\Delta) = \Delta$.

Definition 2.8. The subgroup of $\mathrm{GL}(\mathfrak{h}^*)$ generated by the $s_{\alpha_i^\vee,\alpha_i}$ is called the Weyl Group, W. For $w \in W$ we denote by $l(w)$ the length of w, defined as the least number of reflections $s_{\alpha_i^\vee,\alpha_i}$ that must be composed to get w.

Definition 2.9. We define a few important subsets of $\mathfrak{h}^*$ as follows, (here we include 0 in $\mathbb{N}$).

- The fundamental weights π_i corresponding to the simple roots $\Pi = \{\alpha_1, \dots, \alpha_n\}$ are defined by $\alpha_i^\vee(\pi_j) = \delta_{ij}$ where δ_{ij} is the Kronecker delta.
- The element $\rho = \sum_{i=1}^n \pi_i$ is sometimes called the Weyl vector and plays an important role in representation theory. It is also equal to $\frac{1}{2}\sum_{i=1}^n \alpha_i$, the half-sum of the simple weights.
- Q^+ is the non-negative integral span of the α_i i.e. $\bigoplus \mathbb{N}\alpha_i$.
- P^+ is $\bigoplus \mathbb{N}\pi_i$ which equals $\{\lambda \in \mathfrak{h} \mid \alpha_i^\vee(\lambda) \in \mathbb{N}\}$. These are called the dominant weights.

- The elements of $-P^+$ are called anti-dominant weights.
- The set $\{\lambda \in \mathfrak{h} \mid \alpha_i^\vee(\lambda) < 0\}$ inside $-P^+$ is called the set of regular weights.

The importance of Q^+ is that it defines a partial ordering on $\mathfrak{h}^*$ by $\lambda \geq \mu$ if and only if $\lambda - \mu$ is in Q^+.

Returning to representations, we actually see that for any representation V of $\mathfrak{g}$, we can write $V = \bigoplus V_\lambda$ for characters $\lambda \in \mathfrak{h}^*$. The characters appearing in this decomposition are called the *weights* of the representation V. A striking result about weights is the following:

Theorem 2.10. *Any irreducible $\mathfrak{g}$ representation V always has a unique maximal weight λ and unique minimal weight μ with respect to the partial ordering given by Q^+. The maximal weight is always dominant and the minimal anti-dominant. In fact for a dominant weight $\lambda \in P^+$ there is only one $\mathfrak{g}$ representation up to isomorphism, called $L^+(\lambda)$, which has maximal weight λ. Similarly for μ anti-dominant there is a unique representation $L^-(\mu)$ having lowest weight μ (Section 20.3,* [9]*).*

The universal enveloping algebra, $U(\mathfrak{g})$ of $\mathfrak{g}$ will play an important role in this article. A fact crucial to the study of $U(\mathfrak{g})$ is the following so-called PBW theorem

Theorem 2.11. *(Poincaré-Birkhoff-Witt) If $(x_1, \ldots, x_n)$ is an ordered basis of $\mathfrak{g}$ then the monomials $x_1^{i_1} \ldots x_n^{i_n}$, for nonnegative integers i_j, form a basis of $U(\mathfrak{g})$ (p.92,* [9]*).*

The universal enveloping algebra carries a natural filtration. This is defined by setting $F_k U(\mathfrak{g})$ equal to the subspace spanned by products of at most k elements of $\mathfrak{g}$, and setting $F_0 U(\mathfrak{g}) = \mathbb{C}$. This allows us to define an associated graded algebra

$$grU(\mathfrak{g}) = \bigoplus_{k=0}^{\infty} F_k U(\mathfrak{g}) / F_k U(\mathfrak{g}).$$

The PBW theorem can equivalently be stated as

Theorem 2.12. *We have a natural isomorphism $grU(\mathfrak{g}) \cong S(\mathfrak{g})$. Here $S(\mathfrak{g})$ is the symmetric algebra of $\mathfrak{g}$. This is isomorphic to the polynomial algebra $\mathbb{C}[x_1, \ldots, x_n]$, where $(x_1, \ldots, x_n)$ is a basis of $\mathfrak{g}$.*

It can be shown that the center $Z \subset U(\mathfrak{g})$ acts by scalars on $L^-(\lambda)$. For $z \in Z$, there is therefore a scalar $\chi_\lambda(z) \in \mathbb{C}$ such that $v \in L^-(\lambda)$, $zv = \chi_\lambda(z)v$. The assignment $z \mapsto \chi_\lambda(z)$ is called the *central character* associated to λ. To understand the central character, we first need the Harish-Chandra homomorphism.

Definition 2.13. The Harish-Chandra homomorphism is constructed as follows. We first define a map $f : U(\mathfrak{h}) \to U(\mathfrak{h})$ as the algebra homomorphism induced by the linear map $\mathfrak{h} \to U(\mathfrak{h})$, $h \mapsto h + \rho(h)1$. By the PBW theorem, $U(\mathfrak{g}) = U(\mathfrak{h}) \oplus (\mathfrak{n}U(\mathfrak{g}) + U(\mathfrak{g})\mathfrak{n}^-)$. Thus $z \in Z$ can be written as $z = h + n$ for $h \in U(\mathfrak{h})$, $n \in (\mathfrak{n}U(\mathfrak{g}) + U(\mathfrak{g})\mathfrak{n}^-)$. The Harish-Chandra homomorphism is now the map $\gamma : Z \to U(\mathfrak{h})$ given by $\gamma(z) = f(h)$.

Using this notation, it is known that the central character takes the form

$$\chi_\lambda(z) = \lambda(\gamma(z)).$$

An important result about the Harish-Chandra homomorphism is the following

Theorem 2.14. *(Harish-Chandra) The Harish-Chandra homomorphism is injective and its image is $U(\mathfrak{h})^W$.*

The Weyl group acts on $\mathfrak{h}$ by an action dual to that on $\mathfrak{h}^*$ and this lifts to an action on $U(\mathfrak{h})$. The theorem states that the image of γ is exactly the set of fixed points of this action. This is a powerful tool for understanding the center of $U(\mathfrak{g})$.

3. Borel-Weil-Bott

The start of the whole story of geometric representation theory is the Borel-Weil-Bott theorem. This theorem constructs line bundles on so-called flag varieties whose sheaf cohomology groups realize every highest weight representation.

Let G be a simply-connected semisimple algebraic group over $\mathbb{C}$ with Lie algebra $\mathfrak{g}$. All we need to know about semi-simple Lie groups is that their Lie algebras are semi-simple. Because we assume G is simply-connected, we have the full force of the Lie group-Lie algebra correspondence. In particular, the correspondence between subgroups of G and subalgebras of $\mathfrak{g}$ leads to the following: H corresponding to $\mathfrak{h}$ is called a maximal torus of G, B corresponding to $\mathfrak{b}$ is a Borel subgroup (containing H), and $N \subseteq B$ corresponding to $\mathfrak{n}$ has the property that $B/N \cong H$. We will also need the subgroup N^- such that $Lie(N^-) = \mathfrak{n}^-$.

The flag varieties mentioned above get their name from the case of $SL_n(\mathbb{C})$. For $SL_n(\mathbb{C})$ we can associate a complete flag to any element $g \in SL_n(\mathbb{C})$. Namely if the columns of g are the vectors $v_1, \dots, v_n \in \mathbb{C}^n$ then the flag is

$$0 \subseteq \mathrm{Span}(v_1) \subseteq \mathrm{Span}(v_1, v_2) \subseteq \dots, \subseteq \mathrm{Span}(v_1, \dots, v_n) = \mathbb{C}^n.$$

The identity matrix gives a flag whose stabilizer is the group B of upper triangular matrices, where SL_n acts on flags by transforming each of the subspaces like usual. Because SL_n is a group, it is straightforward to see that it acts transitively on complete flags and therefore the set of complete flags in $\mathbb{C}^n$ is in natural bijection with the quotient SL_2/B. B is a Borel subgroup, and in analogy, any quotient G/B of an algebraic group by a Borel subgroup is called a flag variety.

Definition 3.1. A flag variety of a semi-simple algebraic group G is the quotient G/B of G by a Borel subgroup. These have a natural structure of projective variety that will not be explained in this paper, but can be found in [10], section 21.3.

Usually a Borel subgroup is fixed and G/B is called *the* flag variety of G. The general theory says that different choices of Borel subgroup don't matter, and the choice of Borel subgroup is basically the same as a choice of positive roots for the Lie algebra.

Other than flag varieties, the most important concept needed to state Borel-Weil-Bott is that of a G-equivariant vector bundle.

Definition 3.2. Let X be a topological space with a continuous G-action. A vector bundle $\pi : V \to X$ is called G-equivariant if V also has a continuous G action such that $g \in G$ sends the fiber V_x to V_{gx}, and the map $g\cdot : V_x \to V_{gx}$ is a linear isomorphism.

Given a G-equivariant vector bundle, V, we can define an action of G on the global sections by requiring that the sections be G-equivariant maps. That is, for any $g \in G$ the following diagrams should commute

$$\begin{array}{ccc} V & \xrightarrow{g\cdot} & V \\ {\scriptstyle s}\uparrow & & {\scriptstyle s}\uparrow \\ X & \xrightarrow{g\cdot} & X \end{array}$$

In formulas we simply need to define $(g \cdot s)(x) = gs(g^{-1}x)$.

There is a corresponding notion of a G-equivariant sheaf which we will not define or use, but they are important to the theory.

The action of G on sections gives an action of the Lie algebra on sections by differentiation, giving the formula

$$a \cdot s(x) = \frac{d}{dt}\Big|_{t=0} exp(ta)s(exp(-ta)x)$$

This action of the Lie algebra is fundamental to the localization theorem.

To define the line bundles in Borel-Weil-Bott we need to establish the following.

Proposition 3.3. *The G-equivariant vector bundles on the flag variety G/B (which will generally be called X from here on out) are in 1-1 correspondence with representations of B.*

Proof. Given a representation V of B (with the action written as $b \cdot v$), we can construct a G-equivariant vector bundle on X as follows. Take the trivial bundle $G \times V$, which has an action of B given by

$$b \cdot (g, v) = (gb^{-1}, b \cdot v).$$

Then take the quotient of this action to get a space called $G \times_B V$. This space is equipped with a map

$$\pi : G \times_B V \to X$$

which is induced by the projection to the first factor. The equivalence relation given by the B action is the same as

$$(gb, v) \sim (g, b \cdot v)$$

and people who are familiar with principal bundles will see that this quotient is the usual $G \times_B V$, the fiber bundle with fiber V associated to the principal B-bundle $G \to G/B$.

The action of G on $G \times_B V$ is given by

$$g \cdot (g', v) = (gg', v)$$

and it is straightforward that this action is well defined and that it gives $G \times_B V$ the structure of a G-equivariant vector bundle.

Now for a B-representation, V, and a G-equivariant vector bundle, E, our correspondence is

$$V \mapsto G \times_B V$$

$$E_{eB} \leftarrow\!\shortmid E.$$

Given a G-equivariant vector bundle $\pi : E \to X$ we can recover the representation of B on the fiber E_{eB}. This is done by noting that this fiber is invariant under the action of B because B is the stabilizer of eB.

It is up to the reader to check that we have an isomorphism $(G \times_B V)_{eB} \cong V$ of B-representations.

To establish that this correspondence is 1-1, we just need to check that given an equivariant vector bundle $\mathcal{V}$ over G/B we can take V to be the fiber $\mathcal{V}_{eB}$ (thought of as a representation of B) and find an isomorphism of bundles

$$\begin{tikzcd} G \times_B V \arrow[rr, "\varphi"] \arrow[dr, "p"'] & & \mathcal{V} \arrow[dl, "p"] \\ & G/B & \end{tikzcd}$$

To define this φ we simply set

$$\varphi(g, v) = g \cdot v.$$

This makes the diagram commute because if $v \in \mathcal{V}_{eB}$, then $g \cdot v \in \mathcal{V}_{gB}$ due to the equivariance of $\mathcal{V}$. The inverse isomorphism is given by $v \in \mathcal{V}_{gB} \mapsto (g, g^{-1}v)$. The inverse is well defined because if $gB{=}g'B$ then $g = g'b$ and $g'^{-1} = bg^{-1}$ so

$$(g, g^{-1}v) = (g'b, g^{-1}v) = (g', b \cdot g^{-1}v) = (g', g'^{-1}v)$$

□

The one dimensional B-representations can be shown to be naturally in bijection with the characters of H ([7] p. 256). In turn these are in bijection with characters of $\mathfrak{h}$ by the Lie group-Lie algebra correspondence. Throughout this paper we will not distinguish between the 1-dimensional representation of B and the corresponding character of $\mathfrak{h}$. Under this correspondence, if $\lambda \in \mathfrak{h}^*$ then $n\lambda$ corresponds to the character λ^n of B for each integer n.

Given a weight λ, we can associate a one dimensional B-representation $\mathbb{C}_\lambda$ whose associated G-equivariant line bundle we call $\mathcal{L}(\lambda)$.

The last concept needed to state the Borel-Weil-Bott theorem is the shifted action of the Weyl group.

Definition 3.4. We define a shifted action of W by the formula $w{\star}\lambda = w(\lambda{-}\rho){+}\rho$.

Definition 3.5. This action leads us to define some new sets of weights:

- The weight lattice P is defined to be the integral span of the fundamental weights π_i. This coincides with the set$\{\lambda \in \mathfrak{h} \mid \alpha_i^\vee(\lambda) \in \mathbb{Z}\}$.
- $P_{sing} = \{\, \lambda \in P \mid \alpha^\vee(\lambda - \rho) = 0 \text{ for some root } \alpha \in \Delta \,\}$.
- $P_{reg} = P \setminus P_{sing}$.

It can be shown that

Proposition 3.6. *(1) This action sends P_{sing} to itself i.e. $W \star P_{sing} = P_{sing}$*
(2) The set of antidominant weights $-P^+$ is a fundamental domain for the action of W on P_{reg}.

Now we can finally state

Theorem 3.7. *Borel-Weil-Bott: If $\lambda \in P_{sing}$ then $\mathcal{L}(\lambda)$ has no cohomology at all,*

$$H^i(X, \mathcal{L}(\lambda)) = 0 \textit{ for every } i \geq 0.$$

If $\lambda \in P_{reg}$ then there is a unique $w \in W$ such that $w \star \lambda$ is anti-dominant and we have that

$$H^i(X, \mathcal{L}(\lambda)) = \begin{cases} 0 & i \neq l(w) \\ L^-(w \star \lambda) & i = l(w). \end{cases}$$

Further, $\mathcal{L}(\lambda)$ is ample if and only if λ is regular. [5]

Example 3.8. Working with $G = SL_2(\mathbb{C})$ we can clarify this abstract theory. The flag variety for $SL_2(\mathbb{C})$ is the set of complete flags in $\mathbb{C}^2$ which is just the set of lines in $\mathbb{C}^2$, so it is $\mathbb{P}^1$. The stabilizer of the "x-axis" is the Borel subgroup

$$B = \left\{ \begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix} \middle| \, a \in \mathbb{C}^*, b \in \mathbb{C} \right\}.$$

Letting $[a : b]$ denote the homogeneous coordinates on $\mathbb{P}^1$, the projection map $SL_2(\mathbb{C}) \to \mathbb{P}^1$ is given by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto [a : c].$$

This is because the first column in the matrix is what represents the line that we remember when passing to the flag variety.

The situation with weights for $\mathfrak{sl}_2$ is straightforward. The vector space has a basis

$$h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \quad f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

with commutation relations

$$[h, e] = 2e \quad [h, f] = -2f \quad [e, f] = h.$$

From this we see that ad_h has eigenspaces $h\mathbb{C}$, $e\mathbb{C}$, and $f\mathbb{C}$ and on these spaces the operators ad_{ah}, $a \in \mathbb{C}$ have eigenvalues 0, $2a$, and $-2a$ respectively. Thus if we define the character

$$\rho : \mathfrak{h} \to \mathbb{C}, \quad \rho(ah) = a$$

then we get that

$$\Delta = \{ \pm 2\rho \} \quad \Delta^+ = \Pi = \{ 2\rho \} \quad \pi = \rho \quad W = \{ \pm 1 \}$$
$$P = \mathbb{Z}\rho \quad Q^+ = \mathbb{N}2\rho \quad P^+ = \mathbb{N}\rho.$$

Choosing 2ρ instead of -2ρ as the positive root gives the Borel subalgebra spanned by e and h. This is because $\mathbb{C}e = \mathfrak{g}_{2\rho}$, and this is the Lie algebra of the Borel subgroup B as defined above.

Now we analyze the Borel-Weil-Bott line bundles on $\mathbb{P}^1$. If we choose a weight $n\rho$, $n \in \mathbb{Z}$, then we refer to the corresponding line bundle as $\mathcal{L}(n\rho)$. If $\mathbb{P}^1$ has coordinates $[u : v]$, then it has the two charts U_u and U_v on which u and v respectively are non-zero. We want to understand our bundle by showing triviality over U_u and U_v and determining the transition functions.

In U_u there is a coordinate z and a coordinate x on U_v with $x = z^{-1}$ when $z \neq 0$. With respect to these coordinates, generic elements in U_u and U_v respectively are represented by the matrices

$$\begin{pmatrix} 1 & 0 \\ z & 1 \end{pmatrix} \qquad \begin{pmatrix} x & -1 \\ 1 & 0 \end{pmatrix}.$$

The crucial point here is that no element of B fixes either of these matrices except the identity. Thus if we have an element of $G \times_B \mathbb{C}_{n\rho}$ given by

$$\left[\begin{pmatrix} 1 & 0 \\ z & 1 \end{pmatrix}, t \right]$$

there is no other element in its equivalence class with the same first component.

Because t is just an arbitrary element of $\mathbb{C}$ we have shown that the bundle is trivial over U_u, and similarly also over U_v.

Then the equivalence relation on $G \times_B \mathbb{C}_{n\rho}$ gives

$$\left[\begin{pmatrix} 1 & 0 \\ z & 1 \end{pmatrix}, t\right] \sim \left[\begin{pmatrix} 1 & 0 \\ z & 1 \end{pmatrix}\begin{pmatrix} z^{-1} & -1 \\ 0 & z \end{pmatrix}, \begin{pmatrix} z & 1 \\ 0 & z^{-1} \end{pmatrix} \cdot t\right]$$
$$= \left(\begin{pmatrix} z^{-1} & -1 \\ 1 & 0 \end{pmatrix}, z^n t\right).$$

Now because the general element in U_v has homogeneous coordinates $[1 : x]$ (where $x = z^{-1}$ on $U_u \cap U_v$), we have calculated the transition function to be multiplication by z^n which matches exactly with the sheaf $\mathcal{O}_{\mathbb{P}^1}(-n)$, so we must have the isomorphism

$$\mathcal{L}(n\rho) \cong \mathcal{O}_{\mathbb{P}^1}(-n).$$

We can calculate the the action of G explicitly on the vector bundles associated to the sheaves $\mathcal{L}(n\rho)$ as follows. We know that G acts on pairs by $g(g'B, v) = (gg'B, v)$. If we are working over the chart U_v then we can calculate

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\left(\begin{pmatrix} x & -1 \\ 1 & 0 \end{pmatrix}, v\right) = \left(\begin{pmatrix} ax+b & -a \\ cx+d & -c \end{pmatrix}, v\right)$$
$$= \left(\begin{pmatrix} ax+b & -a \\ cx+d & -c \end{pmatrix}\begin{pmatrix} (cx+d)^{-1} & c \\ 0 & cx+d \end{pmatrix}, \begin{pmatrix} (cx+d) & -c \\ 0 & cx+d^{-1} \end{pmatrix} \cdot v\right)$$
$$= \left(\begin{pmatrix} \frac{ax+b}{cx+d} & -1 \\ 1 & 0 \end{pmatrix}, (cx+d)^n v\right).$$

Or, more concisely:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}(x, v) = \left(\frac{ax+b}{cx+d}, (cx+d)^n v\right).$$

Food for thought for those with a background in number theory: this should resemble the transformation law for modular forms.

For $n \geq 0$, $\mathcal{L}(-n\rho) = \mathcal{O}_X(n)$ has non-zero global sections, and we can now calculate the action of G on these global sections. We write

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

and work in the chart U_v with coordinate x. Then the section $u^k v^{n-k}$ becomes the section x^k when restricted to U_v . The global sections of $\mathcal{O}(n)$ are $\bigoplus_{k=0}^n \mathbb{C}x^k$ so we can calculate the action of G on any section just by considering x^k. We can calculate

$$g^{-1}x = \frac{dx-b}{-cx+a}, \quad s(g^{-1}x) = \left(\frac{dx-b}{-cx+a}, \left(\frac{dx-b}{-cx+a}\right)^k\right),$$
$$gs(g^{-1}x) = \left(x, \left(c\left(\frac{dx-b}{-cx+a}\right) + d\right)^{-n}\left(\frac{dx-b}{-cx+a}\right)^k\right)$$
$$= \left(x, (-cx+a)^n\left(\frac{dx-b}{-cx+a}\right)^k\right) = \left(x, (dx-b)^k(-cx+a)^{n-k}\right).$$

To express this new section in the coordinates u and v we need to write $x = \frac{u}{v}$ and multiply by v^n. This yields the homogeneous polynomial

$$g \cdot u^k v^{n-k} = (du - bv)^k(-cu + av)^{n-k}.$$

To understand the cohomology of these bundles through Borel-Weil-Bott, we need to first identify P_{sing}. $(2\rho)^\vee$ is just multiplication by $1/2$, and so $2\rho^\vee(\lambda-\rho)=0$ if and only if $\lambda=\rho$. The shifted action of $-1 \in W$ is given by

$$-1 \star n\rho = -(n\rho-\rho)+\rho = (2-n)\rho.$$

Thus we see that ρ is a fixed point and that the weights $n\rho$, $n \leq 0$, give a fundamental domain for the action on P_{reg}. Thus by Borel-Weil-Bott, for $n \leq 0$ the lowest weight representation for $n\rho$ is

$$L^-(n\rho) = \mathrm{H}^0(\mathbb{P}^1, \mathcal{O}(-n)).$$

We see that $\mathcal{O}(-n)$ for $n<0$ are exactly the ample line bundles.

For $n \geq 2$, we need to use the element $w=-1 \in W$ to get the antidominant $-1 \star n\rho = (-n+2)\rho$ and then we have, using Serre duality,

$$\begin{aligned}\mathrm{H}^1(\mathbb{P}^1, \mathcal{L}(n\rho)) = \mathrm{H}^1(\mathbb{P}^1, \mathcal{O}(-n)) &\cong \mathrm{H}^0(\mathbb{P}^1, \mathcal{O}(n-2)) \\ \cong H^0(\mathbb{P}^1, \mathcal{L}((2-n)\rho)) \cong L^-((2-n)\rho) &= L^-(-1 \star n\rho).\end{aligned}$$

4. Differential Operators and D-modules

The Beilinson-Bernstein Localization theorem generalizes the Borel-Weil-Bott theorem in that it realizes every representation of $\mathfrak{g}$ as the space of global sections of a certain type of sheaf on X, known as a D-module. We need to establish some basic facts about D-modules before moving onto the theorem.

Definition 4.1. For a smooth algebraic variety X over $\mathbb{C}$ we define the sheaf of differential operators recursively by setting

$$D_X = \bigcup_{p=-\infty}^{\infty} F_p D_X, \quad F_p D_X = 0 \text{ for } p<0$$

where

$$F_p D_X(U) = \{\phi \in \mathcal{E}nd(\mathcal{O}_X)(U) \mid \phi f - f\phi \in F_{p-1}D_X \text{ for all } f \in \mathcal{O}_X(U)\}$$

Here the functions f are considered as elements of $\mathcal{E}nd(\mathcal{O}_X)(U)$ by thinking of them as multiplication operators, i.e. f acts on a function g by sending it to the product fg. In particular, ϕf is an operator that takes g to $\phi(fg)$, it is not the function defined by $\phi(f)$.

This is a perfectly nice definition but it is not clear how it could be used or really what it is. To elucidate it we think about the tangent sheaf, Θ_X, and note that it can be expressed as the subsheaf of derivations in $\mathcal{E}nd(\mathcal{O}_X)$, i.e.

$$\Theta_X(U) = \{\phi \in \mathcal{E}nd(\mathcal{O}_X)(U) \mid \phi(fg) = f\phi(g)+g\phi(f) \text{ for all } f \in \mathcal{O}_X(U)\}$$

It can actually be shown that D_X is the subsheaf of $\mathcal{E}nd(\mathcal{O}_X)$ generated as a sheaf of $\mathcal{O}_X$-algebras by $\mathcal{O}_X$ (thought of as multiplication operators) and Θ_X.

Example 4.2. The Weyl Algebra:

Take X to be $\mathbb{C}^n$, with coordinates $x_1, \ldots, x_n$. Then the tangent sheaf is generated by the global sections (constant vector fields) ∂_{x_i}, $i=1,\ldots,n$, and $\mathcal{O}_X$ is generated by the global sections x_i. Therefore for an open set $U \subseteq \mathbb{C}^n$, the general element of $D_\lambda(U)$ is of the form

$$\sum f_{\alpha_1,\ldots,\alpha_n}(x_1,\ldots,x_n)\partial_{x_i}^{\alpha_1}\cdots\partial_{x_n}^{\alpha_n}.$$

where $f_{\alpha_1,\ldots,\alpha_n} \in \mathcal{O}_X(U)$. For simplicity we will now denote an expression like this with multi-index notation:

$$\sum_{|\alpha|=0}^{k} f_\alpha \partial^\alpha.$$

where the norm of a multi-index $\alpha = \{\alpha_1, \ldots, \alpha_n\}$ is $|\alpha| = \sum_{i=1}^{n} \alpha_i$

Definition 4.3. The Weyl algebra D_n is the algebra of global sections of $D_{\mathbb{C}^n}$. Its sections are all of the form

$$\sum f_\alpha \partial^\alpha$$

where each f_α is a polynomial. It is generated as a $\mathbb{C}$-algebra by $x_1, \ldots, x_n, \partial_{x_1}, \ldots, \partial_{x_n}$ and these generators have commutation relations

$$[x_i, x_j] = 0, \quad [\partial_{x_i}, x_j] = \delta_{ij}, \quad [\partial_{x_i}, \partial_{x_i}] = 0$$

where δ_{ij} is the Kronecker delta

$$\delta_{ij} = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$$

As the ∂_{x_i} are derivations, we have that

$$[\partial_{x_i}, f] = \frac{\partial f}{\partial x_i}$$

for any polynomial f.

We can better understand the definition of D_X locally. Each point of a smooth variety X has an affine neighborhood U such that Θ_U is generated as a sheaf of $\mathcal{O}_U$ modules by vector fields $\partial_1, \ldots, \partial_n$, where n is the dimension of X (p. 30 [13]). With respect to these vector fields, we can see that D_U behaves similarly to $D_{\mathbb{C}^n}$. In particular, as a $\mathcal{O}_U$ module we have

$$F_k D_U = \bigoplus_{|\alpha|=0}^{k} \mathcal{O}_U \partial^\alpha.$$

We refer to the sections of $F_k D_X$ as k-th order differential operators.

Definition 4.4. A (left) D-module is a sheaf of (left) modules over D_X which is defined exactly like a sheaf of $\mathcal{O}_X$-modules. This means that for any open set U, we have a (left) $D_X(U)$-module, $M(U)$, as well as for any $V \subseteq U$, a restriction map $\rho : M(U) \to M(V)$ which is a homomorphism of abelian groups and is compatible with scalar multiplication. If $r \in D_X(U)$ and $m \in M(U)$ then this compatibility means $\rho(rm) = \rho'(r)\rho(m)$ where $\rho' : D_X(U) \to D_X(V)$ is the correspoding restriction map for D_X.

Examples of D-modules include $\mathcal{O}_X$, with the action that D_X inherits as a subsheaf of $\mathcal{E}nd(\mathcal{O}_X)$. Further it can be shown that vector bundles with flat connections correspond exactly to D-modules that are coherent as $\mathcal{O}_X$-modules (p. 27, Theorem 1.4.10 of [7]).

Crucial to the Beilinson-Bernstein theory is the concept of a ring of twisted differential operators. For a given line bundle $\mathcal{L}$ we have

Definition 4.5. We can define a sheaf $D^{\mathcal{L}}$ of differential operators on $\mathcal{L}$ like we did on D_X, by recursively defining a filtration

$$0 \subseteq F_0 D^{\mathcal{L}} \subseteq F_1 D^{\mathcal{L}} \subseteq ...$$

where

$$F_k D^{\mathcal{L}}(U) = \{\phi \in \mathcal{E}nd(\mathcal{L})(U) \mid \phi f - f\phi \in F_{k-1}D^{\mathcal{L}} \text{ for all } f \in \mathcal{O}_X(U)\}.$$

Here we define

$$F_k D^{\mathcal{L}} = 0 \text{ for } k < 0.$$

From this we see that $F_0 D^{\mathcal{L}} = \mathcal{O}_X$.

Letting $\mathcal{L}^\vee$ denote the dual sheaf $\mathcal{H}om(\mathcal{L}, \mathcal{O}_X)$ then it can be shown that we have an isomorphism

$$D^{\mathcal{L}} \cong \mathcal{L} \otimes_{\mathcal{O}_X} D_X \otimes_{\mathcal{O}_X} \mathcal{L}^\vee.$$

Here a local section, $s \otimes P \otimes \phi$, of this sheaf acts on a local section $s' \in \mathcal{L}(U)$ by

$$(s \otimes P \otimes \phi)(s') = P(\phi(s'))s.$$

Because $\mathcal{L}$ is locally trivial we have filtration-preserving isomorphisms $D^{\mathcal{L}}\big|_U \cong D_X\big|_U$ over affine opens U. Therefore we can think of the filtration on $D^{\mathcal{L}}$ in the same way we think about it for D_X, locally in terms of k-th order operators.

Definition 4.6. With this filtration defined we can define the associated graded sheaf of algebras,

$$gr(D^{\mathcal{L}}) = \bigoplus_{k=0}^{\infty} gr_k(D^{\mathcal{L}}),$$

where the graded pieces are

$$gr_k(D^{\mathcal{L}}) = F_k D^{\mathcal{L}} / F_{k-1} D^{\mathcal{L}}.$$

It is not hard to see that

$$(F_k D^{\mathcal{L}})(F_l D^{\mathcal{L}}) = F_{l+k} D^{\mathcal{L}}.$$

This lets us define a product on $gr(D^{\mathcal{L}})$. If $\overline{P} = P + F_{p-1}D^{\mathcal{L}} \in gr_p(D^{\mathcal{L}})$ and $\overline{Q} = Q + F_{q-1}D^{\mathcal{L}} \in gr_q(D^{\mathcal{L}})$ then $\bar{P}\bar{Q} \in gr_{l+k}(D^{\mathcal{L}})$ is defined by

$$\bar{P}\bar{Q} = \overline{PQ} = PQ + F_{p+q-1}D^{\mathcal{L}}$$

Using the Leibniz rule, we can also see that

$$[P, Q] \in F_{p+q-1}D^{\mathcal{L}} \text{ if } P \in F_p D^{\mathcal{L}} \text{ and } Q \in F_q D^{\mathcal{L}}.$$

This implies $grD^{\mathcal{L}}$ is commutative.

For Beilinson-Bernstein we actually want to define a twisted ring of differential operators for each weight of $\mathfrak{g}$.

Definition 4.7. For a weight $\lambda \in \mathfrak{h}^*$, define D_λ to be the sheaf $D^{\mathcal{L}(\lambda+\rho)}$.

This shift by ρ is a little confusing and it is not apparent why we should do it. The goal is consistency with the Harish-Chandra homomorphism, in particular to make sure that Lemma 6.5 below works as it should.

Note that $D^{\mathcal{O}_X} = D_{-\rho}$.

The basic insight of the Beilinson-Bernstein localization theorem is that $\mathfrak{g}$ (and therefore $U(\mathfrak{g})$) acts on line bundles like twisted differential operators. In particular, the action of $a \in \mathfrak{g}$ defined by

$$a \cdot s(x) = \frac{d}{dt}\Big|_{t=0} exp(ta)s(exp(-ta)x)$$

defines a map $\mathfrak{g} \to \mathcal{E}nd(\mathcal{L}(\lambda + \rho))$.

Proposition 4.8. *This map sends $\mathfrak{g}$ to $F_1 D_\lambda$ and extends to a map $\Phi_\lambda : U(\mathfrak{g}) \to \Gamma(X, D_\lambda)$ which respects the filtration on both spaces.*

This proposition follows from unpacking the definitions of Φ_λ and $F_1 D_\lambda$. Morally it is true because the derivative $\frac{d}{dt}\Big|_{t=0}$ is first order.

5. The Localization Theorem

Now we can finally state the Beilinson-Bernstein Localization theorem.

Definition 5.1. The category of D_λ-modules that are quasi-coherent (resp. coherent) as $\mathcal{O}_X$-modules is denoted by $Mod_{qc}(D_\lambda)$ (resp. $Mod_c(D_\lambda)$).

The category of $U(\mathfrak{g})$-modules (the same as $\mathfrak{g}$-representations) on which the center acts by the central character χ_λ is denoted $Mod(\mathfrak{g}, \chi_\lambda)$, and the category of finitely generated $U(\mathfrak{g})$-modules is denoted $Mod_f(\mathfrak{g}, \chi_\lambda)$

Theorem 5.2. *(Beilinson-Bernstein Localization theorem) If λ is regular, then the global sections functors*
- $\Gamma : Mod_{qc}(D_\lambda) \to Mod(\mathfrak{g}, \chi_\lambda)$
- $\Gamma : Mod_c(D_\lambda) \to Mod_f(\mathfrak{g}, \chi_\lambda)$

Are equivalences of categories with quasi-inverse $D_\lambda \otimes_{U(\mathfrak{g})} (\,\cdot\,)$

It is called the localization theorem because of the parallel between the functors $D_\lambda \otimes_{U(\mathfrak{g})} (\,\cdot\,)$ and $\mathcal{O}_{Spec(R)} \otimes_R (\,\cdot\,)$. In particular, the functor $\mathcal{O}_{Spec(R)} \otimes_R (\,\cdot\,)$ makes a module over a commutative ring R into a sheaf over $Spec(R)$ by associating localizations of the module to open sets. In the same vein, $D_\lambda \otimes_{U(\mathfrak{g})} (\,\cdot\,)$ makes a $\mathfrak{g}$-representation into a sheaf over the flag variety. The idea of considering rings as functions on a geometric space and their modules as sheaves/vector bundles revolutionized number theory and commutative algebra in the 20th century. In turn, the localization theorem provides geometric intuition to Lie algebra representations.

The main effort needed to prove the localization theorem lies in the following:

Lemma 5.3. *For $\lambda \in P$, the map $\Phi_\lambda : U(\mathfrak{g}) \to \Gamma(X, D_\lambda)$ is surjective with kernel $U(\mathfrak{g})Ker(\chi_\lambda)$.*

Lemma 5.4. *On the center $Z \subset U(\mathfrak{g})$ the map Φ_λ agrees with the central character χ_λ, i.e. for $z \in Z, \Phi_\lambda(z) = \chi_\lambda(z)id$.*

Theorem 5.5. *Let λ be a regular weight, and $\mathcal{M}$ be in $Mod_{qc}(D_\lambda)$. Then the cohomology $H^k(\mathcal{M}, X)$ is zero for all non-zero k.*

Theorem 5.6. *If λ is a regular weight and $\mathcal{M} \in Mod_{qc}(D_\lambda)$, then $\mathcal{M}$ is generated as a D_λ-module by its global sections, i.e. the natural map*

$$D_\lambda \otimes_{U(\mathfrak{g})} \Gamma(X, \mathcal{M}) \to \mathcal{M}$$

is surjective.

6. Proofs of Lemmas 6.4 and 6.5

We start the proof of Lemma 6.4 with an easier lemma:

Lemma 6.1. *The algebra $grU(\mathfrak{g})$ is commutative and isomorphic to $S(\mathfrak{g})$, the symmetric algebra on $\mathfrak{g}$. The induced map, $gr\Phi_\lambda : grU(\mathfrak{g}) \to \Gamma(X, grD_\lambda)$ is surjective with kernel the ideal generated by $S(\mathfrak{g})^G_+$. The G signifies elements fixed by the adjoint action of G, and the $+$ signifies elements with no constant term.*

Proof. The first statement is just the PBW theorem. For the second, We need to better understand grD_λ. From the general theory of D-modules we actually have the following

Proposition 6.2. *Let $\pi : T^*X \to X$ be the natural projection of the cotangent bundle. Then there is an isomorphism of sheaves $grD_\lambda \cong \pi_*\mathcal{O}_{T^*X}$.*

Proof. [13] Page 32. □

We can now make the further identifications

$$S(\mathfrak{g}) = \mathbb{C}[\mathfrak{g}^*] \cong \mathbb{C}[\mathfrak{g}] \quad \text{and} \quad \Gamma(X, \pi_*\mathcal{O}_{T^*X}) = \Gamma(T^*X, \mathcal{O}_{T^*X}) = \mathbb{C}[T^*x].$$

The identification between $S(\mathfrak{g})$ and regular functions on $\mathfrak{g}^*$ is natural, but the one between $\mathbb{C}[\mathfrak{g}^*] \cong \mathbb{C}[\mathfrak{g}]$ is accomplished by the identification between $\mathfrak{g}$ and $\mathfrak{g}^*$ given by the Killing form.

With these identifications, the map

$$gr\Phi_\lambda : \mathbb{C}[\mathfrak{g}] \to \mathbb{C}[T^*X]$$

is dual to a special map $\mu : T^*X \to \mathfrak{g}$ called the moment map. This map was studied in the work of Kostant and is crucial to the Localization theorem.

To understand the moment map we need the following definition

Definition 6.3. The nilpotent cone $\mathcal{N} \subseteq \mathfrak{g}$ is the set

$$\{a \in \mathfrak{g} \mid ad_a : \mathfrak{g} \to \mathfrak{g} \text{ is a nilpotent operator } \}$$

It is a cone because for $z \in \mathbb{C}$, ad_a is nilpotent if and only if $ad_{za} = zad_a$ is. The nilpotent cone is always singular, notably the origin will be a singular point.

Proposition 6.4. *The nilpotent cone is an affine subvariety of $\mathfrak{g}$.*

Proof. We can write the characteristic polynomial of ad_a as

$$det(tI - ad_a) = t^n + f_{n-1}t^{n-1} + \ldots + f_0.$$

We notice that ad_a is nilpotent if and only if its characteristic polynomial equals t^n, equivalently if each f_i vanishes. If we choose a basis of $\mathfrak{g}$ then the f_i are polynomials in the coefficients of the vector a, and that is what we were looking for. □

Kostant's most important contribution to this story is the following

Theorem 6.5. *The ideal of polynomials vanishing on $\mathcal{N}$ is exactly the ideal generated by $\mathbb{C}[\mathfrak{g}]^G_+$, which we denote by $(\mathbb{C}[\mathfrak{g}]^G_+)$ ([7], Proposition 11.3.1).*

The nilpotent cone enters into our situation through the following

Theorem 6.6. *The map $\mu : T^*X \to \mathfrak{g}$ factors as*

$$\mu : T^*X \twoheadrightarrow \mathcal{N} \hookrightarrow \mathfrak{g}$$

where the second map is just the inclusion, and the first is surjective and birational.

Proposition 6.7. *If a map $f : X \to Y$ of algebraic varieties is surjective and birational then it induces an isomorphism $\Gamma(Y, \mathcal{O}_Y) \cong \Gamma(X, \mathcal{O}_X)$. This applies in particular to the map $T^*X \twoheadrightarrow \mathcal{N}$.*

Proof. Because f is birational, the map $f^* : \mathbb{C}(Y) \to \mathbb{C}(X)$ is an isomorphism which restricts to a map $f^* : \mathbb{C}[Y] \to \mathbb{C}[X]$. If its inverse sends regular functions to regular functions, i.e. if $f^{*-1}(\mathbb{C}[X]) \subseteq \mathbb{C}[Y]$, then we are done. Imagine there is a regular function $\varphi \in \mathbb{C}[X]$ which is the pullback of a rational function $\psi \in \mathbb{C}(Y)$. Then necessarily, the poles of ψ can not be contained in the image $f(X)$, which is impossible because f is surjective, thus φ has no poles and is regular. $\square$

These results imply that we have a factorization

$$gr\Phi_\lambda : \mathbb{C}[\mathfrak{g}] \twoheadrightarrow \mathbb{C}[\mathfrak{g}]/(\mathbb{C}[\mathfrak{g}]^G_+) \xrightarrow{\sim} \mathbb{C}[\mathcal{N}] \xrightarrow{\sim} \mathbb{C}[T^*X].$$

In particular we have an isomorphism $\mathbb{C}[\mathfrak{g}]/(\mathbb{C}[\mathfrak{g}]^G_+) \xrightarrow{\sim} \Gamma(X, grD_\lambda)$ as we have been looking for.

$\square$

Before finishing Lemma 6.4, we look at Lemma 6.5 about the action of the center Z of $U(\mathfrak{g})$.

First we show that Z acts by scalars. G acts on $U(\mathfrak{g})$ by an action induced by the adjoint representation on $\mathfrak{g}$, and it can be shown that the elements fixed by the action constitute the center, i.e. $Z = U(\mathfrak{g})^G$. Then we see that $\Phi_\lambda(z) \in \Gamma(X, D_\lambda)^G$ for $z \in Z$.

We can understand this on the level of associated graded algebras. In particular we know from the theory of conjugacy classes in $\mathfrak{g}$ ([7] Corollary 10.2.5) that $\mathcal{N}$ contains a G-orbit which is dense in $\mathcal{N}$. Therefore $\mathbb{C}[\mathcal{N}]^G = \mathbb{C}$ because the functions fixed by G have to be constant on G orbits, and as they are constant on a dense set they have to be constant. Therefore $\Gamma(X, grD_\lambda)^G = \mathbb{C}id$.

Now take an element $P \in \Gamma(X, D_\lambda)^G$. When we pass to $\Gamma(X, grD_\lambda)^G$ it could only be a constant. If P had a nonconstant term then its constant term would go to 0 in $\Gamma(X, grD_\lambda)$, but P itself would go to its non-zero highest order term, and therefore P must be constant.

Now we need to show that we actually have $\Phi_\lambda(z) = \chi_\lambda(z)$. Because Z acts like scalars, all we need to show is that they act like the correct scalars for one given non-zero section.

We consider the subgroup $N^- \subseteq G$ which is associated to the lie subalgebra $\mathfrak{n}^- = \bigoplus_{\alpha \in -\Delta^+} \mathfrak{g}_\alpha$. We have that N^-B/B is an open subset of X and that $N^- \cap B = e$. We claim that each coset in N^-B/B has a unique representative uB for $u \in N^-$. If $uB = u'bB$ where u and $u'b$ are both in N^-, then it follows $b \in N^- \cap B$ and therefore $b = e$ and $u = u'$.

Define a section $s \in \mathcal{L}(\lambda + \rho)(N^-B/B)$ by first choosing $v \in \mathcal{L}(\lambda + \rho)_{eB}$ and defining

$$s(uB) = uv \text{ for } u \in N^-.$$

Then for $h \in \mathfrak{h}$ we have $\Phi_\lambda(h)s(eB) = (\lambda + \rho)(h)s(eB)$ because $\mathcal{L}(\lambda + \rho)_{eB}$ is a B representation corresponding to the character $\lambda + \rho$. It can be shown ([7] Lemma 9.4.4) that $Z \subseteq U(\mathfrak{h}) \oplus (\mathfrak{n}U(\mathfrak{g}) \cap U(\mathfrak{g})\mathfrak{n}^-)$. Thus we write an arbitrary $z \in Z$ as

$$z = u + n \text{ for some } u \in U(\mathfrak{h}),\ n \in U(\mathfrak{g})\mathfrak{n}^-.$$

Unpacking the definitions of Φ_λ and s, we see that $\Phi_\lambda(n) = 0$, and the result follows from the definition of χ_λ. The idea is that we showed $\Phi_\lambda(h) = (\lambda+\rho)(h) = \lambda(\gamma(h)) = \chi_\lambda(h)$ for $h \in \mathfrak{h} \cap Z$. Because Φ_λ is an algebra homomorphism, it must agree with χ_λ on all of Z. This finishes the proof of Lemma 6.5. $\square$

Now we finish Lemma 6.4. We want to do it by induction on the graded pieces. First we need some definitions.

Definition 6.8. For any $p \geq 0$ define

- $I_p = ker(\chi_\lambda) \cap F_pU(\mathfrak{g})$.
- $J_p = U(\mathfrak{g})ker(\chi_\lambda) \cap F_pU(\mathfrak{g})$.
- $K_p = (S(\mathfrak{g})^G_+) \cap S(\mathfrak{g})_p$.

Lemma 6.4 will follow if for each $p \geq 0$ we have short exact sequences

$$0 \longrightarrow J_p \longrightarrow F_p(U(\mathfrak{g})) \xrightarrow{\Phi_\lambda} \Gamma(F_pD_\lambda) \longrightarrow 0.$$

For $p = 0$ we have $\Phi_\lambda : F_0U(\mathfrak{g}) \to \Gamma(X, F_0D_\lambda)$ is just the identity $\mathbb{C} \to \mathbb{C}$, and $J_0 = 0$. Now assume we have this sequence for $p-1$. To get it for p we want to compare with the graded case where we have short exact sequences

$$0 \longrightarrow K_p \longrightarrow S(\mathfrak{g})_p \xrightarrow{gr\Phi_\lambda} \Gamma(gr_pD_\lambda) \longrightarrow 0.$$

Putting everything together leads to the following commutative diagram

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & J_{p-1} \arrow[r] \arrow[d] & F_{p-1}(U(\mathfrak{g})) \arrow[r, "\Phi_\lambda"] \arrow[d] & \Gamma(F_{p-1}D_\lambda) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & J_p \arrow[r] \arrow[d] & F_p(U(\mathfrak{g})) \arrow[r, "\Phi_\lambda"] \arrow[d] & \Gamma(F_pD_\lambda) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & K_p \arrow[r] \arrow[d] & gr_pU(\mathfrak{g}) = S\mathfrak{g}_p \arrow[r, "gr\Phi_\lambda"] \arrow[d] & \Gamma(gr_pD_\lambda) \arrow[r] & 0 \\
 & 0 & 0 & &
\end{tikzcd}$$

The goal is to show the middle row is exact. The top row is exact by hypothesis, and the bottom is exact by Lemma 7.1. The rightmost column is exact because it is just the application of the left-exact functor Γ to the short exact sequence

$$0 \longrightarrow F_{p-1}D_\lambda \longrightarrow F_pD_\lambda \longrightarrow gr_pD_\lambda \longrightarrow 0.$$

The middle column is exact by definition.

The left column is harder to see, and it is not immediately clear why J_p should map to K_p in the quotient. We see that I_p (a subset of $Z = U(\mathfrak{g})^G$) is fixed by the adjoint action of G, and so maps to $S(\mathfrak{g})^G$. Also, for $p > 0$ the constant term of each element will go to 0 in $F_pU(\mathfrak{g})/F_{p-1}U(\mathfrak{g})$. Therefore, we have maps $I_p \to K_p$ and therefore maps $J_p \to K_p$. Now we need to show these are surjective.

We have that the projection $\pi : F_pU(\mathfrak{g}) \to S(\mathfrak{g})_p$ is a surjective map of $\mathfrak{g}$ representations. Because G is semi-simple, we can write

$$F_pU(\mathfrak{g}) = F_pU(\mathfrak{g})^G \oplus \bigoplus L_i$$

for irreducible G-modules L_i, and similarly,

$$S(\mathfrak{g})_p = S(\mathfrak{g})_p^G \oplus \bigoplus L_i'.$$

Schur's Lemma says that $Hom_G(L, L') = 0$ for two nonisomorphic irreducible G-representations L and L'. Therefore only elements with non-zero $F_pU(\mathfrak{g})^G$ components map to non-zero elements of $S(\mathfrak{g})_p^G$. Therefore $F_pU(\mathfrak{g})^G = Z \cap F_pU(\mathfrak{g})$ surjects onto $S(\mathfrak{g})_p^G$. If $a \in S(\mathfrak{g})_p^G$ then choose $z \in Z \cap F_pU(\mathfrak{g})$ such that $\pi(z) = a$. Then we can see that $\pi(z - \chi_\lambda(z)1) = a$ and that $z - \chi_\lambda(z)1$ is in the kernel of χ_λ and therefore is in I_p. This implies I_p surjects onto $S(\mathfrak{g})_p^G$ and therefore that J_p surjects onto K_p.

With all the columns and all but the middle row shown to be exact, the exactness of the middle row is easy diagram chasing and is left to the reader.

7. Lemmas 6.4 and 6.5 for $\mathfrak{sl}_2$

For $\mathfrak{sl}_2$ we can calculate Φ_λ explicitly. Say $\lambda = n\rho$, then $D_{n\rho} = D^{\mathcal{O}_{\mathbb{P}^1}(-1-n)}$. We can understand this sheaf by noting $D_{n\rho}\big|_{U_u} \cong D_{U_u}$ and $D_{n\rho}\big|_{U_v} \cong D_{U_v}$. This means we just need to understand the transition functions.

On U_u we have the coordinate z and vector field ∂_z. Similarly on U_v we have the coordinate x with vector field ∂_x, and on $U_u \cap U_v$ we have $x = 1/z$ and $\partial_x = \frac{\partial z}{\partial x}\partial_z = -\frac{1}{x^2}\partial_z = -z^2\partial_z$ as sections of $D_{\mathbb{P}^1}$.

Now looking at $\mathcal{L}(\lambda+\rho) = \mathcal{O}_{\mathbb{P}^1}(-n-1)$ we have the identification

$$\mathcal{O}_{\mathbb{P}^1}(-n-1)(U_u) = \bigoplus_{i=0}^{-n-1} z^i.$$

Under this identification we have the section 1 which is identified with x^{-n-1} over U_v. This section is denoted by 1_z, and the corresponding section for U_v is 1_x. Now take the section

$$1_x \otimes \partial_x \otimes 1_x^\vee \in \mathcal{L}(\lambda+\rho) \otimes_{\mathcal{O}_X} D_X \otimes_{\mathcal{O}_X} \mathcal{L}(\lambda+\rho)^\vee(U \cap V)$$

which corresponds to ∂_x in the isomorphism $D_{n\rho}\big|_{U_u} \cong D_{U_u}$. We have

$$1_x \otimes \partial_x \otimes 1_x^\vee = z^{-n-1} \otimes -z^2\partial_z \otimes z^{n+1} = 1_z \otimes -z^{-n+1}\partial_z z^{n+1} \otimes 1_z$$
$$= 1_z \otimes -z^{-n+1}((n+1)z^n + z^{n+1}\partial_z) \otimes 1_z = 1_z \otimes -(n+1)z - z^2\partial_z \otimes 1_z.$$

Therefore if we drop the 1's we get the following formula for our twisted differential operators:

$$\partial_x = -z^2\partial_z - (n+1)z.$$

Let us calculate how $e \in \mathfrak{sl}_2$ acts on sections of $\mathcal{O}(-n-1)$. We take the section z^k over U_u and see

$$\Phi_\lambda(e)z^k = \frac{d}{dt}\Big|_{t=0} exp(te)(exp(-te)z)^k = \frac{d}{dt}\Big|_{t=0}(z-t)^k = -kz^{k-1} = -\partial_z z^k.$$

Therefore

$$\Phi_\lambda(e) = -\partial_z.$$

The action of h and f can be calculated in the same way, and we get

$$\Phi_\lambda(h) = -2z\partial_z - (n+1), \quad \Phi_\lambda(f) = z^2\partial_z + (n+1)z.$$

Similarly we can calculate $gr\Phi_\lambda$ on the basis e, h, f of $\mathfrak{sl}_2$ as

$$e \mapsto -\partial_z, \quad f \mapsto z^2\partial_z, \quad h \mapsto -2z\partial_z.$$

If $\overline{U_u} \subseteq T^*\mathbb{P}^1$ is the preimage of U_u then we can denote $p \in \overline{U_u}$ by a pair (a,b) where a is the z coordinate and b the d_z coordinate of p, so that $z(a,b) = a$ and $\partial_z(a,b) = b$. Then the functions e, h, f on $\mathfrak{g}^*$ are dual to the basis e^*, h^*, f^* of $\mathfrak{g}^*$ defined by $e^*(e) = 1$, $e^*(h) = e^*(f) = 0$, etc. (in particular this is not the dual basis defined by the Killing form). With respect to these coordinates on $\overline{U_u}$ and this basis on $\mathfrak{g}^*$ we have that the map $\mu : \overline{U_u} \to \mathfrak{g}^*$ which is dual to Φ_λ is defined by

$$\mu(a,b) = (\Phi_\lambda(e)(a,b), \Phi_\lambda(h)(a,b), \Phi_\lambda(f)(a,b)) = (-b, -2ab, a^2b).$$

We want to identify $\mathfrak{g}$ with $\mathfrak{g}^*$ using the Killing form. If we have $x \in \mathfrak{g}$ then we define $x^\vee \in \mathfrak{g}$ by $x^\vee(y) = Tr(ad_x ad_y)$. That is, $x^\vee$ is the dual of x as defined by the Killing form. By writing out matrices for ad and computing we can find the following

$$e^\vee = 4f^*, \quad h^\vee = 8h^*, \quad f^\vee = 4e^*.$$

Therefore we can identify $\mathfrak{g}$ with $\mathfrak{g}^*$ by $x \mapsto x^\vee$ and get

$$\mu : \overline{U_u} \to \mathfrak{g}$$

$$(a,b) \mapsto \frac{a^2b}{4}e - \frac{ab}{4}h - \frac{b}{4}f.$$

We should also calculate what happens to the fiber over $\infty = [0,1] = \mathbb{P}^1 \setminus U_u$. When $a \neq 0$, the point $(a,b) \in \overline{U_v}$ corresponds to the point $(a^{-1}, a^2b) \in \overline{U_u}$, which then maps to

$$\frac{b}{4}e - \frac{ab}{4}h - \frac{a^2b}{4}f.$$

If we then let a go to 0 we get $\frac{b}{4}e$ so the fiber over ∞ maps bijectively to $\mathbb{C}e$.

We see that a point $p \in \mathfrak{g}$ in the image of this map satisfies the relation

$$(h^{*2} + e^*f^*)(p) = 0. \tag{7.1}$$

Further, if we have an arbitrary $p = xe + yh + zf$, where $x, y, z \in \mathbb{C}$, such that $y^2 + xz = 0$ then we can see p is actually in the image. If $z = 0$ then $y = 0$, and we see that $p = xe$ is in the image of the fiber over ∞. If $z \neq 0$ then consider the point $(y/z, -4z) \in \overline{U_u}$. It is easy to check that this maps to p and is in fact the only point that maps to p. This means that the image is exactly the set of points satisfying equation 8.1.

As desired, we see that equation 8.1 also defines the Nilpotent cone $\mathcal{N} \in \mathfrak{g}$. If we choose an arbitrary $x = ae + bh + cf \in \mathfrak{g}$ then the characteristic polynomial is

$$det(ad_x - \lambda Id) = -\lambda^3 + (4b^2 + 4ac)\lambda$$

and we see that x is nilpotent if and only if $(b^2 + ac) = (h^{*2} + e^*f^*)(x) = 0$.

We want to check that $\mu : T^*\mathbb{P}^1 \to \mathcal{N}$ is actually birational and surjective. The zero section of $T^*\mathbb{P}^1$ maps to 0, i.e. $(a,0) \mapsto 0$ for a either in U_u or U_v, and this zero section is exactly the fiber over 0. The fiber over any other point is in fact just one point, as indicated in the discussion above. There is a theorem in algebraic geometry that says for a rational map $f : X \to Y$ between varieties over $\mathbb{C}$, f is birational if and only if $f^{-1}(y)$ is one point for the general $y \in Y$ ([12] p.77). This is the case in our situation, so μ is birational.

The fibers of these moment maps actually play a special role in representation theory, and are called Springer fibers. The interested reader can read more about this topic in the book by Chriss and Ginzburg [13].

Now we want to understand the non-graded version. The first thing to calculate is the center, Z, of $U(\mathfrak{sl}_2)$. We claim Z is generated by the element $\frac{1}{2}h^2+ef+fe = \frac{1}{2}h^2-h+2ef$. The Harish-Chandra homomorphism sends it to $\frac{1}{2}(h^2-1)$. Because $U(\mathfrak{h}) = \mathbb{C}[h]$, and W acts by $wh = -h$ we see that $U(\mathfrak{h})^W = \mathbb{C}[h^2]$. By Harish-Chandra, $Z \cong U(\mathfrak{h})^W$. Because $U(\mathfrak{h})^W$ is generated as a $\mathbb{C}$ algebra by $\frac{1}{2}(h^2-1)$, Z is generated by $\frac{1}{2}h^2+ef+fe$. We calculate that

$$\chi_{n\rho}(\frac{1}{2}h^2+ef+fe) = n\rho(\frac{1}{2}(h^2-1)) = \frac{1}{2}(n^2-1).$$

We now specialize to the case $\lambda = -\rho$ which corresponds to the standard differential operators $D_{\mathbb{P}^1}$. In this case, we get $ker(\chi_{-\rho}) = Z(\frac{1}{2}h^2+ef+fe)$.

What we want now is an isomorphism $\Gamma(\mathbb{P}^1, D_{\mathbb{P}^1}) \cong U(\mathfrak{sl}_2)/(U(\mathfrak{sl}_2)ker(\chi_\rho))$. Passing through the sequence of identifications

$$gr(U(\mathfrak{sl}_2)) \cong S(\mathfrak{sl}_2) = \mathbb{C}[\mathfrak{sl}_2^*] \cong \mathbb{C}[\mathfrak{sl}_2]$$

we have that

$$(\frac{1}{2}h^2+ef+fe) = (\frac{1}{2}h^2+2ef) \mapsto (\frac{1}{2}h^{\vee 2}+2e^\vee f^\vee) = (32h^{*2}+32e^*f^*).$$

This implies that when passing to the graded algebra, $Z(\frac{1}{2}h^2+ef+fe)$ becomes the ideal in $\mathbb{C}[g]$ generated by $4h^{*2}+4e^*f^*$. This proves the surjectivity $J_p \to K_p$ in the notation of Section 7.

It is left to the reader to use our description of Φ_λ for $\mathfrak{sl}_2$ to show that for $z \in Z$, $\Phi_\lambda(z) = \chi_\lambda(z)$.

Putting everything together we see that Φ_λ induces a map

$$U(\mathfrak{sl}_2)/(U(\mathfrak{sl}_2)ker(\chi_\rho)) \to \Gamma(\mathbb{P}^1, D_{\mathbb{P}^1})$$

which descends to the isomorphism

$$gr\Phi_\lambda : \mathbb{C}[\mathfrak{sl}_2]/(4h^{*2}+4e^*f^*) \to \Gamma(\mathbb{P}^1, D^1_{\mathbb{P}})$$

between associated graded algebras. This implies that in fact the original map gave an isomorphism

$$U(\mathfrak{sl}_2)/(U(\mathfrak{sl}_2)ker(\chi_{-\rho})) \cong \Gamma(\mathbb{P}^1, D_{\mathbb{P}^1}).$$

8. Proofs of Theorems 6.6 and 6.7

Proposition 8.1. *For the line bundle $\mathcal{L}(\lambda)$ associated to a weight λ we have $\mathcal{L}(\lambda)^\vee = \mathcal{L}(-\lambda)$. Similarly if λ is anti-dominant and $L^-(\lambda)$ is the irreducible module of lowest weight λ then $L^-(\lambda)^* = L^+(-\lambda)$.*

Proof. If the Lie algebra representation is written $\rho : \mathfrak{g} \to \mathfrak{gl}(L^-(\lambda))$ then the dual representation on $L^-(\lambda)^\vee$ is defined to be $a \cdot \phi = -\phi \circ \rho(a)$ for $a \in \mathfrak{g}$. Thus the weights will be the negatives of the weights for $(L^-(\lambda))$ and $-\lambda$ will be the highest one.

To define a section of $\mathcal{L}(\lambda) = G \times_B \mathbb{C}_\lambda$ we need a section of the trivial bundle $G \times \mathbb{C}$ which behaves properly with respect to the action of B. In particular we need a function $f : G \to \mathbb{C}$ transforming like $f(gb) = \lambda(b)f(g)$ for $b \in B$ and $g \in G$. A corresponding function for the dual $\mathcal{L}(\lambda)^\vee$ then has the transformation rule $f(gb) = \lambda(b)^{-1}f(g)$ which is what defines the bundle $\mathcal{L}(-\lambda)$ when λ is considered a character of $\mathfrak{b}$.

□

Proof of Theorem 6.6. Choosing an anti-dominant λ, we start with the map (surjective by Borel-Weil-Bott)

$$p_\lambda : \mathcal{O}_X \otimes_{\mathbb{C}} L^-(\lambda) \twoheadrightarrow \mathcal{L}(\lambda)$$

and its dual

$$\mathcal{L}(-\lambda) \hookrightarrow \mathcal{O}_X \otimes_{\mathbb{C}} L^+(-\lambda).$$

We then tensor this with $\mathcal{L}(\lambda)$ and get

$$i_\lambda : \mathcal{O}_X \hookrightarrow \mathcal{L}(\lambda) \otimes_{\mathbb{C}} L^+(-\lambda)$$

which is still injective because $\mathcal{L}(\lambda)$ is a line bundle, and therefore locally free and flat as an $\mathcal{O}_X$-module.

Now, given a D_λ-module $\mathcal{M}$ we can tensor $\mathcal{M}$ with these maps to get

$$\overline{p_\lambda} : \mathcal{M} \otimes_{\mathbb{C}} L^-(\lambda) \twoheadrightarrow \mathcal{M} \otimes_{\mathcal{O}_X} \mathcal{L}(\lambda)$$

$$\overline{i_\lambda} : \mathcal{M} \hookrightarrow \mathcal{M} \otimes_{\mathcal{O}_X} \mathcal{L}(\lambda) \otimes_{\mathbb{C}} L^+(-\lambda)$$

It can be shown that $Im(i_\lambda)$ and $Ker(p_\lambda)$ locally are direct summands of $\mathcal{L}(\lambda) \otimes_{\mathbb{C}} L^+(-\lambda)$ and $\mathcal{L}(\lambda)$ respectively, implying that $\overline{i_\lambda}$ and $\overline{p_\lambda}$ are still injective and surjective respectively.

The goal of these maps is to use what Borel-Weil-Bott tells us about the cohomology of $\mathcal{L}$ to learn about the cohomology of $\mathcal{M}$. Namely we will use the following lemma to say that the induced maps on cohomology are injective or surjective.

Lemma 8.2. *$Im(\overline{i_\lambda})$ and $Ker(\overline{p_\lambda})$ are direct summands of $\mathcal{M} \otimes_{\mathcal{O}_X} \mathcal{L}(\lambda) \otimes_{\mathbb{C}} L^+(-\lambda)$ and $\mathcal{M} \otimes_{\mathcal{O}_X} \mathcal{L}(\lambda)$ respectively and therefore $\overline{i_\lambda}$ and $\overline{p_\lambda}$ respectively have a left inverse and right inverse.*

Proof. Proposition 11.4.1 in [7]. □

Proof of Theorem 6.6. If λ is regular we want to show that the cohomology $\mathrm{H}^i(X, \mathcal{M})$ is 0 for each $i > 0$ and each $\mathcal{M} \in Mod_{qc}(D_\lambda)$. To do this, it is convenient to work with sheaves that are actually coherent over $\mathcal{O}_X$, and luckily we can because for any quasi-coherent sheaf $\mathcal{M}$, we have that $\mathcal{M}$ is the direct limit of its coherent subsheaves:

$$\mathcal{M} = \varinjlim \mathcal{N}, \quad \mathcal{N} \subseteq \mathcal{M} \text{ coherent}.$$

Then from Hartshorne, Proposition 2.9 on page 209 [1], we know that the natural map

$$\mathrm{H}^i(X, \varinjlim \mathcal{N}) \to \mathrm{H}^i(X, \mathcal{M})$$

is an isomorphism. Our strategy therefore is to show that the map

$$\mathrm{H}^i(X, \mathcal{N}) \to \mathrm{H}^i(X, \mathcal{M})$$

is the zero map for each coherent subsheaf $\mathcal{N} \subseteq \mathcal{M}$. Consider the diagram

$$\begin{array}{ccc}
\mathrm{H}^i(X, \mathcal{N}) & \xrightarrow{\overline{i_\lambda}_*} & \mathrm{H}^i(X, \mathcal{N} \otimes_{\mathcal{O}_X} \mathcal{L}(\lambda) \otimes_{\mathbb{C}} L^+(-\lambda)) \\
\downarrow & & \downarrow \\
\mathrm{H}^i(X, \mathcal{M}) & \xrightarrow{\overline{i_\lambda}_*} & \mathrm{H}^i(X, \mathcal{M} \otimes_{\mathcal{O}_X} \mathcal{L}(\lambda) \otimes_{\mathbb{C}} L^+(-\lambda))
\end{array}$$

where the horizontal maps are injective because they have a left inverse by Lemma 5.8.

Notice that because cohomology commutes with direct limits (and in particular direct sums), we get

$$\begin{aligned}\mathrm{H}^i(X, \mathcal{N} \otimes_{\mathcal{O}_X} \mathcal{L}(\lambda) \otimes_{\mathbb{C}} L^+(-\lambda)) &\cong \mathrm{H}^i(X, \mathcal{N} \otimes_{\mathcal{O}_X} \mathcal{L}(\lambda) \otimes_{\mathbb{C}} \mathbb{C}^{dim(L^-(\lambda))}) \\ &\cong \mathrm{H}^i(X, (\mathcal{N} \otimes_{\mathcal{O}_X} \mathcal{L}(\lambda))^{\oplus n}) \cong \mathrm{H}^i(X, (\mathcal{N} \otimes_{\mathcal{O}_X} \mathcal{L}(\lambda))^{\oplus n}.\end{aligned}$$

When λ is regular we know from Borel-Weil-Bott that $\mathcal{L}(\lambda)$ is ample. Thus for any regular λ we have

$$\mathrm{H}^i(X, \mathcal{N} \otimes_{\mathcal{O}_X} \mathcal{L}(\lambda)^{\otimes k}) = 0$$

for all $i > 0$ and $k \geq k_0$ for some integer k_0. We will see that $\mathcal{L}(\lambda)^{\otimes k}$ is itself the line bundle associated to the regular weight $k\lambda$ so we can say there is some regular weight $k\lambda$ such that

$$\mathrm{H}^i(X, \mathcal{N} \otimes_{\mathcal{O}_X} \mathcal{L}(k\lambda) \otimes_{\mathbb{C}} L^+(-k\lambda)) \cong \mathrm{H}^i(X, \mathcal{N} \otimes_{\mathcal{O}_X} \mathcal{L}(k\lambda)) = 0.$$

We could then plug $k\lambda$ in for λ in the diagram above, and use that $i_{k\lambda *}$ is injective to see that the map

$$\mathrm{H}^i(X, \mathcal{N}) \to \mathrm{H}^i(X, \mathcal{M})$$

must be zero for each coherent subsheaf $\mathcal{N}$ of $\mathcal{M}$ and therefore we must have

$$\mathrm{H}^i(X, \mathcal{M}) = 0.$$

Thus all we have left is the following

Proposition 8.3. *$\mathcal{L}(\lambda) \otimes_{\mathcal{O}_X} \mathcal{L}(\mu) \cong \mathcal{L}(\lambda + \mu)$.*

Proof. It can be checked that $\mathcal{L}(\lambda) \otimes_{\mathcal{O}_X} \mathcal{L}(\mu)$ is G-equivariant with the G action $g \cdot (e \otimes v) = g \cdot e \otimes g \cdot v$ where e and v are elements of the fibers $\mathcal{L}(\lambda)_{g'B}$ and $\mathcal{L}(\mu)_{g'B}$ respectively. We know from the definition of a tensor product of Lie-algebra representations that $\mathfrak{h}$ acts on $\mathbb{C}_\lambda \otimes_{\mathbb{C}} \mathbb{C}_\mu$ with the character $\lambda + \mu$. Because we have

$$(\mathcal{L}(\lambda) \otimes_{\mathcal{O}_X} \mathcal{L}(\mu))_{eB} \cong \mathcal{L}(\lambda)_{eB} \otimes_{\mathbb{C}} \mathcal{L}(\mu)_{eB} \cong \mathbb{C}_\lambda \otimes_{\mathbb{C}} \mathbb{C}_\mu$$

and $\mathcal{L}(\lambda + \rho)$ is uniquely specified by the B-representation on $\mathcal{L}(\lambda + \rho)_{eB}$, we are done.

□

Now we move on to Theorem 6.7 that D_λ-modules for λ regular are generated by their global sections. We consider the map

$$D_\lambda \otimes_{U(\mathfrak{g})} \Gamma(X, \mathcal{M}) \to \mathcal{M}$$

and call its image $\mathcal{M}'$ and its cokernel $\mathcal{M}''$. Therefore we have a short exact sequence

$$0 \to \mathcal{M}' \to \mathcal{M} \to \mathcal{M}'' \to 0$$

in the category $Mod_{qc}(D_\lambda)$. By Theorem 6.6, the global sections functor Γ is exact on this category and so we get a short exact sequence

$$0 \to \Gamma(\mathcal{M}') \to \Gamma(\mathcal{M}) \to \Gamma(\mathcal{M}'') \to 0.$$

Now by the definition of $\mathcal{M}'$, the first map is actually an isomorphism, forcing $\Gamma(\mathcal{M}'') = 0$. If we can show that this implies $\mathcal{M}'' = 0$ we will be done.

As in the proof of Theorem 6.6 we can say that for a regular weight λ', $\mathcal{L}(\lambda')$ is ample. By definition this means that $\mathcal{N} \otimes_{\mathcal{O}_X} \mathcal{L}(\lambda')^{\otimes k}$ is generated by global sections for k large enough and $\mathcal{N}$ any coherent sub sheaf of $\mathcal{M}''$. If $\mathcal{M}''$ were non-zero then it would have a non-zero coherent subsheaf $\mathcal{N}$ for which $\mathcal{N} \otimes_{\mathcal{O}_X} \mathcal{L}(\lambda)^{\otimes k}$ is generated by global sections for a fixed k. If we set $\lambda = k\lambda'$ then $\mathcal{L}(\lambda')^{\otimes k} \cong \mathcal{L}(\lambda)$. Now we know $\mathcal{N} \otimes_{\mathcal{O}_X} \mathcal{L}(\lambda)$ has global sections and so $\mathcal{M}'' \otimes_{\mathcal{O}_X} \mathcal{L}(\lambda)$ must also have non-zero global sections. To verify that the non-zero global sections of $\mathcal{N} \otimes_{\mathcal{O}_X} \mathcal{L}(\lambda)$ don't go to 0 in $\mathcal{M}'' \otimes_{\mathcal{O}_X} \mathcal{L}(\lambda)$ we note that Γ and $\cdot \otimes \mathcal{L}(\lambda)$ are exact functors. This means we have exact sequences

$$0 \to \mathcal{N} \to \mathcal{M}'', \quad 0 \to \mathcal{N} \otimes_{\mathcal{O}_X} \mathcal{L}(\lambda) \to \mathcal{M}'' \otimes_{\mathcal{O}_X} \mathcal{L}(\lambda)$$
$$0 \to \Gamma(\mathcal{N} \otimes_{\mathcal{O}_X} \mathcal{L}(\lambda)) \to \Gamma(\mathcal{M}'' \otimes_{\mathcal{O}_X} \mathcal{L}(\lambda)).$$

Now we use the maps

$$p_\lambda : \mathcal{M}'' \otimes_{\mathcal{O}_X} L^-(\lambda) \twoheadrightarrow \mathcal{M}'' \otimes_{\mathcal{O}_X} \mathcal{L}(\lambda)$$
$$p_{\lambda *} : \mathrm{H}^0(\mathcal{M}'' \otimes_{\mathcal{O}_X} L^-(\lambda)) \twoheadrightarrow \mathrm{H}^0(\mathcal{M}'' \otimes_{\mathcal{O}_X} \mathcal{L}(\lambda))$$

and notice that because $\mathrm{H}^0(\mathcal{M}'' \otimes_{\mathcal{O}_X} \mathcal{L}(\lambda)) \neq 0$, necessarily $\mathrm{H}^0(\mathcal{M}'' \otimes_{\mathcal{O}_X} L^-(\lambda)) \neq 0$ but this is just $\mathrm{H}^0(\mathcal{M}'') \otimes_{\mathbb{C}} L^-(\lambda)$. Therefore if $\mathcal{M}'' \neq 0$, then $\mathrm{H}^0(\mathcal{M}'') \neq 0$ which implies that our $\mathcal{M}''$ must be 0, and $\mathcal{M}$ is generated by its global sections.

9. Finishing the Proof of the Localization Theorem

At this point, most of the work is done, and the final step in proving the Localization theorem is to show how our results on the cohomology of D_λ-modules lead to the equivalence of categories of Theorem 6.3. In particular we need to show the functors

$$\Gamma : Mod_{qc}(D_\lambda) \to Mod(\mathfrak{g}, \chi_\lambda)$$

and

$$D_\lambda \otimes_{U\mathfrak{g}} \cdot : Mod(\mathfrak{g}, \chi_\lambda) \to Mod_{qc}(D_\lambda)$$

are quasi-inverses. We need natural isomorphisms

$$\Gamma(D_\lambda \otimes_{U\mathfrak{g}} \cdot) \cong id_{Mod(\mathfrak{g},\chi_\lambda)}, \quad D_\lambda \otimes_{U\mathfrak{g}} \Gamma(X, \cdot) \cong id_{Mod_{qc}(D_\lambda)}.$$

We start by giving natural isomorphisms $\Gamma(D_\lambda \otimes_{U\mathfrak{g}} M) \cong M$ for any $M \in Mod(\mathfrak{g}, \chi_\lambda)$.

For any $\Gamma(X, D_\lambda)$-module, M, we can take a resolution

$$\Gamma(X, D_\lambda)^{\oplus I} \to \Gamma(X, D_\lambda)^{\oplus J} \to M \to 0$$

where I and J are some sets.

Applying the functor $\Gamma(X, D_\lambda \otimes_{U\mathfrak{g}} \cdot)$, which is right-exact by Theorem 6.6, to this short exact sequence, we get

$$\Gamma(X, D_\lambda)^{\oplus I} \to \Gamma(X, D_\lambda)^{\oplus J} \to \Gamma(X, D_\lambda \otimes_{U\mathfrak{g}} M) \to 0$$

The first two terms remain unchanged because we have

$$\Gamma(X, D_\lambda \otimes_{U\mathfrak{g}} \Gamma(X, D_\lambda)^{\oplus I}) \cong \Gamma(X, D_\lambda \otimes_{U\mathfrak{g}} \Gamma(X, D_\lambda))^{\oplus I}$$
$$\cong \Gamma(X, D_\lambda \otimes_{\Gamma(X,D_\lambda)} \Gamma(X, D_\lambda))^{\oplus I} \cong \Gamma(X, D_\lambda)^{\oplus I}.$$

The important step that $D_\lambda \otimes_{U\mathfrak{g}} \Gamma(X, D_\lambda) \cong D_\lambda \otimes_{\Gamma(X,D_\lambda)} \Gamma(X, D_\lambda)$ follows because of the surjectivity of $\Phi_\lambda : U\mathfrak{g} \to \Gamma(X, D_\lambda)$.

Using the natural map

$$M \to \Gamma(X, D_\lambda \otimes_{U\mathfrak{g}} M)$$

which simply sends $m \in M$ to the global section $1 \otimes m \in \Gamma(X, D_\lambda \otimes_{U\mathfrak{g}} M)$, we can get a commutative diagram

$$\begin{tikzcd}
\Gamma(X,D_\lambda)^{\oplus I} \arrow[r] \arrow[d, "="] & \Gamma(X,D_\lambda)^{\oplus J} \arrow[r] \arrow[d, "="] & M \arrow[r] \arrow[d] & 0 \\
\Gamma(X,D_\lambda)^{\oplus I} \arrow[r] & \Gamma(X,D_\lambda)^{\oplus J} \arrow[r] & \Gamma(X, D_\lambda \otimes_{U\mathfrak{g}} M) \arrow[r] & 0
\end{tikzcd}$$

and this implies the isomorphism $\mathcal{M} \cong \Gamma(X, D_\lambda \otimes_{U(\mathfrak{g})} \mathcal{M})$ as desired.

Conversely we need to give natural isomorphisms $\mathcal{M} \cong D_\lambda \otimes_{U\mathfrak{g}} \Gamma(X,\mathcal{M})$. We start with the short exact sequence

$$0 \to \mathcal{K} \to D_\lambda \otimes_{U\mathfrak{g}} \Gamma(X,\mathcal{M}) \to \mathcal{M} \to 0$$

where the kernel, $\mathcal{K}$, is in $Mod_{qc}(D_\lambda)$ and the second map is surjective by Theorem 6.6. We then apply the exact functor Γ to the sequence to get

$$0 \to \Gamma(X,\mathcal{K}) \to \Gamma(X, D_\lambda \otimes_{U\mathfrak{g}} \Gamma(X,\mathcal{M})) \to \Gamma(X,\mathcal{M}) \to 0$$

but because the second map is an isomorphism by the above (where $\Gamma(X,\mathcal{M})$ plays the role of M), we see that $\Gamma(X,\mathcal{K}) = 0$. Therefore $\mathcal{K} = 0$ because it is a quasicoherent D_λ-module and must be generated by global sections. □

The fact that $Mod_c(D_\lambda)$ corresponds to $Mod_f(\mathfrak{g},\chi_\lambda)$ can be found in [7], Corollary 11.2.6. This finishes our proof of the Beilinson-Bernstein Localization theorem.

If the reader is interested in this type of math, a great book to read is *D-Modules, Perverse Sheaves, and Representation Theory* by Hotta, Takeuchi, and Tanisaki, [7]. The book covers Beilinson-Bernstein and its applications, along with a multitude of other topics central to geometric representation theory.

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