BOTT PERIODICITY AND K-THEORY

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Abstract. The homotopy and K-theoretic forms of Bott Periodicity can be shown to be equivalent using heavy machinery. However in this paper, we follow the details of Bott's simple and concrete homotopy linking the two different forms.

Contents

1. Introduction

Bott first made the remarkable observation that the classical groups, that is the orthogonal, unitary, and symplectic groups, have periodic homotopy groups. More precisely in his papers [2, 3], Bott showed that

$$\pi_n(U) \cong \pi_{n+2}(U),$$

$$\pi_n(O) \cong \pi_{n+4}(Sp),$$

$$\pi_n(Sp) \cong \pi_{n+4}(O).$$

We will be concerned with the complex case $\pi_n(U) \cong \pi_{n+2}(U)$. This result would spark the development of K-theory. However, the K-theoretic form of Bott periodicity takes on a different form, stating that the homomorphism on reduced K-theory

$$\widetilde{K}(X) \to \widetilde{K}(\Sigma^2 X),$$

induced by taking the external tensor product of stable equivalence classes of vector bundles over X with the canonical line bundle over S^2 is an isomorphism. It's not immediately clear how this is equivalent to Bott's original theorem. In fact, the direct proof of the K-theoretic form of the theorem found in Atiyah's book [5] uses a careful analysis of clutching functions to show that this map is an isomorphism and makes no reference to the identity $\pi_n(U) \cong \pi_{n+2}(U)$. Meanwhile, Bott's original proof of this identity made use of Morse theory and insight into the topology of the spaces to describe a homotopy equivalence

$$\beta\colon BU \to \Omega^2 BU,$$

where BU is the universal classifying space for U. We may then consider the adjoint map

$$\beta^*\colon BU \wedge S^2 \to BU.$$

As we will see, taking the external tensor product of stable equivalence classes of vector bundles over X with the canonical line bundle over S^2 corresponds to a map

$$\alpha\colon BU \wedge S^2 \to BU.$$

It is possible using heavy machinery to show that these two maps must be homotopic, but this leaves something to be desired on an explicit level. In his paper [1], Bott describes an explicit homotopy between these two maps. This establishes a compatibility between the two forms of Bott periodicity and explicitly shows how these two versions relate to each other. In this paper, we work out the details of the explicit homotopy given by Bott [1], as well as develop some of the necessary theory to make sense of this homotopy. We begin with some basic notation and a quick review of the necessary material to put the homotopy into context.

2. Preliminaries

Throughout this paper, X, Y and Z will mean compact, connected, CW complexes with basepoints. Let $F(X,Y)$ be the space of basepoint preserving maps from X to Y. Then we have the equivalence

$$F(X, F(Y,Z)) \cong F(X \wedge Y, Z),$$

where $X \wedge Y = X \times Y / X \vee Y$ is the smash product taking $X \vee Y$ to be $X \times \{y_0\} \cup \{x_0\} \times Y \subset X \times Y$ with x_0 and y_0 the basepoints of X and Y respectively. In particular $\Sigma^n X = X \wedge S^n$ and $\Omega^n X = F(S^n, X)$ give us

$$F(X, \Omega^n Y) \cong F(\Sigma^n X, Y).$$

We can pass this through π_0 to obtain the equivalence

$$[X, \Omega^n Y] \cong [\Sigma^n X, Y],$$

where $[X,Y]$ represents basepoint preserving homotopy classes of maps from X to Y. We note that $S^n \simeq \Sigma^k S^{n-k}$ gives us the identity

$$\pi_n(X) \cong [S^n, X] \cong [S^{n-k}, \Omega^k X] \cong \pi_{n-k}(\Omega^k X).$$

We will assume familiarity with vector bundle theory and the classification of vector bundles. For an introduction to vector bundles we reference Atiyah [5] or chapter 23 of May [4]. We let $Vect(X)$ be the set of equivalence classes of complex vector bundles over X and let $Vect_n(X)$ be the set of equivalence classes of n-dimensional complex vector bundles over X. Recall that we have a universal n-plane bundle $\gamma_n\colon E_n \to BU(n)$, such that we have an isomorphism of pointed sets

$$[X_+, BU(n)] \cong Vect_n(X),$$

induced by pullbacks of γ_n. We also have inclusions $i\colon BU(n) \to BU(n+1)$ such that $i^*(\gamma_{n+1}) \cong \gamma_n \oplus \varepsilon$, where ε is the trivial line bundle. We then have the universal bundle $BU = \mathrm{colim} BU(n)$. It is shown in May [4] that

$$U \simeq \Omega BU.$$

This gives us the identity

$$\pi_n(U) \cong \pi_n(\Omega BU) \cong \pi_{n+1}(BU).$$

Because of this isomorphism, we have the following equivalent homotopic form of Bott periodicity.

Theorem 2.1. *$\pi_q(BU) \cong \pi_{q+2}(BU)$, for $q \geq 1$*

As shown in May [4] we may construct $BU(n)$ as a colimit of Grassmannians. Let $G_n(\mathbb{C}^k)$ be the space of n-dimensional complex subspaces of $\mathbb{C}^k$. We may take

$$E_n^k = \{(A, z) : A \in G_n(\mathbb{C}^k) \text{ and } z \in A\}$$

and let $\gamma_n^k\colon E_n^k \to G_n(\mathbb{C}^k)$ be the projection $(A, z) \mapsto A$, called the canonical bundle. Then the standard inclusion $\mathbb{C}^k \hookrightarrow \mathbb{C}^{k+1}$ gives us inclusions $G_n(\mathbb{C}^k) \hookrightarrow G_n(\mathbb{C}^{k+1})$ and $E_n^k \hookrightarrow E_n^{k+1}$. We define $BU(n) = \mathrm{colim} G_n(\mathbb{C}^k) = G_n(\mathbb{C}^\infty)$ and $E_n = E_n^\infty$. To get the desired inclusions $i_n\colon BU(n) \to BU(n+1)$, we may fix an isomorphism $\mathbb{C}^\infty \oplus \mathbb{C} \cong \mathbb{C}^\infty$ which then induces a homeomorphism between $G_k(\mathbb{C}^\infty \oplus \mathbb{C})$ and $G_k(\mathbb{C}^\infty)$. We then define $i_n\colon G_n(\mathbb{C}^\infty) \to G_{n+1}(\mathbb{C}^\infty \oplus \mathbb{C})$ by taking $A \mapsto A \oplus \mathbb{C}$, which gives our desired inclusion.

Definition 2.2. We define $\mathcal{EU}(X)$ to be the set of equivalence classes of *Vect(X) under the relation that $\xi \sim \eta$ if $\xi \oplus \varepsilon^n \cong \eta \oplus \varepsilon^m$, for some n and m, where ε^n is the n-dimensional trivial bundle. We let $[\xi]_s$ denote the equivalence class of ξ in $\mathcal{EU}(X)$ and call it the stable equivalence class of ξ.*

We observe that if we take the inclusions $Vect_n(X) \hookrightarrow Vect_{n+1}(X)$ obtained by mapping $\xi \mapsto \xi \oplus \varepsilon$, then, since X is connected, $\mathcal{EU}(X) = \mathrm{colim}\, Vect_n(X) = \mathrm{colim}[X_+, BU(n)]$. Since we are assuming X is compact, we have that

$$\mathcal{EU}(X) = \mathrm{colim}[X_+, BU(n)] = [X_+, \mathrm{colim} BU(n)] = [X_+, BU].$$

We wish to consider nondegenerately based spaces. To do this, we can consider the map $f\colon S^0 \to X_+$ which sends the base point of S^0 to the disjoint basepoint of X_+ and the other point of S^0 to the desired basepoint of X. We then note that the homotopy cofiber $Cf = X \cup_f CS^0$ is just homotopy equivalent to X. Taking maps into BU, the cofiber sequence $S^0 \to X_+ \to X \to \Sigma S^0$ induces an exact sequence

$$[S^1, BU] \to [X, BU] \to [X_+, BU] \to [S^0, BU],$$

which gives us the isomorphism

$$[X, BU] \cong [X_+, BU],$$

since BU is simply connected. This follows from $BU \simeq \Omega U$ and the fact that U is path-connected. Throughout the rest of this paper whenever we consider homotopies of maps into Z we will always be working with simply connected spaces Z. So we will have the equivalence

$$[X, Z] \cong [X_+, Z].$$

Thus we will not worry whether our homotopies fix basepoints or not as a class in $[X_+, Z]$ corresponds to a unique class in $[X, Z]$.

As will become clear later in the paper, we will be especially concerned with the Grassmannians of the form

$$\Gamma_n = G_n(\mathbb{C}^{2n}).$$

For $n \leq k$, we may take inclusions $\Gamma_n \hookrightarrow \Gamma_k$ by first taking an inclusion $\mathbb{C}^{2n} \hookrightarrow \mathbb{C}^{2k}$ and then fixing a $(k-n)$-dimensional plane P in the remaining $2k-2n$ coordinates and taking $A \mapsto A \oplus P$ for every $A \in \Gamma_n$. Any two of these inclusions $i_1, i_2 \colon \Gamma_n \hookrightarrow \Gamma_k$ differ by an automorphism of Γ_k induced by a change of basis in $\mathbb{C}^{2k}$. Let $T \in GL_{2k}(\mathbb{C})$ be the corresponding linear isomorphism. Then because $GL_{2k}(\mathbb{C})$ is path-connected, we may choose a path between the identify transformation and T in $GL_{2k}(\mathbb{C})$ which then induces a homotopy between i_1 and i_2. This gives us a unique inclusion up to homotopy. We see that up to homotopy, $BU \simeq \mathrm{hocolim}\Gamma_n$ is the homotopy colimit.

2.1. **K-Theory.**

Changing focus slightly, we note that *Vect*(X) forms a commutative monoid with respect to Whitney sums. Given any commutative monoid A, we can form an abelian group $K(A)$ called the Grothendieck group of A, with the universal property that there exists a semigroup homomorphism $f\colon A \to K(A)$ such that if G is any group with a semigroup homomorphism $g\colon A \to G$, then there is a unique group homomorphism $h\colon K(A) \to G$ such that $h \circ f = g$.

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "g"'] & K(A) \arrow[dl, dashed, "h"] \\ G & \end{tikzcd}$$

This guarantees us that $K(A)$ must be unique and essentially says that $K(A)$ is the smallest group containing A. If A is a semiring then multiplication in A induces multiplication in $K(A)$, which gives it a ring structure. The tensor product turns *Vect*(X) into a semiring.

Definition 2.3. The *K-Theory of* X, written $K(X)$, is the Grothendieck ring $K($*Vect(X)*$)$. We call elements of $K(X)$ *virtual bundles* over X.

For $\xi \in$ *Vect(X)*, we let $[\xi]$ denote the element of $K(X)$ represented by ξ. We can represent general elements of $K(X)$ by formal differences, $[\xi] - [\eta]$, where $\xi, \eta \in$ *Vect(X)*, with the relation that $[\xi_1] - [\eta_1] = [\xi_2] - [\eta_2]$ iff $\xi_1 \oplus \eta_2 \oplus \varepsilon^n \cong \xi_2 \oplus \eta_1 \oplus \varepsilon^n$, for some n. Addition is then given by

$$[\xi_1] - [\eta_1] + [\xi_2] - [\eta_2] = [\xi_1 \oplus \xi_2] - [\eta_1 \oplus \eta_2],$$

and multiplication is given by

$$([\xi_1] - [\eta_1])([\xi_2] - [\eta_2]) = [\xi_1 \otimes \xi_2] + [\eta_1 \otimes \eta_2] - [\xi_1 \otimes \eta_2] - [\xi_2 \otimes \eta_1].$$

Given $[\xi] - [\eta] \in K(X)$, there is some $\eta' \in$ *Vect(X)* such that $\eta \oplus \eta' \cong \varepsilon^n$ for some trivial bundle ε^n. Then $[\xi] - [\eta] = [\xi \oplus \eta'] - [\eta \oplus \eta'] = [\xi'] - [\varepsilon^n]$, where $\xi' \cong \xi \oplus \eta'$. From this we see that every virtual bundle can be written in the form $[\xi] - [\varepsilon^n]$.

A map $f\colon X \to Y$ induces via pullback a semiring homomorphism $f^*\colon Vect(Y) \to Vect(X)$ by taking $\xi \mapsto f^*\xi$ which then induces a ring homomorphism $f^*\colon K(Y) \to K(X)$ by taking $[\xi] - [\eta] \mapsto [f^*\xi] - [f^*\eta]$.

We may consider the induced homomorphism from the inclusion of the basepoint $i\colon \{x_0\} \hookrightarrow X$ given by $i^*\colon K(X) \to K(x_0)$. This map corresponds to the dimension map $d\colon K(X) \to \mathbb{Z}$ since any vector bundle over a point must be trivial and is thus characterized by its dimension. We then define $\widetilde{K}(X)$ to be $\ker(d)$ and thus $\widetilde{K}(X)$ is an ideal of $K(X)$ and therefore a ring without identify. This short exact sequence splits giving us the equivalence $K(X) \cong \widetilde{K}(X) \times \mathbb{Z}$. We call $\widetilde{K}$(X) the *reduced* K-theory of X.

Proposition 2.4. $\mathcal{EU}(X) \cong \widetilde{K}(X)$

Proof. Take $[\xi]_s \leftrightarrow [\xi] - [\varepsilon^n]$, where $n = \dim(\xi)$. □

2.2. **Kunneth-Type Formula.** For any X and Y we have an external product, $\mu\colon K(X) \otimes K(Y) \to K(X \times Y)$ given by $\mu(a \otimes b) = p_1^*(a)p_2^*(b)$ where $p_1\colon X \times Y \to X$ and $p_2\colon X \times Y \to Y$ are the projections. We will write $a * b$ in place of $\mu(a \otimes b)$.

Theorem 2.5 (Unreduced Bott Periodicity)**.** *The external product map*

$$\mu\colon K(X) \otimes K(S^2) \to K(X \times S^2),$$

is an isomorphism.

A priori, it's not clear how this is equivalent to the homotopy theoretic form of Bott periodicity. Before we can show how these are related, we first wish to relate this to a map on reduced K-theory. To do this, we note that since $K(-)$ is a representable functor and since X and Y are CW complexes, we have a splitting short exact sequence as mentioned in May [4]

$$0 \to \widetilde{K}(X \wedge Y) \to \widetilde{K}(X \times Y) \to \widetilde{K}(X \vee Y) \to 0,$$

which comes from the cofiber sequence

$$X \vee Y \to X \times Y \to X \wedge Y.$$

We also have an isomorphism

$$\widetilde{K}(X \vee Y) \cong \widetilde{K}(X) \oplus \widetilde{K}(Y)$$

since X and Y are retracts of $X \times Y$. This gives us

$$\widetilde{K}(X \times Y) \cong \widetilde{K}(X \wedge Y) \oplus \widetilde{K}(X) \oplus \widetilde{K}(Y).$$

We see that if $a \in \widetilde{K}(X)$ and $b \in \widetilde{K}(Y)$, then $a * b \in K(X \times Y)$ such that $a * b$ is 0 over $X \vee Y$. This gives us that $a * b \in \widetilde{K}(X \times Y)$ and corresponds to a unique element of $\widetilde{K}(X \wedge Y)$. From this we obtain a map $\mu\colon \widetilde{K}(X) \otimes \widetilde{K}(Y) \to \widetilde{K}(X \wedge Y)$, which we also denote by $a * b$.

Theorem 2.6 (Reduced Bott Periodicity)**.** *The map* $\mu\colon \widetilde{K}(X) \otimes \widetilde{K}(S^2) \to \widetilde{K}(\Sigma^2 X)$ *is an isomorphism.*

Proof. This map on reduced K-theory is essentially just the restriction of the map on K-theory given by

$$\begin{tikzcd}
K(X) \otimes K(Y) \arrow[r, equal] \arrow[d, "\mu"'] & \widetilde{K}(X) \otimes \widetilde{K}(Y) \oplus \widetilde{K}(X) \oplus \widetilde{K}(Y) \oplus \mathbb{Z} \arrow[d, "\mu \,\oplus\, \| \,\oplus\, \| \,\oplus\, \|"] \\
\widetilde{K}(X \wedge Y) \arrow[r, equal] & \widetilde{K}(X \wedge Y) \oplus \widetilde{K}(X) \oplus \widetilde{K}(Y) \oplus \mathbb{Z}.
\end{tikzcd}$$

Because of this, it is equivalent to prove the isomorphism in either the reduced or unreduced case. □

We note that $\widetilde{K}(S^2) \cong \pi_2(BU) \cong \mathbb{Z}$. We may think of S^2 as $\mathbb{CP}^1$.

Theorem 2.7. *The equivalence class of the canonical line bundle, $b = [\gamma_1^2] - [\varepsilon]$, generates $\widetilde{K}(S^2)$.*

Theorem 2.8 (Bott Periodicity)**.** *The map $\widetilde{K}(X) \to \widetilde{K}(X) \otimes \widetilde{K}(S^2) \to \widetilde{K}(\Sigma^2 X)$ given by*

$$a \mapsto a \otimes b \mapsto a * b,$$

is an isomorphism.

Proof. Since the first map is always an isomorphism, this map being an isomorphism is equivalent to the second map being an isomorphism which is just Theorem 2.6. $\square$

Throughout the rest of this paper we will be working with the reduced case. Using our identification of $\mathcal{EU}(X)$ with $\widetilde{K}(X)$ we wish to define $*$ on vector bundles in such a way that it is well defined on equivalence classes in $\mathcal{EU}(X)$ and agrees with how we defined $*$ in $\widetilde{K}(X)$ under our identification. If $[\xi]_s, [\eta]_s \in \mathcal{EU}(X)$, then let $n = \dim(\xi)$ and let $m = \dim(\eta)$. We can consider the external tensor product $\xi \otimes \eta$ as a vector bundle over $X \times Y$, but this is not well defined on equivalence classes. Using the reduced external product we see that

$$\begin{aligned}
([\xi] - [\varepsilon^n]) * ([\eta] - [\varepsilon^m]) &= [\xi \otimes \eta] - [\xi \otimes \varepsilon^m] - [\varepsilon^n \otimes \eta] + [\varepsilon^n \otimes \varepsilon^m] \\
&= [\xi \otimes \eta] + [\xi^\perp \otimes \varepsilon^m] + [\varepsilon^n \otimes \eta^\perp] + [\varepsilon^n \otimes \varepsilon^m] \\
&\quad - [\xi \otimes \varepsilon^m \oplus \xi^\perp \otimes \varepsilon^m] - [\varepsilon^n \otimes \eta \oplus \varepsilon^n \otimes \eta^\perp] \\
&= [\xi \otimes \eta] + [\xi^\perp \otimes \varepsilon^m] + [\varepsilon^n \otimes \eta^\perp] - [\varepsilon^k],
\end{aligned}$$

where $\xi^\perp \in -[\xi]$. From this we see that the desired product under our identification is

$$\xi * \eta = \xi \otimes \eta \oplus \xi^\perp \otimes \varepsilon^m \oplus \varepsilon^n \otimes \eta^\perp.$$

Since the map $\widetilde{K}(X) \otimes \widetilde{K}(Y) \to \widetilde{K}(X \times Y)$ gives a unique map $\widetilde{K}(X) \otimes \widetilde{K}(Y) \to \widetilde{K}(X \wedge Y)$, the map we've just defined gives us a map $\mathcal{EU}(X) \otimes \mathcal{EU}(Y) \to \mathcal{EU}(X \times Y)$ which induces a map $\mathcal{EU}(X) \otimes \mathcal{EU}(Y) \to \mathcal{EU}(X \wedge Y)$. It is this explicit product on vector bundles that we will be working with in the last section.

3. Bott Periodicity

If we assume the K-theoretic form of Bott periodicity, then in particular the case when X is a sphere gives us the equivalence

$$\pi_n(BU) = [S^n, BU] \cong \widetilde{K}(S^n) \cong \widetilde{K}(S^{n+2}) \cong [S^{n+2}, BU] = \pi_{n+2}(BU).$$

So the homotopy theoretic form of complex Bott periodicity

$$\pi_n(U) = \pi_{n+2}(U),$$

follows immediately from the K-theoretic form of Bott periodicity. Bott's original proof used Morse theory to describe a homotopy equivalence

$$\beta \colon BU \to \Omega^2 BU.$$

This immediately induces an isomorphism

$$\widetilde{K}(X) \cong [X, BU] \cong [X, \Omega^2 BU] \cong [X \wedge S^2, BU] \cong \widetilde{K}(X \wedge S^2).$$

However, we have no immediate guarantee that this isomorphism is in any way related to the tensor product of bundles. The tensor product of stable equivalence classes of vector bundles gives us a map

$$\otimes\colon BU \wedge BU \to BU.$$

The generator of $\tilde{K}(S^2)$ also corresponds to a map

$$b\colon S^2 \to BU.$$

We may then consider the composition

$$\alpha\colon BU \wedge S^2 \xrightarrow{id \wedge b} BU \wedge BU \xrightarrow{\otimes} BU,$$

Which induces the reduced external tensor product. It turns out that this map α is actually homotopic to $\beta^*\colon BU \wedge S^2 \to BU$, the adjoint of β. This is what we prove in the final section of this paper.

The external product on reduced K-theory with the canonical line bundle of S^2 then takes the form

$$\widetilde{K}(X) \cong [X, BU] \xrightarrow{- \wedge id} [X \wedge S^2, BU \wedge S^2] \xrightarrow{\alpha \circ -} [X \wedge S^2, BU] \cong \widetilde{K}(\Sigma^2 X).$$

By showing that α and β^* are homotopic we show that the external tensor product of stable vector bundles over X with the canonical line bundle of S^2 and the map on K-theory coming from β are actually the same by the following commutative diagram

$$\begin{tikzcd}
\widetilde{K}(X) \cong [X, BU] \arrow[r, "- \wedge id"] \arrow[drr, "\beta \circ -"'] & {[X \wedge S^2, BU \wedge S^2]} \arrow[r, "\alpha \circ -", "\beta^* \circ -"'] & {[X \wedge S^2, BU] \cong \widetilde{K}(\Sigma^2 X)} \arrow[d, equal, "*"] \\
 & & {[X, \Omega^2 BU].}
\end{tikzcd}$$

The K-theoretic form of periodicity then follows immediately from Bott's work which showed that β is a homotopy equivalence. We will spend the rest of the paper establishing the homotopy.

3.1. **The Finite Step.** To show that these maps are homotopic, we will construct explicit maps on finite Grassmannians which pass to the colimit to give the desired maps α and β.

Implicitly in the work of Bott [2, 3] a map

$$f\colon \Gamma_n \to \Omega U(2n)$$

is given for each n. The primary work of Bott's proof is showing that this map induces an isomorphism on homotopy groups as $n \to \infty$.

By identifying Γ_n with $U(2n)/U(n) \times U(n)$, We can describe a map coming from the suspension in the long exact sequence of homotopy groups induced by the fiber sequence

$$U(n) \to U(2n)/U(n) \to U(2n)/U(n) \times U(n),$$

of the form

$$\lambda\colon U(n) \to \Omega\Gamma_n.$$

Then since $U(2n)/U(n)$ is $2n$-connected, the map λ becomes an isomorphism on homotopy groups as $n \to \infty$.

We then consider the composition to get the map

$$\beta = \Omega\lambda \circ f \colon \Gamma_n \to \Omega^2\Gamma_{2n},$$

which by the work of Bott induces isomorphisms on homotopy groups as $n \to \infty$. We then consider its adjoint

$$\beta^* \colon \Gamma_n \wedge S^2 \to \Gamma_{2n}.$$

By the classification of vector bundles, we know that the equivalence class $[\gamma_n^{2n} * \gamma_1^2]_s$ is given as the pullback via some map

$$\alpha \colon \Gamma_n \wedge S^2 \to \Gamma_m,$$

for some m. We will see that α commutes with inclusions $\Gamma_n \hookrightarrow \Gamma_k$ up to homotopy since $\alpha \circ i_1$ and $i_2 \circ \alpha$ differ by an automorphism of Γ_k induced by a change of basis. This then induces a map on the homotopy colimit $\alpha \colon BU \wedge S^2 \to BU$, which is unique up to homotopy and represents the tensor product of stable vector bundles with the canonical line bundle over S^2. If we then take an inclusion $\Gamma_{2n} \hookrightarrow \Gamma_m$, we may show that $\alpha \colon \Gamma_n \wedge S^2 \to \Gamma_m$ and $\beta^* \colon \Gamma_n \wedge S^2 \to \Gamma_m$ are actually homotopic and thus when we pass these maps to the homotopy colimit they represent the same homotopy class.

3.2. **Explicit Maps.** We will work in Γ_{4n} for our homotopies. Because the sequence

$$0 \to [\Gamma_n \wedge \Gamma_1, \Gamma_{4n}] \to [\Gamma_n \times \Gamma_1, \Gamma_{4n}] \to [\Gamma_n \vee \Gamma_1, \Gamma_{4n}] \to 0,$$

is exact we know that the homotopy classes of $[\Gamma_n \wedge \Gamma_1, \Gamma_{4n}]$ are exactly those homotopy classes of $[\Gamma_n \times \Gamma_1, \Gamma_{4n}]$ which are nullhomotopic over $\Gamma_n \vee \Gamma_1$. This allows us to work with maps $\Gamma_n \times \Gamma_1 \to \Gamma_{4n}$.

Let $A_0 \in \Gamma_n$ and $L_0 \in \Gamma_1$ be the basepoints of the spaces. And let $A^\perp$ represent the orthogonal compliment of A inside of $\mathbb{C}^{2n}$ and $L^\perp$ be the orthogonal compliment of L inside of $\mathbb{C}^2$. We can use the identification

$$\mathbb{C}^{8n} = (\mathbb{C}^{2n} \otimes \mathbb{C}^2) \oplus (\mathbb{C}^{2n} \otimes \mathbb{C}^2)$$

to define our map $\alpha \colon \Gamma_n \times \Gamma_1 \to \Gamma_{4n}$ explicitly by

$$\alpha(A, L) = (A \otimes L \oplus A^\perp \otimes L_0) \oplus (A_0 \otimes L^\perp \oplus A_0 \otimes L_0),$$

where the direct sums inside the parentheses take place inside the two $\mathbb{C}^{2n} \otimes \mathbb{C}$ terms respectively and the direct sum outside the parentheses takes place in the direct sum of the two terms. In the formula for α, the first three terms correspond to the reduced tensor product $\gamma_n^{2n} * \gamma_1^2$ and the last term corresponds to a trivial bundle so we see that

$$[\alpha^*(\gamma_{4n}^{8n})]_s = [\gamma_n^{2n} * \gamma_1^2]_s.$$

Bott's work [1, 2, 3] gives us a map $f \colon \Gamma_n \to \Omega U(2n)$ defined by sending each n-plane $A \in \Gamma_n$ to a loop of unitary transformations where at each time $0 \le \theta \le 2\pi$ the transformation $f(A)(\theta)$ is defined by taking

$$f(A)(\theta)z = \begin{cases} e^{i\theta}z, & z \in A \text{ and } 0 \le \theta \le \pi, \\ e^{-i\theta}z, & z \in A^\perp \text{ and } 0 \le \theta \le \pi, \\ e^{i\theta}z, & z \in A_0 \text{ and } \pi \le \theta \le 2\pi, \\ e^{-i\theta}z, & z \in A_0^\perp \text{ and } \pi \le \theta \le 2\pi. \end{cases}$$

Thinking of $\mathbb{C}^{2n}$ as $\mathbb{C}^n \otimes \mathbb{C}^2$ we can give $\lambda\colon U(n) \to \Omega\Gamma_n$ explicitly by first defining

$$\lambda(a)(\phi) = Span\{z \otimes e_1 \cos\phi + a(z) \otimes e_2 \sin\phi : z \in \mathbb{C}^n\}.$$

for $0 \le \phi \le \pi/2$, where $\{e_1, e_2\}$ is a basis for $\mathbb{C}^2$. This gives us a map into the space of paths in Γ_n with fixed basepoints $\mathbb{C}^{2n} \otimes e_1$ and $\mathbb{C}^{2n} \otimes e_2$. We can then identify this space with the space $\Omega\Gamma_n$ of loops in Γ_n with basepoint $\mathbb{C}^n \otimes e_1$ in such a way that we have a homotopy equivalence between these two spaces. For instance we may take

$$\lambda(a)(\phi) = Span\{z \otimes e_1 \cos\phi - e_2 \sin\phi : z \in \mathbb{C}^n\}.$$

for $\pi/2 \le \phi \le \pi$.

3.3. **The Desired Homotopy.**

Theorem 3.1. *The maps α and β^* are homotopic.*

Proof. We will start by identifying $\Gamma_1 = \mathbb{CP}^1$ with S^2 in the following way. Let I^2 be the rectangle defined by $0 \le \theta \le 2\pi$ and $0 \le \phi \le \pi$. We can then describe a map $\rho\colon I^2 \to \Gamma_1$ by defining

$$\rho(\theta, \phi) = Span\{e_1 \cos\phi/2 + e_2 e^{i\theta} \sin\phi/2\}.$$

We note that $\rho(0, \phi) = \rho(2\pi, \phi)$. We also observe that both $\rho(\theta, 0)$ and $\rho(\theta, \pi)$ are independent of θ. This then defines a map $\rho\colon S^2 \to \Gamma_1$ and gives us a way of identifying Γ_1 with S^2, where we think of S^2 as the quotient of I^2 obtained by identifying $(0, \phi) \sim (2\pi, \phi)$ and collapsing $(\theta, 0)$ and (θ, π) to two separate points. Let $\bar{L}$ be the image of L under the complex conjugate map. We see the set of lines L such that $L = \bar{L}$ is given by the equator $\theta = \pi$. (We are assuming that $\{e_1, e_2\}$ is a real basis for $\mathbb{C}^2$.) This divides Γ_1 into two hemispheres, D^+ and D^-, which we may choose so that D^+ corresponds to $\theta < \pi$ and D^- corresponds to $\theta > \pi$.
We first wish to find

$$\beta^*\colon \Gamma_n \times \Gamma_1 \to \Gamma_{2n}.$$

We have that

$$\beta^*(A, \theta, \phi) = \lambda(f(A)(\theta))(\phi),$$

by definition. And since for $\phi \ge \pi/2$ we have that λ is constant in terms of θ we may homotope β^* to be the map

$$(A, \theta, \phi) \mapsto \lambda(f(A)(\theta))(\phi/2).$$

This was the result of our identification earlier. We now wish to compute this homotopic map which we will still call β^*.
For $\theta < \pi$, we may decompose $\mathbb{C}^{2n} = A \oplus A^\perp$. Then we have that

$$\beta^*(A, (\theta, \phi)) = Span\{z \otimes e_1 \cos\phi/2 + f(A)(\theta)(z) \otimes e_2 \sin\phi/2 : z \in \mathbb{C}^n\}.$$

Splitting this up over $A \oplus A^\perp$, we get

$$\begin{aligned}\beta^*(A, (\theta, \phi)) = {} & Span\{z \otimes e_1 \cos\phi/2 + e^{i\theta} z \otimes e_2 \sin\phi/2 : z \in A\} \\ & \oplus Span\{z \otimes e_1 \cos\phi/2 + e^{-i\theta} z \otimes e_2 \sin\phi/2 : z \in A^\perp\}.\end{aligned}$$

And this is just

$$\beta^*(A, L) = A \otimes L \oplus A^\perp \otimes \bar{L}$$

When $\theta > \pi$, we may do a similar thing only using the decomposition $\mathbb{C}^{2n} = A_0 \oplus A_0^\perp$ to get

$$\beta^*(A, L) = A_0 \otimes L \oplus A_0^\perp \otimes \bar{L}.$$

Putting all this together we have that

$$\beta^*(A,L) = \begin{cases} A\otimes L \oplus A^\perp \otimes \bar{L}, & \text{when } L \in D^+,\\ A_0 \otimes L \oplus A_0^\perp \otimes \bar{L}, & \text{when } L \in D^-.\end{cases}$$

We can take an inclusion $\Gamma_{2n} \hookrightarrow \Gamma_{4n}$ to get

$$\beta^*(A,L) = \begin{cases} (A\otimes L \oplus A^\perp \otimes \bar{L}) \oplus A_0 \otimes \mathbb{C}^2, & \text{when } L \in D^+,\\ (A_0 \otimes L \oplus A_0^\perp \otimes \bar{L}) \oplus A_0 \otimes \mathbb{C}^2, & \text{when } L \in D^-.\end{cases}$$

We now make the observation that

$$A_0 \otimes \mathbb{C}^2 = A_0 \otimes (L \oplus L^\perp) = A_0 \otimes L \oplus A_0 \otimes L^\perp,$$

for all $L \in \Gamma_1$. Then we may find a path $h\colon I \to \Gamma_1$ from A_0 to $A_0^\perp$, which gives us a homotopy $H\colon \Gamma_n \times \Gamma_1 \times I \to \Gamma_{4n}$ between β^* and the map given by

$$(A,L) \mapsto \begin{cases} (A\otimes L \oplus A^\perp \otimes \bar{L}) \oplus (A_0^\perp \otimes L \oplus A_0 \otimes L^\perp), & \text{when } L \in D^+,\\ (A_0 \otimes L \oplus A_0^\perp \otimes \bar{L}) \oplus (A_0^\perp \otimes L \oplus A_0 \otimes L^\perp), & \text{when } L \in D^-.\end{cases}$$

This homotopy can be explicitly given by

$$H(A,L,t) = \begin{cases} (A\otimes L \oplus A^\perp \otimes \bar{L}) \oplus (h(t) \otimes L \oplus A_0 \otimes L^\perp), & \text{when } L \in D^+,\\ (A_0 \otimes L \oplus A_0^\perp \otimes \bar{L}) \oplus (h(t) \otimes L \oplus A_0 \otimes L^\perp), & \text{when } L \in D^-.\end{cases}$$

For the next step we wish to find a homotopy between this map and the map

$$(A,L) \mapsto \begin{cases} (A\otimes L \oplus A^\perp \otimes \bar{L}) \oplus (A_0^\perp \otimes L \oplus A_0 \otimes L^\perp), & \text{when } L \in D^+,\\ (A_0 \otimes L \oplus A_0^\perp \otimes L) \oplus (A_0^\perp \otimes \bar{L} \oplus A_0 \otimes L^\perp), & \text{when } L \in D^-,\end{cases}$$

obtained by switching the two middle terms on the bottom hemisphere and leaving the map unchanged on the top hemisphere. We will do this via a homotopy which is fixed on $\Gamma_n \times D^+$ and which on $\Gamma_n \times D^-$ acts as a rotation, allowing us to switch the middle two terms.

To explicitly describe this rotation we start with the decomposition

$$\begin{aligned}\mathbb{C}^{8n} &\cong (\mathbb{C}^{2n} \oplus \mathbb{C}^{2n}) \oplus (\mathbb{C}^{2n} \oplus \mathbb{C}^{2n})\\ &= (A_0 \otimes \mathbb{C}^2 \oplus A_0^\perp \otimes \mathbb{C}^2) \oplus (A_0^\perp \otimes \mathbb{C}^2 \oplus A_0 \otimes \mathbb{C}^2)\end{aligned}$$

We may choose an orthonormal basis $\{s_1, \dots, s_{2n}, u_1, \dots, u_{2n}, v_1, \dots, v_{2n}, w_1, \dots, w_{2n}\}$ for $\mathbb{C}^{8n}$ such that $\{r_1, \dots, r_{2n}\}$ is an orthonormal basis for the first $\mathbb{C}^{2n}$ term if $r = s$, the second $\mathbb{C}^{2n}$ term if $r = u$, the third $\mathbb{C}^{2n}$ term if $r = v$, and the fourth $\mathbb{C}^{2n}$ term if $r = w$. Then for $0 \le t \le 1$ we may define a linear isometry $T_t\colon \mathbb{C}^{8n} \to \mathbb{C}^{8n}$ by fixing s_i and w_i for each i and taking $u_i \mapsto \cos(t)u_i - \sin(t)v_i$ and $v_i \mapsto \sin(t)u_i + \cos(t)v_i$. This gives us a map $P_t\colon \Gamma_{4n} \to \Gamma_{4n}$ by rotating any subspace as induced by T_t. We note that for $L \in \partial D^-$, since $L = \bar{L}$, the plane

$$(A_0 \otimes L \oplus A_0^\perp \otimes L) \oplus (A_0^\perp \otimes \bar{L} \oplus A_0 \otimes L^\perp)$$

is fixed by P_t. This then induces the desired homotopy $H_t\colon \Gamma_n \times \Gamma_1 \to \Gamma_{4n}$ given by

$$H_t(A,L) = \begin{cases} (A\otimes L \oplus A^\perp \otimes \bar{L}) \oplus (A_0^\perp \otimes L \oplus A_0 \otimes L^\perp), & \text{when } L \in D^+,\\ P_t((A_0 \otimes L \oplus A_0^\perp \otimes \bar{L}) \oplus (A_0^\perp \otimes L \oplus A_0 \otimes L^\perp)), & \text{when } L \in D^-.\end{cases}$$

After performing this rotation we note that

$$A_0 \otimes L \oplus A_0^\perp \otimes L = \mathbb{C}^{2n} \otimes L = A \otimes L \oplus A^\perp \otimes L,$$

for any $A \in \Gamma_n$. So our map is now in the form

$$(A, L) \mapsto (A \otimes L \oplus A^{\perp} \otimes \Phi(L)) \oplus (A_0^{\perp} \otimes \Psi(L) \oplus A_0 \otimes L^{\perp}),$$

where $\Phi, \Psi \colon \Gamma_1 \to \Gamma_1$ are given by

$$\Phi(L) = \begin{cases} \bar{L}, & L \in D^+, \\ L & L \in D^-, \end{cases}$$

and

$$\Psi(L) = \begin{cases} L, & L \in D^+, \\ \bar{L}, & L \in D^-. \end{cases}$$

The first map Φ sends both hemispheres into D^- and Ψ sends both hemispheres into D^+. Thus they are both nullhomotopic. This gives us a homotopy to the map

$$\begin{aligned}(A, L) \mapsto &(A \otimes L \oplus A^{\perp} \otimes L_0) \oplus (A_0^{\perp} \otimes L_0 \oplus A_0 \otimes L^{\perp}) \\ &= (A \otimes L \oplus A^{\perp} \otimes L_0) \oplus (A_0 \otimes L^{\perp} \oplus A_0^{\perp} \otimes L_0).\end{aligned}$$

We may then take a path $I \to \Gamma_n$ from $A_0^{\perp}$ to A_0 to get a homotopy to

$$(A, L) \mapsto (A \otimes L \oplus A^{\perp} \otimes L_0) \oplus (A_0 \otimes L^{\perp} \oplus A_0 \otimes L_0),$$

which is just α, thus completing the proof.

□

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