CONVERGENCE OF THE FOURIER SERIES

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Abstract. The Fourier series is an expression of a $2\pi$ periodic, integrable function as a sum of a basis of trigonometric polynomials. In the following, we first discuss basic definitions and operations pertaining the Fourier Series. Then, we discuss the Abel summability of the Fourier series of Riemann integrable, $2\pi$ periodic functions. Finally, we prove the mean square convergence of the Fourier series and Parseval’s identity.

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1. Introduction

The Fourier series of a $2\pi$ periodic, integrable function provides a representation of the function as the sum of trigonometric functions. The representation of a more elaborate function through a basis of well understood trigonometric functions is useful in a similar way to how representing an infinitely differentiable function as a Taylor series is useful in analysis and number theory. The generalization of Fourier series to different $2\pi$ periodic bases is called Harmonic Analysis, which has further applications in algebra and analysis. This paper will show how the Fourier series of a function converges to the original function. Section 2 introduces basic definitions of the Fourier series such as the Fourier series and the partial Fourier sum. Section 3 introduces the Dirichlet Kernel and convolution, two concepts that are important when manipulating different Fourier series. Section 4 introduces the Abel mean and proves the Abel summability of the Fourier series to the original function. Section 5 introduces the inner product space of Riemann integrable functions to prove the mean square convergence of the Fourier series and the related Parseval’s identity.
2. Preliminaries

Let \( f: [a, b] \to \mathbb{C} \) where \( a, b \in \mathbb{R} \) be a (Riemann) integrable function such that \( f(b) = f(a) \). The domain of this function can be expanded to all \( x \in \mathbb{R} \) to make a periodic function by allowing \( f(x + m(b - a)) = f(x + n(b - a)) \) for all \( m, n \in \mathbb{Z} \). Often in this paper we will assume \( (a, b, L) = (-\pi, \pi, 2\pi) \), creating a \( 2\pi \) periodic function with \( f(-\pi) = f(\pi) \) or a function on a circle. This substitution of \( a, b, L \) for \(-\pi, \pi, 2\pi\) respectively can be done without loss of generality as it is a mere translation and scaling of the function.

Definition 2.1. Let \( n \in \mathbb{Z} \). The \( n \)th Fourier coefficient of \( f \) is

\[
\hat{f}(n) := \frac{1}{L} \int_a^b f(\theta)e^{-2\pi in\theta/L}d\theta.
\]

Definition 2.3. Let \( N \) be a positive integer. The \( N \)th partial Fourier sum of \( f \) is

\[
S_N(f)(\theta) := \sum_{n=-N}^{N} \hat{f}(n)e^{2\pi in\theta/L}.
\]

Definition 2.5. The Fourier series of \( f \) is

\[
\lim_{N \to \infty} (S_N(f)(\theta)) = \sum_{n=-\infty}^{\infty} \hat{f}(n)e^{2\pi in\theta/L}.
\]

The convergence of this limit will be discussed in later sections. If we assume the Fourier series converges to \( f \), Equation (2.2) for the \( n \)th Fourier coefficient can be derived from the Fourier series, giving the intuition for why Equation (2.2) involves an integral. For the following proof, assume that we do not know Equation (2.2) as the definition of a Fourier coefficient.

Theorem 2.7. Suppose \( f \) is a \( 2\pi \) periodic function that is integrable from \([-\pi, \pi]\), and the Fourier series of \( f \) given by Equation (2.6) converges to \( f \). Then,

\[
\hat{f}(n) = \int_a^b f(\theta)e^{-in\theta/L}d\theta
\]

Proof. From Equation (2.6),

\[
f(\theta) = \sum_{n=-\infty}^{\infty} \hat{f}(n)e^{in\theta}.
\]

Then, manipulating Equation (2.6) for any \( m \in \mathbb{Z} \)

\[
\frac{1}{2\pi} \int_{-\pi}^{\pi} f(\theta)e^{-im\theta}d\theta = \frac{1}{2\pi} \int_{-\pi}^{\pi} \sum_{n=-\infty}^{\infty} \hat{f}(n)e^{in\theta}e^{-im\theta}d\theta
\]

\[
= \frac{1}{2\pi} \int_{-\pi}^{\pi} \sum_{n=-\infty}^{\infty} \hat{f}(n)e^{i(n-m)\theta}d\theta.
\]

When \( n = m \),

\[
\frac{1}{2\pi} \int_{-\pi}^{\pi} \hat{f}(n)d\theta = \hat{f}(n).
\]
When \( n \neq m \), the expression can be evaluated using \( e^{i\theta} = \cos \theta + i \sin \theta \)

\[
\frac{1}{2\pi} \int_{-\pi}^{\pi} \hat{f}(n)e^{i(n-m)\theta} \, d\theta = \frac{1}{2\pi} \left. \hat{f}(n) e^{i(n-m)\theta} \right|_{\theta = -\pi}^{\pi} = 0.
\]

This proof used the orthogonality of the family of functions \( \{e^{inx}\}_{n \in \mathbb{Z}} \). This will be elaborated upon when discussing the convergence of the Fourier series.

It is important to note that these definitions relating to Fourier analysis are often stated as representing a real valued \( f \) as a sum of sine and cosine functions. One can interchange between exponential form and the sinusoidal form by applying Euler’s formula. The expression for the Fourier series for example, would be

\[
c_0 + \sum_{n=1}^{\infty} c_n \cos(n \theta) + \sum_{n=1}^{\infty} s_n \sin(n \theta).
\]

This sinusoidal form will not be used in this paper.

3. Convolution

**Definition 3.1.** Let \( N \) be a positive integer. The \( N \)th Dirichlet kernel is a function

\[
D_N(\theta) = \sum_{n=-N}^{N} e^{inx}.
\]

In other words, the Dirichlet kernel is a partial Fourier series where its \( n \)th Fourier coefficient \( \hat{f}(n) \) is 1 if \(|n| \leq N\) and 0 if \(|n| > N\).

**Theorem 3.3.** The closed form equation for the \( N \)th Dirichlet kernel is

\[
D_N(\theta) = \frac{\sin((N + 1/2)x)}{\sin(x/2)}.
\]

**Proof.** By Equation (3.2), the \( N \)th Dirichlet kernel is

\[
D_N(\theta) = \sum_{n=-N}^{N} e^{inx}.
\]

By splitting the sum into sums with negative and nonnegative \( N \), we obtain

\[
D_N(\theta) = \sum_{n=-N}^{-1} (e^{i\theta})^n + \sum_{n=0}^{N} (e^{i\theta})^n.
\]

As \( \theta \in \mathbb{R}, \left| e^{inx} \right| \leq 1 \) and the sum can be solved as a geometric progression with common ratio \( e^{i\theta} \):

\[
D_N(\theta) = \frac{(e^{i\theta})^{-N} - 1}{1 - e^{i\theta}} + \frac{1 - (e^{i\theta})^{N+1}}{1 - e^{i\theta}}.
\]

Manipulating further,

\[
D_N(\theta) = \frac{(e^{i\theta})^{-N} - (e^{i\theta})^{N+1}}{1 - e^{i\theta}} = \frac{(e^{i\theta})^{-N-1/2} - (e^{i\theta})^{N+1/2}}{(e^{i\theta})^{-1/2} - (e^{i\theta})^{1/2}}.
\]

Finally, apply the expression \( e^{i\theta} - e^{-i\theta} = 2i \sin(\theta) \) to obtain Equation (3.4) \( \square \)
Definition 3.5. Let \( f, g \) be integrable on \([-\pi, \pi]\) with \( f(-\pi) = f(\pi) \). The convolution of \( f \) with \( g \) on \([-\pi, \pi]\) is

\[
(f * g)(\theta) := \frac{1}{2\pi} \int_{-\pi}^{\pi} f(\theta - \phi)g(\phi)d\phi.
\]

The following is a collection of some important properties of convolution.

Theorem 3.7. Suppose \( f, g, h \) are \( 2\pi \) periodic functions integrable on \([-\pi, \pi]\).

1. \( f * (g + h) = (f * g) + (f * h) \) (Distributivity of Convolution over Addition)
2. \( (cf) * g = c(f * g) = f * (cg) \) for any \( c \in \mathbb{C} \) (Homogeneity of Convolution)
3. \( f * g = g * f \) (Commutativity of Convolution)
4. \( (f * g) * h = f * (g * h) \) (Associativity of Convolution)
5. \( f * g \) is continuous
6. \( \hat{f} * \hat{g}(n) = \hat{f} * \hat{g}(n) \)

A proof of these properties can be found in any standard source on Fourier analysis (for example, see page 44 of [2]).

Theorem 3.8. Suppose \( f \) is integrable on \([-\pi, \pi]\) with \( f(-\pi) = f(\pi) \) Then,

\[
S_N(f) = f * D_N.
\]

Proof. Using Equation (2.4)

\[
S_N(f) = \sum_{n=-N}^{N} \hat{f}(n)e^{in\phi}
\]

\[=
\sum_{n=-N}^{N} \left( \frac{1}{2\pi} \int_{-\pi}^{\pi} f(\theta)e^{-in\theta}d\theta \right)e^{in\phi}.
\]

Finally, pass constants through and commute the sum and integral:

\[
\frac{1}{2\pi} \int_{-\pi}^{\pi} f(\theta) \left( \sum_{n=-N}^{N} e^{in(\phi-\theta)} \right)d\theta = f * D_N.
\]

\[
\square
\]

4. Convergence using the Abel mean

The issues surrounding the convergence of the Fourier series are not straightforward. The Fourier series of a function integrable on \([-\pi, \pi]\) does not converge pointwise to the function itself since the derivation of Fourier coefficients is done through integration. For example, consider this piecewise-defined function

\[
f(\theta) = \begin{cases} 
1 & \theta = k\pi \text{ for all } k \in \mathbb{Z} \\
0 & \text{otherwise.}
\end{cases}
\]

The Fourier coefficients for this Riemann integrable function, \( \hat{f}(n) \), are 0 for all \( n \). This result is an example of two complications surrounding Fourier series. First, we find that the partial Fourier sum \( S_N(f) \) is 0 for all \( N \), but the value of the function is 1. The Fourier series and the actual function can disagree at countably many points. As uniform convergence is stricter than pointwise convergence, the Fourier series of a function also does not uniformly converge to the function. Second, we find that the Fourier series of a function is not unique to the function, as the
function $f \equiv 0$ would also have the same Fourier series. Thus, we prove a weaker result concerning convergence and uniqueness at points of continuity.

**Theorem 4.1.** Suppose $f$ is a $2\pi$ periodic function that is integrable on $[-\pi, \pi]$ with $f(n) = 0$ for all $n \in \mathbb{Z}$. If $f$ is continuous at $\theta_0 \in \mathbb{R}$, then $f(\theta_0) = 0$.

**Proof.** Without loss of generality, assume $\theta_0 = 0$. Suppose for contradiction that $f(0) > 0$. Since $f$ is continuous when $\theta = 0$, there exists $0 < \delta < \pi/2$ such that if $|\theta| < \delta$ then $f(\theta) > f(0)/2$. Also, as $f$ is integrable, there exists a bound $B$ such that $|f(\theta)| \leq B$ for all $\theta$. Now consider the function

$$g(\theta) = \epsilon + \cos(\theta)$$

with small $\epsilon > 0$ chosen such that if $\delta \leq |\theta| \leq \pi$, then $|g(\theta)| < 1 - \epsilon/2$. Note that, $|g(\theta)|$ reaches its maximum value when $\theta = 0$ and not when $|\theta| = \pi$. Thus, the choice of this $\epsilon$ depends on $\delta$. Now, choose a positive $0 < \eta < \delta$ such that if $|\theta| < \eta$, then $g(\theta) \geq 1 + \epsilon/2$.

Next, define a family of trigonometric polynomials based on $g(\theta)$:

$$g_k(\theta) = (g(\theta))^k;$$

where $k \in \mathbb{N}$. $|g_k(\theta)|$ reaches its maximum value when $\theta = 0$, and $g_k(0)$ increases as $k$ increases. Notice that as $g_k$ is a trigonometric polynomial for all $k$, we obtain that the Fourier coefficients $\hat{f}(n) = 0$ for all $n$ from our assumption. Thus,

$$\int_{-\pi}^{\pi} f(\theta)g_k(\theta)d\theta = 0 \quad \text{for all } k. \tag{4.2}$$

Remembering $0 < \delta \leq \pi$, $|f(\theta)| \leq B$, and that $|g_k(\theta)| \leq (1 - \epsilon/2)^k$ when $\delta \leq |\theta|;$

$$\left| \int_{\delta \leq |\theta|} f(\theta)g_k(\theta)d\theta \right| \leq 2\pi B(1 - \epsilon/2)^k.$$

This equation gives an approximation for the integral when $|\theta|$ is large.

Also, because $0 < f(\theta)$ and $0 < g(\theta)$ when $|\theta| < \delta$:

$$\int_{\eta \leq |\theta| < \delta} f(\theta)g_k(\theta)d\theta \geq 0.$$

Finally, if $|\theta| < \eta$, then $\frac{f(0)}{2} \leq f(\theta)$ and $(1 + \epsilon/2)^k \leq g_k(\theta)$, resulting in

$$\int_{|\theta| < \eta} f(\theta)g_k(\theta)d\theta \geq 2\eta \frac{f(0)}{2}(1 + \epsilon/2)^k.$$

This equation gives an approximation for the integral when $|\theta|$ is small. It forms a lower bound for the innermost portion of the integral when the values are positive.

Taking the sum of the two approximations and the constant gives a lower bound of the whole integral.

$$\int_{-\pi}^{\pi} f(\theta)g_k(\theta)d\theta \geq 2\eta \frac{f(0)}{2}(1 + \epsilon/2)^k - 2\pi B(1 - \epsilon/2)^k. \tag{4.3}$$

When $k \to \infty$, the right side goes to infinity, so $\int f(\theta)g_k(\theta)d\theta$ also goes to infinity, leading to a contradiction with Equation (4.2). This proves the theorem if $f(\theta)$ is...
real. For complex numbers, split the integral into functions of real and complex \( \theta \) and apply a similar argument.

Next, we will attempt to prove that the Fourier series is Abel summable to the function at a continuous point. Abel summable can be thought of as a weaker form of pointwise convergence.

**Definition 4.4.** Let \( 0 \leq r < 1 \). The **Poisson kernel** \( P_r : \mathbb{R} \to \mathbb{R} \) is a \( 2\pi \) periodic function defined as

\[
P_r(\theta) = \sum_{n=-\infty}^{\infty} r^{|n|} e^{i n \theta}.
\]

In other words, the Poisson kernel is a Fourier series that has a geometric sequence with common ratio \( r \) as its Fourier coefficients.

**Theorem 4.6.** The closed form equation for the Poisson kernel is

\[
P_r(\theta) = \frac{1 - r^2}{1 - 2r \cos(\theta) + r^2}.
\]

**Proof.** First, split the sum from Equation (4.5) into nonnegative and negative \( n \)

\[
P_r(\theta) = \sum_{n=0}^{\infty} r^{|n|} e^{i n \theta} + \sum_{n=1}^{\infty} r^{|n|} e^{-i n \theta}.
\]

As both sums are absolutely convergent geometric progressions,

\[
P_r(\theta) = \frac{1}{1 - re^{i \theta}} + \frac{re^{-i \theta}}{1 - re^{-i \theta}} = \frac{1 - r e^{i \theta}}{(1 - re^{i \theta})(1 - re^{-i \theta})} = \frac{1 - r^2}{1 - (e^{i \theta} + e^{-i \theta}) + r^2}.
\]

Finally, apply Euler’s formula to obtain the final result in Equation (4.7) \( \square \)

**Definition 4.8.** Let \( \sum_{k=0}^{\infty} c_k \) be a series of complex numbers. The series is **Abel summable** to \( s \) if for every \( 0 \leq r < 1 \) the **Abel means**

\[
A_r(c_k) = \sum_{k=0}^{\infty} c_k r^k
\]

converge, and the limit

\[
\lim_{r \to 1} A_r(c_k) = s.
\]

**Theorem 4.11.** Suppose \( f \) is an integrable, \( 2\pi \) periodic function. Then,

\[
A_r(S_k(f)(\theta)) = f * P_r.
\]

**Proof.** This proof is nearly identical to the proof of Equation 3.9 on partial Fourier sums and convolution of the Dirichlet kernel. The infinite sum of the Poisson kernel can pass through the integral of convolution because the sum from Equation (4.5) is uniformly convergent as \( r \) goes to 1. \( \square \)

Now we introduce the concept of an approximation to the identity. The concept will be useful in proving the Abel summability of the Fourier series through convolution.

**Definition 4.13.** A sequence of trigonometric polynomials \( \{K_n(\theta)\}_{n=1}^{\infty} \) is an **approximation to the identity** if it satisfies the following three properties:
(1) For all $n \geq 1$

(4.14) \[ \frac{1}{2\pi} \int_{-\pi}^{\pi} K_n(\theta) \, d\theta = 1. \]

(2) There exists $M > 0$ such that for all $n \geq 1$

(4.15) \[ \int_{-\pi}^{\pi} |K_n(\theta)| \, d\theta \leq M. \]

(3) For every $\delta > 0$

(4.16) \[ \int_{\delta \leq |\theta| \leq \pi} |K_n(\theta)| \, d\theta \to 0 \text{ as } n \to \infty. \]

Now we will prove a theorem which will show why these trigonometric polynomials are referred to as “approximations” of the identity for convolution.

**Theorem 4.17.** Let $\{K_n\}_{n=1}^{\infty}$ be an approximation of the identity and $f$ be a $2\pi$ periodic function integrable on $[-\pi, \pi]$. Then

(4.18) \[ \lim_{n \to \infty} (f \ast K_n)(\theta) = f(\theta). \]

whenever $f$ is continuous at $\theta$.

**Proof.** Suppose $f$ is continuous at $\theta$ and let $\epsilon > 0$. Then there exists $\delta > 0$ such that if $|\phi| < \delta$, then $|f(\theta - \phi) - f(\phi)| < \epsilon$. From the definition of convolution and Equation (4.14) we obtain:

\[
(f \ast K_n)(\theta) - f(\theta) = \frac{1}{2\pi} \int_{-\pi}^{\pi} K_n(\phi) f(\theta - \phi) \, d\phi - f(\theta) = \\
= \frac{1}{2\pi} \int_{-\pi}^{\pi} K_n(\phi) (f(\theta - \phi) - f(\theta)) \, d\phi.
\]

Thus,

\[ |(f \ast K_n)(\theta) - f(\theta)| = \left| \frac{1}{2\pi} \int_{-\pi}^{\pi} K_n(\phi) (f(\theta - \phi) - f(\theta)) \, d\phi \right|. \]

Splitting the integral and applying the triangle inequality,

\[
|(f \ast K_n)(\theta) - f(\theta)| \leq \frac{1}{2\pi} \left( \int_{|\phi| < \delta} |K_n(\phi)||f(\theta - \phi) - f(\theta)||d\phi + \int_{\delta \leq |\phi| \leq \pi} |K_n(\phi)||f(\theta - \phi) - f(\theta)||d\phi \right).
\]

Because $f$ is integrable, $f$ is bounded by some bound $B$, so

\[
|(f \ast K_n)(\theta) - f(\theta)| \leq \frac{\epsilon}{2\pi} \int_{-\pi}^{\pi} |K_n(\phi)| \, d\phi + \frac{2B}{2\pi} \int_{\delta \leq |\phi| \leq \pi} |K_n(\phi)| \, d\phi.
\]

From Equation (4.15), there exists a bound $M$ such that

\[
|(f \ast K_n)(\theta) - f(\theta)| \leq \frac{M \epsilon}{2\pi} + \frac{2B}{2\pi} \int_{\delta \leq |\phi| \leq \pi} |K_n(\phi)| \, d\phi.
\]

From Equation (4.16), there exists $n$ where the right integral is bounded by $M\epsilon/2B$ so

\[
|(f \ast K_n)(\theta) - f(\theta)| \leq \frac{2M \epsilon}{2\pi}.
\]
As \( M/\pi \) is constant and \( \epsilon \) can be arbitrarily small, \( f \ast K_n \) converges to \( f \).  

Now, we will show that the Poisson kernel is an approximation to the identity so that we can apply Equation (4.18).

**Theorem 4.19.** The Poisson kernel \( P_r \) is an approximation to the identity as \( r \to 1 \).

**Proof.** From the definition of the Poisson kernel in Equation (4.5)

\[
\frac{1}{2\pi} \int_{-\pi}^{\pi} P_r(\theta) d\theta = \frac{1}{2\pi} \int_{-\pi}^{\pi} \sum_{n=-\infty}^{\infty} r^{|n|} e^{i n \theta} d\theta.
\]

As the Poisson kernel is absolutely convergent,

\[
\frac{1}{2\pi} \int_{-\pi}^{\pi} P_r(\theta) d\theta = \sum_{n=-\infty}^{\infty} r^{|n|} \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{i n \theta} d\theta.
\]

When \( n \) is nonzero, the integral can be solved using \( e^{i \theta} = \cos(\theta) - i \sin(\theta) \)

\[
\frac{r^{|n|}}{2\pi} \frac{e^{i n \theta}}{n} \bigg|_{\theta=-\pi}^{\theta=\pi} = 0.
\]

Thus, the sum is equal to the value when \( n = 0 \):

\[
\frac{1}{2\pi} \int_{-\pi}^{\pi} P_r(\theta) d\theta = \frac{1}{2\pi} \int_{-\pi}^{\pi} 1 d\theta = 1.
\]

Therefore, Equation (4.14) is verified.

From Equation (4.7)

\[
P_r(\theta) = \frac{1 - r^2}{1 - 2r \cos(\theta) + r^2}.
\]

Manipulating the denominator:

\[
1 - 2r \cos(\theta) + r^2 = (1 - r)^2 + 2r(1 - \cos(\theta)).
\]

Thus, the numerator and denominator are both positive and \( P_r = |P_r| \) so

\[
\int_{-\pi}^{\pi} |P_r(\theta)| d\theta \leq 2\pi.
\]

and Equation (4.15) is verified.

We can also see that if \( \frac{1}{2} \leq r < 1 \) and \( \delta \leq |\theta| \leq \pi \), the denominator is bounded from below by \( c_\delta \) where \( 1 - 2r \cos(\theta) + r^2 \geq c_\delta > 0 \). Thus, \( P_r \leq (1 - r)^2/c_\delta \) and because the right hand side converges to 0 as \( n \to 0 \), Equation (4.16) holds.  

**Theorem 4.20.** The Fourier series of an integrable, \( 2\pi \) periodic function is Abel summable to \( f \) at all points of continuity. Moreover, if \( f \) is continuous on \([-\pi, \pi]\), then the Fourier series of \( f \) is uniformly Abel summable to \( f \).

**Proof.** From Equation (4.12)

\[
A_r(S_k(f)(\theta)) = f \ast P_r.
\]

From Theorem 4.19 we know that the Poisson kernel is an approximation of the identity so at points of continuity:

\[
\lim_{r \to 1} (f \ast P_r)(\theta) = f(\theta).
\]
Therefore, the Fourier series is Abel summable to $f$ at points of continuity.

5. **Mean Square Convergence**

Now we will prove the convergence of the partial Fourier sums to the overall function as opposed to pointwise Abel summability of the Fourier series. First, we will introduce the concepts related to an inner product space.

**Definition 5.1.** An **inner product** on a vector space $V$ over $\mathbb{C}$ is a map $(\cdot, \cdot) : V \times V \to \mathbb{C}$ such that the following properties hold for all $X, Y, Z \in V$ and $\alpha, \beta \in \mathbb{C}$:

1. $(X, Y) = (Y, X)$ (Hermetian),
2. $(\alpha X + \beta Y, Z) = \alpha (X, Z) + \beta (Y, Z)$ (Linearity),
3. $(X, \alpha Y + \beta Z) = \bar{\alpha} (X, Z) + \bar{\beta} (X, Z)$ (Conjugate-linearity),
4. $(X, X) \geq 0$ (Positive semidefinite).

An **inner product space** is a vector space with an inner product. In most definitions, the inner product is positive definite, where $(X, X) = 0$ implies $X = 0$. We define the inner product as positive semidefinite in this paper due to issues surrounding our norm of $\mathcal{R}$ with functions that are 0 except at countably many points.

**Definition 5.2.** Let $V$ be an inner product space. The **norm** of $X \in V$ is

$$\|X\| = \sqrt{(X, X)}.$$

**Definition 5.3.** Let $V$ be an inner product space. Elements $X, Y \in V$ are orthogonal if $(X, Y) = 0$.

Now we will prove the generalized Pythagorean theorem for inner product spaces.

**Theorem 5.4.** Suppose $V$ is an inner product space and $X, Y \in V$ are orthogonal. Then,

$$\|X + Y\|^2 = \|X\|^2 + \|Y\|^2.$$

**Proof.** Using Definition 5.2 of the norm, linearity, and Definition 5.3 of orthogonality

$$\|X + Y\|^2 = (X + Y, X + Y).$$

$$= (X, X) + (X, Y) + (Y, X) + (Y, Y).$$

$$= (X, X) + (Y, Y).$$

$$= \|X\|^2 + \|Y\|^2.$$

\[\square\]

Now we will define two other important relations for inner product spaces.

**Definition 5.6.** Suppose $V$ is an inner product space and $X, Y \in V$. Then

1. The **Cauchy Schwarz inequality** states that

$$|\langle X, Y \rangle| \leq \|X\| \|Y\|.$$
The triangle inequality states that
\[ \|X + Y\| \leq \|X\| + \|Y\|. \]

Now we will provide the space within which we will work.

**Theorem 5.9.** The set of complex-valued Riemann integrable functions from \([-\pi, \pi]\), labeled \(R\), forms an inner product space. Specifically, for \(\lambda \in \mathbb{C}\) and functions \(f, g\):

1. \((f + g)(\theta) = f(\theta) + g(\theta)\) vector addition,
2. \((\lambda f)(\theta) = \lambda \cdot f(\theta)\) scalar multiplication,
3. \((f, g) = \frac{1}{2\pi} \int_{0}^{2\pi} f(\theta) \overline{g(\theta)} d\theta\) inner product,

**Proof.** The axioms of a vector space follow from the definitions of vector addition and scalar multiplication. The proof of the first three properties from Definition 5.1 follow from splitting \(f, g\) into functions of real and complex \(\theta\) and performing manipulations. The fourth property follows from the fact that \(f(\theta) \overline{f(\theta)} = |f(\theta)|^2\). \(\square\)

**Theorem 5.10.** The Cauchy Schwarz and triangle inequalities hold for the space \(R\)

**Proof.** Suppose \(a, b\) are positive real numbers. Then,
\[ 0 \leq (a - b)^2. \]

Expanding the right hand side, we see
\[ ab \leq \frac{1}{2}(a^2 + b^2). \]

Now let \(f, g \in R\) and set \(a, b\) such that for \(\lambda > 0\)
\[ a = \lambda^{1/2}|f(\theta)| \quad \text{and} \quad b = \lambda^{-1/2}|g(\theta)|. \]

Then, we find from Equation (5.11) and from \(|g(\theta)| = |\overline{g(\theta)}|\) that
\[ |f(\theta) \overline{g(\theta)}| \leq \frac{1}{2}(\lambda |f(\theta)|^2 + \lambda^{-1} |g(\theta)|^2). \]

Then, by integrating both sides on \(\theta\) and applying the definition of the norm,
\[ \|(f, g)\| \leq \frac{1}{2\pi} \int_{0}^{2\pi} |f(\theta) \overline{g(\theta)}| d\theta \leq \frac{1}{2}(\lambda \|f(\theta)\|^2 + \lambda^{-1} \|g(\theta)\|^2). \]

Finally by substituting \(\lambda\) with \(\|g\|/\|f\|\), we obtain the Cauchy Schwarz inequality when \(\|f\| \neq 0\). When \(\|f\| = 0\) except at countably many points because the norm is positive semi-definite. Therefore, \(f \overline{g} = 0\) except at countably many points and the Cauchy Schwarz inequality holds as \(0 = 0\)
\[ \|f + g\|^2 = (f + g, f + g) = (f, f) + (f, g) + (g, f) + (g, g). \]

By definition of the norm,
\[ (X, X) = \|X\|^2 \quad \text{and} \quad (Y, Y) = \|Y\|^2. \]

From the Cauchy Schwarz inequality
\[ |(X, Y) + (Y, X)| \leq 2\|X\|\|Y\|. \]
Adding Equation (5.12) and (5.13), we find
\[ \|X + Y\|^2 \leq \|X\|^2 + 2\|X\|\|Y\| + \|Y\|^2 = (\|X\| + \|Y\|)^2. \]
This proves the triangle inequality for \( \mathcal{R} \). □

Now we introduce the concept of orthonormality.

**Theorem 5.14.** The family of functions \( \{e^{in\theta}\}_{n \in \mathbb{Z}} \) is orthonormal; that is,
\[
(5.15) \quad (e^{in\theta}, e^{im\theta}) = \begin{cases} 
1 & \text{if } n = m \\
0 & \text{if } n \neq m.
\end{cases}
\]

**Proof.** First, take the definition of the inner product from Definition 5.9. Then the integral can be solved in a similar manner to the solution of Equation (2.9) or proving the first property for Theorem 4.19 to obtain the desired result. □

This family of functions \( \{e^{in\theta}\}_{n \in \mathbb{Z}} \) should seem familiar
\[
(5.16) \quad (f, e^{in\theta}) = \frac{1}{2\pi} \int_0^{2\pi} f(\theta)e^{-in\theta} d\theta = \hat{f}(n).
\]

Next, we will prove a theorem using the orthonormality of this family of functions.

**Theorem 5.17.** If \( b_n \in \mathbb{C} \) for all \( n \in \mathbb{N} \), then
\[
(5.18) \quad \left( f - S_N(f), \sum_{|n| \leq N} b_ne^{in\theta} \right) = 0.
\]

**Proof.** From the definition of the partial Fourier sum
\[
\left( f - S_N(f), \sum_{|n| \leq N} b_ne^{in\theta} \right) = \left( f - \sum_{|n| \leq N} \hat{f}(n)e^{in\theta}, \sum_{|n| \leq N} b_ne^{in\theta} \right).
\]

From the linearity of the inner product,
\[
\left( f - S_N(f), \sum_{|n| \leq N} b_ne^{in\theta} \right) = \left( f, \sum_{|n| \leq N} b_ne^{in\theta} \right) - \left( \sum_{|n| \leq N} \hat{f}(n)e^{in\theta}, \sum_{|n| \leq N} b_ne^{in\theta} \right).
\]

From linearity and conjugate linearity, Equation (5.17), and orthonormality we obtain
\[
\left( f - S_N(f), \sum_{|n| \leq N} b_ne^{in\theta} \right) = (\hat{f}(0)b_0 + \cdots + \hat{f}(n)b_n + \hat{f}(-n)b_{-n}) - (\hat{f}(0)b_0 + \cdots + \hat{f}(n)b_n + \hat{f}(-n)b_{-n}) = 0.
\]

The result helps us solve the mean square convergence of the Fourier series.

**Theorem 5.19. (Best Approximation Lemma)** If \( f \) is a 2\( \pi \) periodic function integrable from \([-\pi, \pi]\), then for all \( c_n \in \mathbb{C} \)
\[
(5.20) \quad \|f - S_N(f)\| \leq \left\| f - \sum_{|n| \leq N} c_ne^{in\theta} \right\|.
\]
Proof. First, add \( S_N(f) - S_N(f) \) to \( f - \sum_{|n| \leq N} c_n e^{in\theta} \) while noting Equation 3.9:
\[
f - \sum_{|n| \leq N} c_n e^{in\theta} = f - S_N(f) + \sum_{|n| \leq N} (\hat{f}(n) - c_n) e^{in\theta}.
\]
As \( f - S_N(f) \) and \( \sum_{|n| \leq N} (\hat{f}(n) - c_n) e^{in\theta} \) are orthogonal, the Pythagorean theorem can be applied
\[
\| f - \sum_{|n| \leq N} c_n e^{in\theta} \|^2 = \| f - S_N(f) \|^2 + \| \sum_{|n| \leq N} (\hat{f}(n) - c_n) e^{in\theta} \|^2.
\]
As the norm of the function is positive semidefinite, Equation (5.20) holds. \( \square \)

**Theorem 5.21. Mean Square Convergence.** Suppose \( f \) is continuous and 2\( \pi \) periodic. Then as \( N \to \infty \)
\[
\| S_N(f) - f \| \to 0.
\]

**Proof.** As \( f \) is continuous, from Theorem 4.20 we know there exists a trigonometric polynomial \( P \) of degree \( M \) such that
\[
f(\theta) - P_M(\theta) < \epsilon \text{ for all } \theta.
\]
Taking the square and integrating the inequality gives, by our definition for the norm
\[
\| f - P \| < \epsilon.
\]
As \( \| f - S_N(f) \| \leq \| f - P \| \) from the best approximation lemma, through orthornormality we obtain Equation (5.22) when \( N \geq M \). \( \square \)

Finally, we will prove the Parseval’s identity for a continuous function.

**Theorem 5.23.** Suppose \( f \) is a continuous, 2\( \pi \) periodic function. Then,
\[
\sum_{n = -\infty}^{\infty} |\hat{f}(n)|^2 = \| f \|^2.
\]

**Proof.** First, we add and subtract the \( S_N(f) \) from \( f \)
\[
f = f - S_N(f) + S_N(f).
\]
As \( f - S_N(f) \) and \( S_N(f) \) are orthogonal from Theorem 4.17, apply the Pythagorean theorem:
\[
\| f \|^2 = \| f - S_N(f) \|^2 + \| S_N(f) \|^2.
\]
By applying Definition 2.3 for the partial Fourier sum,
\[
\| f \|^2 = \| f - S_N(f) \|^2 + \sum_{|n| \leq N} |\hat{f}(n)|^2 \| e^{in\theta} \|^2.
\]
From the orthonormality of \( \{ e^{in\theta} \}_{n \in \mathbb{Z}} \)
\[
\| f \|^2 = \| f - S_N(f) \|^2 + \sum_{|n| \leq N} |\hat{f}(n)|^2.
\]
Therefore, applying the Mean Square Convergence gives us the desired result as \( n \to \infty \). \( \square \)
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