THE FUNDAMENTALS OF COMPLEX ANALYSIS AND ITS IMMEDIATE APPLICATIONS

SKULI GUDMUNDSSON

ABSTRACT. This paper is an exposition on the basic fundamental theorems of complex analysis. Given is a brief introduction to analyticity and path integration, and from there the theorems regarding the Cauchy Integral, power series functions, and meromorphic functions are stated and proved. The paper concludes with an immediate application of the proved theorems - an inquiry into Bernoulli polynomials and a proof of Wallis’ product for π.

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1. INTRODUCTION

In complex analysis, integrals of the type
\[ \int_{\gamma} f(z) \, dz \]
where \( \gamma \) is a complex path, represent a different geometric challenge than in one-variable real analysis where the only possible paths are closed intervals or points. To talk about complex integration in a meaningful way, however, we must first introduce complex differentiability and give proper definitions of the tools of complex integration - paths, contours, and homotopies. This is done in Section 2. In Section 3, the focus is on the Cauchy Integral Formula and interesting consequences of it - e.g., that a complex differentiable function is holomorphic (infinitely many times complex differentiable) - whereas in Section 4, the results of Section 3 are developed to define meromorphic functions and to relate analyticity of a function to its representability as a power series.

Date: Submitted August 26, 2018.
The fifth and final section deals with an interesting application of all the theorems previously developed. Investigated are the Bernoulli polynomials:

\[ B_k(x) = \frac{\partial^k u}{\partial z^k} \big|_{z=0} \]

where \( u \) is given by

\[ u(x, z) = \begin{cases} 
1 & \text{if } z = 0 \\
\frac{ze^{zx}}{e^z-1} & \text{if } z \neq 0 
\end{cases} \]

These are then used to derive representations of \( \sin(x) \), \( \csc(x) \) and other trigonometric identities, which immediately give Wallis' product representation of \( \pi \):

\[ \frac{\pi}{2} = \frac{2 \cdot 2 \cdot 4 \cdot 4 \cdot 6 \cdot 6 \cdot ...}{1 \cdot 3 \cdot 3 \cdot 5 \cdot 5 \cdot 7 \cdot ...} \]

2. Basic Definitions in Complex Analysis

2.1. Analyticity. A brief view of the basics of complex analysis is presented in this section as a preamble to more complicated investigations. We begin with the definition of analyticity (complex differentiability), which is analogous to its real counterpart:

**Definition 2.1.** Let \( \Omega \) be an open subspace of \( \mathbb{C} \), and let \( f \) be defined over \( \Omega \). The function \( f \) is said to be **analytic** over \( \Omega \) if, for \( h \in \mathbb{C} \), the following limit

\[ \lim_{h \to 0} \frac{f(z + h) - f(z)}{h} \]

exists for all \( z \in \Omega \).

It is natural to identify the complex-valued plane with \( \mathbb{R}^2 \). Let \( f \) be a complex-valued function defined on an open subset \( \Omega \subset \mathbb{C} \). Furthermore, let \( \hat{\Omega} \) be an open subset of \( \mathbb{R}^2 \) defined as:

\[ \hat{\Omega} = \{ (x, y) \in \mathbb{R}^2 : x + iy \in \Omega \} \]

Thus we can identify \( f \) with real valued functions \( u \) and \( v \), defined on \( \hat{\Omega} \), according to the rule:

\[ f(x + iy) = u(x, y) + iv(x, y) \]

Letting \( z = x + iy \), we have:

\[
\frac{\partial f}{\partial z}(x + iy) = \left( \frac{1}{2} \frac{\partial}{\partial x} - \frac{i}{2} \frac{\partial}{\partial y} \right)(u(x, y) + iv(x, y)) \\
= \frac{1}{2} \left( \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right)(x, y) + i \frac{1}{2} \left( \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \right)(x, y) \\
\frac{\partial f}{\partial \bar{z}}(x + iy) = \left( \frac{1}{2} \frac{\partial}{\partial x} + \frac{i}{2} \frac{\partial}{\partial y} \right)(u(x, y) + iv(x, y)) \\
= \frac{1}{2} \left( \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right)(x, y) + i \frac{1}{2} \left( \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \right)(x, y)
\]
2.2. Paths and Homotopies. A short discussion of paths and homotopies is also necessary for the formulation of basic theorems in complex analysis. A path in an open subset $U \subset \mathbb{C}$ is a continuous function from a closed interval to $U$. If $\gamma$ is a path, then $\gamma$ is said to be closed if $\gamma(a) = \gamma(b)$. Using this knowledge, we can make sense of the definition of a homotopy:

**Definition 2.2.** Let $\gamma_1 : [0, 1] \rightarrow \mathbb{C}$ and $\gamma_2 : [0, 1] \rightarrow \mathbb{C}$ be paths. A continuous function $\sigma : [0, 1] \times [0, 1] \rightarrow \mathbb{C}$ is called a homotopy between $\gamma_1$ and $\gamma_2$ if, for all $0 \leq x \leq 1$,

$$
\sigma(0, x) = \gamma_1(x) \quad \sigma(1, x) = \gamma_2(x)
$$

A path is contractible if it is homotopic to a constant path.

Heuristically, a homotopy between two closed paths in the complex plane can be thought of as a continuous deformation of one closed path into another closed path, as in the figure below:

![Figure 1. A homotopy exists between the two closed paths in $\mathbb{C}$ since it is possible to continuously deform/expand one to get the other.](image)

When scrutinizing homotopic reductions, however, the domain in question matters. In the case of Figure 1, a homotopy does exist between the two paths on $\mathbb{C}$, but it would not exist on $\mathbb{C} \setminus \{a\}$ if $a \in \mathbb{C}$ were a point "in between" the two paths. This conceptualization of a homotopy, while not rigorous, is extremely useful for the purposes of this paper. For a more rigorous treatment of homotopies, see Hatcher’s *Algebraic Topology* [3].

Homotopies will be vital to our discussion of complex integration. We define the integral along $\gamma$ of a complex-valued function $f$ (that is defined at least on the image of $\gamma$) by:

$$
\int_{\gamma} f(z) \, dz = \int_{a}^{b} f(\gamma(t)) \gamma'(t) \, dt
$$
The integral on the right is independent of parameterization. This means that if \( \tau : [a, b] \to [c, d] \) is monotone increasing and continuously differentiable along with its inverse, and the path \( \hat{\gamma} \) is defined by

\[
\hat{\gamma}(s) = \gamma(\tau^{-1}(s))
\]

then we have:

\[
\int_a^b f(\gamma(t))\gamma'(t) \, dt = \int_c^d f(\hat{\gamma}(s))\hat{\gamma}'(s) \, ds
\]

This we can see by making the real change of variables \( s = \tau(t) \) in the integral on the right. It can be helpful to assign a unique name to classes of paths independent of parameterization. These are called contours, and we say that a contour \( C \) corresponds to a path \( \gamma \). Since the integral depends only on the contour, we write the integral as:

\[
\int_C f(z) \, dz
\]

The number

\[
\frac{1}{2\pi i} \int_C \frac{dz}{z - \zeta}
\]

is called the index of a contour \( C \) with respect to the point \( \zeta \), and is denoted by \( \text{Ind}(C, \zeta) \). This concept is useful for theorems in which paths are closed. For closed paths, the index is 1 if the point \( \zeta \) lies in the interior (or convex hull) of \( C \), and it is 0 if the \( \zeta \) lies in the exterior of \( C \).

We conclude this subsection with the intuitive notion of the length of a path, which is defined as follows:

**Definition 2.4.** Let \( \gamma : [a, b] \to \mathbb{C} \) be a path in the complex plane. If \( \gamma \) is continuously differentiable, then the length of \( \gamma \), represented by \( L \), is given by the following integral:

\[
L = \int_a^b |\gamma'(t)| \, dt
\]

2.3. **Basic Theorems.** Armed with the definitions from Sections 2.1 and 2.2, we can state the following theorem:

**Theorem 2.5.** Suppose that \( u \) and \( v \) are continuously differentiable on \( \tilde{\Omega} \). The following conditions are equivalent:

1. \( \lim_{h \to 0} \frac{f(z + h) - f(z)}{h} \) exists for all \( z \in \Omega \)
2. \( \frac{\partial f}{\partial \bar{z}} = 0 \) throughout \( \Omega \).
3. The functions \( u \) and \( v \) satisfy the Cauchy-Riemann equations throughout \( \tilde{\Omega} \):

\[
\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}
\]

4. \( \int_C f(z) \, dz = 0 \) along any closed, contractible contour \( C \) in \( \Omega \).
5. \( \int_C f(z) \, dz = \int_{C'} f(z) \, dz \) whenever \( C' \) and \( C'' \) are homotopic.
Proof. See pages 86-87 of [1]. □

The next theorem is akin to a sixth condition of Theorem 2.5, but, due to its complex structure, is better stated as a separate theorem:

**Theorem 2.6.** If for each sequence \( \{z_n\} \) such that \( z_n \in \Omega \setminus \{z_0\} \) and \( z_n \to z_0 \) as \( n \to \infty \) we can write

\[
f(z) = f(z_0) + (z - z_0)(\lambda + \epsilon(z))
\]

for all \( z \in \Omega \), where \( \epsilon : \Omega \to \mathbb{C} \) is continuous at \( z_0 \) with \( \epsilon(z_0) = 0 \), then \( f \) is analytic at \( z_0 \).

Proof. See Chapter 1 of [2]. □

As in the real case, from these basic theorems we can generalize about compositions of complex differentiable functions, as the following theorem states:

**Theorem 2.7.** Let \( f \) and \( g \) be analytic functions on \( \Omega \).
1. If \( a, b \in \mathbb{C} \), then \( af + bg \) is analytic on \( \Omega \).
2. The product \( fg \) is analytic on \( \Omega \).
3. If \( g \) has no zeroes in \( \Omega \), then \( f/g \) is analytic on \( \Omega \).
4. If \( U \in \Omega \) is open, then \( f \) restricted to \( U \), written \( f|_U \), is analytic.
5. If \( \Omega = \bigcup_{\alpha} U_{\alpha} \) and \( h|_{U_{\alpha}} \) is analytic on \( U_{\alpha} \) for every \( \alpha \), then \( h \) is analytic on \( \Omega \).

Proof. The proof is analogous to proofs in the real case. □

With Theorems 2.5, 2.6 and 2.7 at hand, we can check whole classes of functions with respect their complex differentiability:

**Example 2.8.** Consider \( f(z) = z^n \), with \( n \in \mathbb{Z} \). If \( n \neq -1 \), then by the first condition of Theorem 2.5:

\[
\lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0} = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0} = \lim_{z \to z_0} (z^{n-1} + z_0 z^{n-2} + \ldots + z_0^{n-2} z + z_0^{n+1}) = n z_0^{n-1}
\]

for all \( z_0 \in \mathbb{C} \). By condition 1 of Theorem 2.7, all complex polynomials are analytic on \( \mathbb{C} \). If, however, \( n = -1 \), we find that by choosing \( \gamma(t) = e^{it} \) for \( 0 \leq t \leq 2\pi \)

\[
\int_\gamma \frac{dz}{z} = \int_0^{2\pi} i dt = 2\pi i \neq 0
\]

By condition 4, \( f \) is not holomorphic on all of \( \mathbb{C} \).

**Example 2.9.** Consider \( f(x + iy) = x + xi \). We have that

\[
\frac{\partial u}{\partial x} = 1, \quad \frac{\partial v}{\partial y} = 0
\]

By condition 3, the function is nowhere differentiable on \( \mathbb{C} \).

**Example 2.10.** Consider \( f(z) = |z|^2 = x^2 + y^2 \). Here the function \( f \) is a real-valued polynomial, and thus \( f \) is real-differentiable on \( \mathbb{C} \), yet

\[
\frac{\partial u}{\partial x} = 2x, \quad \frac{\partial v}{\partial y} = 0
\]
\[
\frac{\partial u}{\partial y} = 2y, \quad \frac{\partial v}{\partial x} = 0
\]

By condition 3, the function is nowhere holomorphic on \( \mathbb{C} \) except when \( x = y = 0 \).

The final theorem is an immediate consequence of the definitions in Section 2.2:

**Theorem 2.11** (The \( ML \)-Theorem). Let \( \gamma : [a, b] \to \mathbb{C} \) be a path, and let \( f \) be a continuous function on \( \gamma \). Suppose that \( |f(z)| \leq M \) for all \( z \) in the image of \( \gamma \). Then the following holds:

\[
\left| \int_{\gamma} f(z) \, dz \right| \leq ML
\]

where \( L \) is defined as in Definition 2.4.

**Proof.** We know from real analysis that

\[
\left| \int_{\gamma} f(z) \, dz \right| = \left| \int_{a}^{b} f(\gamma(t))\gamma'(t) \, dt \right| \leq \int_{a}^{b} |f(\gamma(t))||\gamma'(t)| \, dt \\
\leq M \int_{a}^{b} |\gamma'(t)| \, dt = ML
\]

\( \square \)

3. The Cauchy Integral

The goal of this section is to prove the Cauchy Integral Formula - a key result in complex analysis. We begin by proving Cauchy’s Theorem for Triangles. This theorem, while not central to the arguments of this paper, is included because of its importance in proving results in complex analysis, such as Rouche’s Theorem and others, which are powerful tools for counting poles and zeros. Although Rouche’s Theorem is beyond the scope of this paper, poles and techniques for counting them are discussed in Section 4.

**Theorem 3.1** (Cauchy’s Theorem for Triangles). Suppose that \( f \) is analytic on \( \Omega \) and \( T = [z_1, z_2, z_3] \) is any triangle such that \( \hat{T} \subset \Omega \) (where \( \hat{T} \) is the convex hull of \( T \)). Then \( \int_{T} f(z) \, dz = 0 \).

**Proof.** Let \( a, b, c \) be the midpoints of \([z_1, z_2]\), \([z_2, z_3]\), and \([z_3, z_1]\), respectively, and let us consider the triangles \([z_1, a, c]\), \([z_2, b, a]\), and \([a, b, c]\):
The integral of $f$ over $T$ is the sum of the integrals over the four triangles. Moreover, by the triangle inequality, if $T_1$ is one of these triangles and $\left| \int_{T_1} f(z) \, dz \right|$ is as large as possible, then

$$\left| \int_T f(z) \, dz \right| \leq 4 \left| \int_{T_1} f(z) \, dz \right|$$

Also, if $L(X)$ measures the length of the sides of a geometric object $X$, then $L(T_1) = \frac{1}{2} L(T)$, because a line joining two midpoints of a triangle is half as long as the opposite side. Proceeding in this way, we obtain a sequence $\{T_n : n = 1, 2, ...\}$ of triangles such that

$$L(T_n) = 2^{-n} L(T)$$

with $\hat{T}_{n+1} \subset \hat{T}_n$, and so we obtain

$$\left(3.2\right) \quad \left| \int_T f(z) \, dz \right| \leq 4^{-n} \left| \int_{T_n} f(z) \, dz \right|$$

Now the $\hat{T}_n$ form a decreasing sequence of nonempty closed and bounded (hence compact) sets in $\mathbb{C}$ whose diameters approach 0 as $n \to \infty$. Hence, there is a point $z_0 \in \cap_{n=1}^{\infty} \hat{T}_n$. (If the intersection is empty, then by compactness, some finite collection of $T_i$’s would have an empty intersection.) Since $f$ is analytic at $z_0$, there is a continuous function $\epsilon : \Omega \to \mathbb{C}$ with $\epsilon(z_0) = 0$ (see Theorem 2.6) and such that

$$\left(3.3\right) \quad f(z) = f(z_0) + (z - z_0) [f'(z_0) + \epsilon(z)], \quad z \in \Omega$$

Therefore we have that

$$\left(3.4\right) \quad \int_{T_n} f(z) \, dz = \int_{T_n} (z - z_0) \epsilon(z) \, dz, \quad n \in \mathbb{N}$$

Now by the $ML$-Theorem,

$$\left(3.5\right) \quad \left| \int_{T_n} (z - z_0) \epsilon(z) \, dz \right| \leq \sup_{z \in T_n} \left| \epsilon(z) \right| \left| (z - z_0) \right| L(T_n)$$

$$\leq \sup_{z \in T_n} \left| \epsilon(z) \right| \left( L(T_n) \right)^2$$

$$\leq \sup_{z \in T_n} \left| \epsilon(z) \right| 4^{-n} \left( L(T) \right)^2$$

Thus we have that

$$\left(3.6\right) \quad \left| \int_T f(z) \, dz \right| \leq \sup_{z \in T_n} \left| \epsilon(z) \right| \left( L(T) \right)^2$$

By continuity of $\epsilon$ at $z_0$, and since $\epsilon(z_0) = 0$, the right side of Equation (3.6) tends to zero as $n$ tends to infinity. We conclude that $\int_T f(z) \, dz = 0$. □
Theorem 3.7 (The Cauchy Integral Formula). Let $\Omega \subset \mathbb{C}$ be open, $f$ an analytic function on $\Omega$, $z$ a point in $\Omega$ and $C$ a closed, contractible contour of index 1 with respect to $z$. Then

\[(3.8)\]
\[f(z) = \frac{1}{2\pi i} \int_{C} \frac{f(\zeta)}{\zeta - z} \, d\zeta\]

Proof. Let $w$ be a point in $\Omega$, and let $\overline{B}(w, r)$ be contained in $\Omega$ with $z \in B(w, r)$. Since any closed, contractible contour $C$ with index 1 with respect to $z$ is homotopic to some circle $C(w, r')$ in $\Omega \setminus \{z\}$, by part 5 of Theorem 2.5 we have that

\[\int_{C} \frac{f(\zeta)}{\zeta - z} \, d\zeta = \int_{C(w, r')} \frac{f(\zeta)}{\zeta - z} \, d\zeta\]

Thus we need only prove the Cauchy Integral Formula for $C(w, r)$. Therefore, let $z \in C(w, r)$, and let $C_1$ be the contour corresponding to

\[(3.9)\]
\[\gamma_1(t) = w + re^{2\pi it}\]

We will show that for all $\varepsilon > 0$

\[\left| \frac{1}{2\pi i} \int_{C_1} \frac{f(\zeta)}{\zeta - z} - f(z) \, d\zeta \right| \leq \varepsilon\]

First, we define a function $g$ in the following way:

\[g(\zeta) = \frac{f(\zeta) - f(z)}{\zeta - z}\]

Since $f$ is analytic on $\Omega$, and so by condition 1 of Theorem 2.5, there exist positive $M$ and $R$ such that

\[|g(\zeta)| = \left| \frac{f(\zeta) - f(z)}{\zeta - z} \right| \leq M\]

for $\zeta \in B(z, R)$. Let therefore

\[0 < \rho < \min\{R, r - |z - w|, \frac{\varepsilon}{M}\}\]

and furthermore, let $C_2$ be the contour corresponding to the path

\[(3.10)\]
\[\gamma_2(t) = z + \rho e^{2\pi it}\]

The contours $C_1$ and $C_2$ are homotopic in $\Omega \setminus \{z\}$, with the function

\[\sigma(t) = s\gamma_2(t) + (1 - s)\gamma_1(t)\]

serving as a homotopy. What’s more, $g$ is analytic on $\Omega \setminus \{z\}$, and so condition 5 of Theorem 2.5 gives us that

\[\frac{1}{2\pi i} \int_{C_1} g(\zeta) \, d\zeta = \frac{1}{2\pi i} \int_{C_2} g(\zeta) \, d\zeta\]

Now by the $ML$-Theorem, we have that
\[ \left| \frac{1}{2\pi i} \int_{C_2} g(\zeta) \, d\zeta \right| \leq M \rho < \varepsilon \]

Since \( C_1 \) is of index 1 with respect to \( z \), we have that
\[ \frac{1}{2\pi i} \int_{C_1} g(\zeta) \, d\zeta = \frac{1}{2\pi i} \int_{C_1} \frac{f(\zeta) - f(z)}{\zeta - z} \, d\zeta = \frac{1}{2\pi i} \int_{C_1} \frac{f(\zeta)}{\zeta - z} \, d\zeta - f(z) \]

And thus, for all \( \varepsilon > 0 \), we have that
\[ \left| \frac{1}{2\pi i} \int_{C_1} \frac{f(\zeta)}{\zeta - z} - f(z) \, d\zeta \right| \leq \varepsilon \]

□

The following theorem will almost immediately allow us to see that an analytic function is not just once complex differentiable, but infinitely many times complex differentiable:

**Theorem 3.11.** Let \( \gamma \) be a path and let \( g \) be a complex-valued analytic function on \( \bar{\gamma} \) (the image of \( \gamma \)). Define a function \( F \) on the open set \( \Omega = \mathbb{C} \setminus \bar{\gamma} \) by

\[ F(z) = \int_{\gamma} \frac{g(w)}{w - z} \, dw \]

Then \( F \) has derivatives of all orders on \( \Omega \), and

\[ F^{(n)}(z) = n! \int_{\gamma} \frac{g(w)}{(w - z)^{n+1}} \, dw \]

for all \( z \in \Omega \) and all \( n \in \mathbb{N} \cup \{0\} \).

**Proof.** The proof is by induction. The formula for \( F^{(n)}(z) \) is valid for \( n = 0 \) by hypothesis, which gives us our base case. Let us assume that the formula holds for a given \( n \) and all \( z \in \Omega \). We fix \( z_1 \in \Omega \) and choose \( r > 0 \) small enough that \( D(z_1, r) \subset \Omega \). For any point \( z \in D(z_1, r) \) with \( z \neq z_1 \), we have

\[ \frac{F^{(n)}(z) - F^{(n)}(z_1)}{z - z_1} = (n + 1)! \int_{\gamma} \frac{g(w)}{(w - z_1)^{n+2}} \, dw \]

\[ = \frac{n!}{z - z_1} \int_{\gamma} \frac{(w - z_1)^{n+1} - (w - z)^{n+1}}{(w - z_1)^{n+1}(w - z)^{n+1}} g(w) \, dw - (n + 1)! \int_{\gamma} \frac{g(w)}{(w - z_1)^{n+2}} \, dw \]

\[ = \frac{n!}{z - z_1} \int_{\gamma} (z - z_1) \sum_{k=0}^{n} (w - z_1)^{n-k}(w - z)^k \frac{g(w)}{(w - z_1)^{n+1}(w - z)^{n+1}} \, dw - (n + 1)! \int_{\gamma} \frac{g(w)}{(w - z_1)^{n+2}} \, dw \]

Here the numerator of the first integral of the last line is obtained by the algebraic identity

\[ a^{n+1} - b^{n+1} = (a - b) \sum_{k=0}^{n} a^{n-k} b^k \]

where \( a = w - z_1 \) and \( b = w - z \). Therefore, we have that
\[ \left| \frac{F^{(n)}(z) - F^{(n)}(z_1)}{z - z_1} - (n + 1)! \int_{\gamma} \frac{g(w)}{(w - z_1)^{n+2}} \, dw \right| = n! \left| \int_{\gamma} \frac{\sum_{k=0}^n (w - z_1)^{n-k+1}(w - z)^k - (n + 1)(w - z)^{n+1}}{(w - z_1)^{n+2}(w - z)^{n+1}} g(w) \, dw \right| \]  

(3.12)

\[ \leq n! \left[ \max_{z \in \gamma^*} \left( \frac{\sum_{k=0}^n (w - z_1)^{n-k+1}(w - z)^k - (n + 1)(w - z)^{n+1}}{(w - z_1)^{n+2}(w - z)^{n+1}} g(w) \right) \right] L(\gamma) \]

The inequality we obtain from the ML-Theorem. Furthermore, since

\[ \lim_{z \to z_1} \sum_{k=0}^n (w - z_1)^{n-k+1}(w - z)^k = \sum_{k=0}^n (w - z_1)^{n+1} = (n + 1)(w - z_1)^{n+1} \]

the max in Equation (3.12) approaches 0 as \( z \) tends to \( z_1 \), and thus we obtain that

\[ \lim_{z \to z_1} \frac{F^{(n)}(z) - F^{(n)}(z_1)}{z - z_1} = (n + 1)! \int_{\gamma} \frac{g(w)}{(w - z_1)^{n+2}} \, dw \]

\[ \square \]

Combining Theorems 3.7 and 3.11, we get the following corollary:

**Corollary 3.13.** If \( f \) is analytic on \( \Omega \), then \( f \) has derivatives of all orders on \( \Omega \). Moreover, if \( C \) is a closed, contractible contour in \( \Omega \), then

(3.14) \[ f^{(n)}(z) = \frac{n!}{2\pi i} \int_{C} \frac{f(w)}{(w - z)^{n+1}} \, dw, \quad z \in \hat{C} \]

**Proof.** Apply Theorem 3.11 to the Cauchy Integral Formula. \( \square \)

**Remark 3.15.** Interestingly enough, a converse to Cauchy’s Theorem for Triangles holds as well. That is, if \( f \) is continuous on \( \Omega \) with \( \int_{T} f(z) \, dz = 0 \) for each triangle \( T \) s.t. \( \hat{T} \subset \Omega \), then \( f \) is analytic. We already have the tools for a proof - Theorems 3.7 and 3.11. However, such a proof would require a thorough discussion of primitives, which is beyond the scope of this paper.

## 4. Power Series Expansion and Meromorphic Functions

The Cauchy Integral Formula allows us to prove that two fundamental notions in complex analysis - analyticity and power series expansion - are in fact equivalent. To prove this fact, however, we must first state the following lemma:

**Lemma 4.1.** Let \( \Omega \subset \mathbb{C} \) be open and \( f_n \) a locally uniformly convergent sequence of functions analytic on \( \Omega \). Then their limit \( f \) is an analytic function and the successive derivatives of the \( f_n \) converge locally uniformly to \( f \).

**Proof.** The proof employs techniques similar to those used in real analysis. \( \square \)
Theorem 4.2. Let $\Omega \subset \mathbb{C}$ be open and $f$ a function on $\Omega$. The following two conditions are equivalent.

1. $f$ is analytic on $\Omega$.
2. For every $w \in \Omega$, there is a neighborhood $U_w$ of $w$ and a sequence $a_n$, depending on $w$, such that for all $z \in U_w$, we have

$$f(z) = \sum_{n=0}^\infty a_n (z - w)^n$$

Proof. Assume that condition 2 is satisfied, and let $x \in U_w \setminus \{w\}$. Then

$$\sum_{n=0}^\infty a_n (x - w)^n$$

converges, so we have

$$|a_n||x - w|^n \leq C$$

where $C$ is a constant depending on $w$ but not on $n$. If $0 < r < |x - w|$, then for all $z \in \overline{B}(w, r)$ the sum

(4.3) $$\sum_{n=0}^\infty a_n (z - w)^n$$

converges uniformly by comparison with the series

$$C \sum_{n=0}^\infty \left( \frac{r}{|x - w|} \right)^n$$

Therefore, we see that the sum in Equation (4.3) converges (locally) uniformly on $B(w, |x - w|)$. The partial sums are all polynomials, and thus analytic. Lemma 4.1 tells us that the series converges to an analytic function on $B(w, |x - w|)$. Taking the union over all $x \in U_w$, and using condition 5 of Theorem 2.5, we see that the series represents an analytic function at least on the smallest open disc about $w$ containing $U_w$, and thus $f$ is analytic on $U_w$. Applying condition 5 of Theorem 2.5 shows that $f$ is analytic on the entirety on $\Omega$.

Now suppose condition 1 is satisfied. We define

$$\rho = \sup \{ r \in \mathbb{R} : \overline{B}(w, r) \subset \Omega \}$$

Since $\Omega$ is open, we have that $\rho > 0$. Therefore, let $0 < r < \rho$ and let $C$ be the circle of radius $r$ about $w$, oriented counterclockwise. We then expand $(\zeta - z)^{-1}$ in powers of $(z - w)$:

$$\frac{1}{\zeta - w} = \sum_{n=0}^\infty \frac{(z - w)^n}{(\zeta - z)^{n+1}}$$

Inserting this into the Cauchy Integral Formula, we get that

(4.4) $$f(z) = \frac{1}{2\pi i} \int_C \frac{f(\zeta) \, d\zeta}{\zeta - z} = \sum_{n=0}^\infty \frac{(z - w)^n}{2\pi i} \int_C \frac{f(\zeta) \, d\zeta}{(\zeta - w)^{n+1}}$$
The term by term integration is justified by uniform convergence for $|z - w| < r$. Using the differentiation formula given by Corollary 3.13, we can write $f$ as

$$f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(w)}{n!} (z - w)^n$$

This is valid for all $|z - w| < r$ and all $r < \rho$, and so for all $|z - w| < \rho$. Therefore condition 2 is satisfied with $U_w$ and

$$a_n = \frac{f^{(n)}(w)}{n!}$$

\[ \square \]

The equivalence of holomorphicity and power series expansion provides an elegant basis for the definition of meromorphic functions. The next two theorems, Local Factorization and Rigidity Principle, are continuations of this relation, and easily solidify the various definitions of meromorphy (Theorem 4.8).

**Theorem 4.5 (Local Factorization).** Let $\Omega \subset \mathbb{C}$ be open, and $f$ analytic on $\Omega$. Every point $w$ in $\Omega$ has a neighborhood $U_w$ where either

1. $f(z) = 0$ identically, or
2. $f(z) = (z - w)^{v_w(f)} u(z)$

where $v_w(f)$ is the order of vanishing of $f$ at $w$, and $u$ is an analytic function with no zeroes in $U_w$. In particular, $f$ has no zeroes in the punctured neighborhood $U_w \setminus \{w\}$.

**Proof.** If $f^{(n)}(w) = 0$ for all $n$ then the proof of Theorem 4.1 shows that condition 1 of Theorem 4.2 is true throughout the largest disc about $w$ contained in $\Omega$. Otherwise, there exists an integer $k$ such that $f^{(k)} \neq 0$, while $f^{(n)} = 0$ for $n < k$. This $k$ is the order of vanishing at $w$, and so $k = v_w(f)$. By Taylor’s theorem

$$f(z) = f^{(k)}(w)(z - w)^k + o(|z - w|^k)$$

for $z$ near $w$. If we define the function $u$ by

$$u(z) = (z - w)^{-k} f(z)$$

for $z \in \Omega \setminus \{w\}$, we find

$$u(z) = f^{(k)}(w) + o(1)$$

for $z$ near $w$. By condition 3 of Theorem 2.5, $u$ is analytic on $\Omega \setminus \{w\}$ and continuous at $w$. From this, we can conclude that $u$ is analytic on all of $\Omega$, although for brevity’s sake, the proof is omitted (see Chapter 2 of [2]). Equation (4.6) shows that $u(w) \neq 0$ and, by continuity, $u(z) \neq 0$ in some neighborhood $U_w$ of $w$. \[ \square \]

**Theorem 4.7 (Rigidity Principle).** Let $\Omega$ be an open, connected subset of $\mathbb{C}$ and $f$ an analytic function on $\Omega$. Either

1. $f(z) = 0$ for all $z \in \Omega$, or
2. the zeroes of $f$ are all of finite order and are discrete; that is, each has a neighborhood containing no other zeroes.
Proof. Define

\[ V_n = \{ z \in \Omega : f^{(n)}(z) = 0 \} \]

and

\[ V = \bigcap_{n=0}^{\infty} V_n \]

By Lemma 4.1, \( f^{(n)} \) is continuous for all \( n \), and thus \( V_n \) is closed. As the intersection of a collection of closed sets, \( V \) is closed as well. On the other hand, if \( w \in V \) then Theorem 4.5 gives a whole neighborhood \( U_w \) of \( w \) which is contained in \( V \), so \( V \) is open as well. The subset \( \Omega \) was assumed to be connected, so either \( V = \Omega \) or \( V = \emptyset \). In the former case \( \Omega \subset V_0 \), so condition 1 holds. In the latter case, every zero is of finite order. Theorem 4.5 then shows that each of these zeroes has a neighborhood containing no others. \( \Box \)

**Theorem 4.8** (Definitions of Meromorphicity). Let \( \Omega \) be an open subset of \( \mathbb{C} \) and \( f \) a function defined on a subset of \( \Omega \). The following conditions are equivalent:

1. Every point \( w \in \Omega \) has a punctured neighborhood \( U_w \setminus \{w\} \) where \( f \) is defined and can be represented as a quotient

\[ f(z) = \frac{g_w(z)}{h_w(z)} \]

where the functions \( g_w \) and \( h_w \) are analytic on \( U_w \).

2. Every point \( w \in \Omega \) has a punctured neighborhood \( U_w \setminus \{w\} \) where \( f \) is defined and can be represented by an infinite series,

\[ f(z) = \sum_{n=v_w(f)}^{\infty} a_n(z - w)^n \]

with some integer \( v_w(f) \) and a sequence \( a_n \), depending on \( w \) but not on \( z \), with \( a_{v_w(f)} \neq 0 \). This series is called the Laurent series expansion of \( f \) about \( w \).

3. Every point \( w \in \Omega \) has a punctured neighborhood \( U_w \setminus \{w\} \) where \( f \) is defined and can be represented as

\[ f(z) = (z - w)^{v_w(f)} u(z) \]

with some integer \( v_w(f) \) and a function \( u \) which is analytic on \( U_w \) with no zeroes there.

Proof. An immediate application of Theorems 4.5 and 4.7. \( \Box \)

The points where \( v_w(f) < 0 \) are called **poles**, the integer \( -v_w(f) \) the **order** of the pole, and the coefficient \( a_{-1} \) the **residue** of \( f \) at the pole \( w \), denoted \( \text{Res}_w f(z) \).

With meromorphic functions defined, we can now state the **Residue Theorem**, which enables us to compute integrals of meromorphic functions along closed, contractible contours with ease:

**Theorem 4.9.** Let \( \Omega \subset \mathbb{C} \) be open and \( f \) a meromorphic function on \( \Omega \). If \( C \) is a closed contractible contour in \( \Omega \) then
\[
\frac{1}{2\pi i} \int_C f(z) \, dz = \sum_\zeta \text{Ind}(C, \zeta) \text{Res}(f(z))
\]

Proof. We provide only a sketch of the proof. Since \( C \) is a closed, contractible contour in \( \Omega \), it is homotopic to a small circle around each pole contained in the convex hull of \( C \). Let \( n \) be the number of such poles, and let \( C_i \) denote a small circle centered at each pole, \( \zeta_i \), with \( 1 \leq i \leq n \). Then we have that:

\[
\frac{1}{2\pi i} \int_C f(z) \, dz = \sum_{i=1}^{n} \frac{1}{2\pi i} \int_{C_i} f(z) \, dz
\]

By choosing each \( C_i \) small enough (which is again possible using homotopies), we can on each circle take the Laurent series expansion of \( f \), and thus obtain

\[
\int_{C_i} f(z) \, dz = \int_{C_i} \sum_{n=\nu_w(f)}^\infty a_n (z - \zeta_i)^{n} dw(f) \, dz = 2\pi i \cdot a_{-1}
\]

Thus we obtain that the integral

\[
\frac{1}{2\pi i} \int_C f(z) \, dz
\]

is equal to the sum of the residues, which lie in the convex hull of \( C \). Notice that this proves the equality

\[
\frac{1}{2\pi i} \int_C f(z) \, dz = \sum_\zeta \text{Ind}(C, \zeta) \text{Res}(f(z))
\]

since for those poles outside the convex hull the index will be zero. \( \square \)

An immediate consequence of the Residue Theorem is the following:

**Theorem 4.10.** Let \( \Omega \subset \mathbb{C} \) be open and \( f \) an analytic function on \( \Omega \). If \( C \) is a closed contractible contour in \( \Omega \) then

\[
\frac{1}{2\pi i} \int_C \frac{\partial}{\partial z} \log f(z) \, dz = \sum_\zeta \text{Ind}(C, \zeta) \nu_\zeta(f)
\]

where \( \nu_\zeta(f) \) is the order of vanishing of \( f \) at \( \zeta \) and the sum is taken over the zeroes of \( f \).

Proof. In general, \( \log f \) is not well defined on \( \Omega \), but if we take

\[
\frac{\partial}{\partial z} \log f(z) = \frac{\partial f(z)}{f(z)}
\]

then \( \frac{\partial}{\partial z} \log f(z) \) is a well defined meromorphic function on \( \Omega \). The result then follows by using condition 3 of Theorem 4.8, and then applying Theorem 4.9. \( \square \)

With these powerful theorems at hand, we can prove a basic formula for evaluating the residues of certain functions at simple poles, one that will be very useful in the coming Section:
Theorem 4.11. Let be $f$ an analytic function near $w$ and $g$ a meromorphic function near $w$. Then $fg$ is meromorphic near $w$, and

$$\text{Res}_{z=w}(f(z)g(z)) = f(w)\text{Res}_{z=w}(g(z))$$

if $g$ has a simple pole at $w$.

Proof. Let $h = fg$. It can be quickly verified that $h$ is meromorphic by using the analyticity of $f$ and condition 3 of Theorem 4.8 on $g$. The formula is derived by taking the Laurent series expansions of $h$ and $g$. \qed

5. Bernoulli Polynomials and Wallis’ Product

The Bernoulli Polynomials are defined by

$$B_k(x) = \frac{\partial^k u}{\partial z^k} |_{z=0}$$

where the function $u$ is given by

$$u(x, z) = \begin{cases} 1 & \text{if } z = 0 \\
\frac{ze^{zx}}{e^{z}-1} & \text{if } z \neq 0 \end{cases}$$

(5.1)

For $x \in \mathbb{C}$, the function $u$ has simple poles at the points $z = 2\pi in$, where $n$ is a nonzero integer. The residues at these poles are (see page 115 of [1]):

$$\text{Res}_{z=2\pi in}(u(x, z)) = 2\pi in e^{2\pi in x}$$

What’s more, $u(x, z)$ is analytic for $|z| < 2\pi$, so by Theorem 4.2:

(5.2)

$$u(x, z) = \sum_{k=0}^{\infty} \frac{B_k(x)}{k!} z^k$$

Since $u$ is analytic for $|z| < 2\pi$, we can write it in terms of the Cauchy integral, and furthermore, we can differentiate the integrand, yielding

$$B_k(x) = \frac{k!}{2\pi i} \int_C z^{-k-1} u(x, z) \, dz$$

Here, the contour $C$ must be contractible and have index 1 with respect to 0. With this in mind, we let $C_N$ be a counterclockwise-oriented square contour with sides of length $(4N + 2)\pi$ parallel to the axes, and center 0. Since $C_0$ satisfies the requirements, we have

$$B_k(x) = \frac{k!}{2\pi i} \int_{C_0} z^{-k-1} u(x, z) \, dz$$

Next, we consider the following integral for $N > 0$:

$$\frac{k!}{2\pi i} \int_{C_N} z^{-k-1} u(x, z) \, dz$$

This has index 1 with respect to the poles $z = 2\pi in$ with $-N \leq n \leq N$, and index 0 with respect to the others. By Theorem 4.9 we know that
The first integral is simply $B_k$, and the sum can be evaluated by Theorem 4.11:

$$\frac{k!}{2\pi i} \int_{C_N} z^{-k-1} u(x, z) \, dz = k! \sum_{n \neq 0}^{\pm N} (2\pi in)^{-k} e^{2\pi inx}$$

Therefore, we have that:

$$B_k(x, z) = \frac{k!}{2\pi i} \int_{C_N} z^{-k-1} u(x, z) \, dz - k! \sum_{n \neq 0}^{\pm N} (2\pi in)^{-k} e^{2\pi inx}$$

Next, we show that the integral tends to zero. For this, we break up $C_N$ into its sides: $C_{\text{right}}$, $C_{\text{left}}$, $C_{\text{upper}}$ and $C_{\text{lower}}$. Now, on every side of $C_N$, we have the following (see page 117 of [1]):

$$\left| e^{z^{-k}} e^{z^{-1}} \right| \leq e^{\pi} e^{\pi - 1}$$

Furthermore, since

$$|z|^{-k} \leq (2N + 1)^{-k} \pi^{-k}$$

we obtain

$$|u(x, z) z^{-k-1}| \leq \frac{\pi^{-k} e^\pi}{e^\pi - 1} (2N + 1)^{-k}$$

uniformly for $z \in C_N$. The length of the contour is $8(2N + 1)\pi$, so by the ML-Theorem, we have that:

$$\left| \frac{k!}{2\pi i} \int_{C_N} z^{-k-1} u(x, z) \, dz \right| \leq \frac{4\pi^{-k} e^\pi}{e^\pi - 1} (2N + 1)^{1-k}$$

By assumption $k > 1$, so the right side tends to zero as $N$ tends to infinity. Therefore, we find

$$B_k(x) = -k! \lim_{N \to \infty} \frac{k!}{2\pi i} \int_{C_N} z^{-k-1} u(x, z) \, dz$$

By grouping the terms with opposite signs, we obtain that:

$$B_k(x) = -k! \lim_{N \to \infty} \sum_{n=1}^{N} (2\pi in)^{-k} \left[ e^{2\pi inx} + (-1)^k e^{2\pi inx} \right]$$

This we can write as:
\[(5.4)\quad B_k(x) = \begin{cases} 
(-1)^{(k+2)/2} \cdot k! \cdot 2(2\pi)^{-\frac{k}{2}} \sum_{n=1}^{\infty} \cos(2\pi nx) & \text{if } k \text{ is even} \\
(-1)^{(k+1)/2} \cdot k! \cdot 2(2\pi)^{-\frac{k}{2}} \sum_{n=1}^{\infty} \sin(2\pi nx) & \text{if } k \text{ is odd} 
\end{cases} \]

Now, we have proved Equation (5.4) for \(k > 1\) and \(0 \leq x \leq 1\). But in fact, it remains true for \(k = 1\), so long as \(0 < x < 1\). This, too, we can prove by contour integration, with slightly different estimates (see page 119 of [1]):

\[\left| \int_{C_{\text{left}}} u(x,z) \cdot z^{-2} \, dz \right| \leq \frac{2e^{\pi}}{e^{\pi} - 1} e^{-2(2N+1)\pi(x-1)}\]

\[\left| \int_{C_{\text{right}}} u(x,z) \cdot z^{-2} \, dz \right| \leq \frac{2e^{\pi}}{e^{\pi} - 1} e^{-(2N+1)\pi x}\]

\[\left| \int_{C_{\text{lower}}} u(x,z) \cdot z^{-2} \, dz \right| \leq (2N+1)^{-1} \int_{-\infty}^{\infty} \frac{e^{\xi}}{e^{\xi} + 1} \, d\xi\]

For \(C_{\text{upper}}\) and \(C_{\text{lower}}\), the right side tends to 0 as \(N\) tends to infinity, since \(0 < x < 1\). For \(C_{\text{upper}}\) and \(C_{\text{lower}}\), again since \(0 < x < 1\), the integral on the right side converges, and thus the right side approaches 0 as \(N\) tends to infinity. Based on the same arguments as before, we conclude that Equation (5.2) holds for \(k = 1\) if \(0 < x < 1\). The equations do not hold for \(k = 1\) if \(x = 0\) or \(x = 1\). Indeed, \(B_1(1) = \frac{1}{2}\) and \(B_1(0) = -\frac{1}{2}\), where Equation (5.4) would give zero in both cases.

Further interesting results occur when in the power series expansion for \(u(x,z)\) - that is, Equation (5.2) - we replace \(B_k(x)\) with Equation (5.3):

\[(5.5)\quad u(x,z) = 1 + a(x)z - \sum_{k=1}^{\infty} \lim_{N \to \infty} \sum_{N \leq n \leq N} \left( \frac{z}{2\pi inx} \right)^k e^{2\pi inx}\]

Here the term \(a(x)\) is defined as follows:

\[(5.6)\quad a(x) = \begin{cases} 
-\frac{1}{2} & \text{if } x = 0 \\
0 & \text{if } 0 < x < 1 \\
\frac{1}{2} & \text{if } x = 1 
\end{cases} \]

This is done to make Equation (5.5) correct in the cases \(x = 1\) and \(x = 0\). Here, switching the order of summations will be enormously beneficial, yet the double sum is not absolutely convergent. This can be circumvented by splitting off the \(k = 1\) term, switching the order of the sums, and then restoring the \(k = 1\) term:
\[
\begin{align*}
    u(x, z) &= 1 + a(x)z - \lim_{N \to \infty} \sum_{-N \leq n \leq N, n \neq 0} \left( \frac{z}{2\pi i n} \right) e^{2\pi i nx} - \sum_{k=2}^{\infty} \lim_{N \to \infty} \sum_{-N \leq n \leq N, n \neq 0} \left( \frac{z}{2\pi i n} \right)^k e^{2\pi i nx} \\
    &= 1 + a(x)z - \lim_{N \to \infty} \sum_{-N \leq n \leq N, n \neq 0} \left( \frac{z}{2\pi i n} \right)^2 e^{2\pi i nx} - \sum_{k=2}^{\infty} \lim_{N \to \infty} \sum_{-N \leq n \leq N, n \neq 0} \left( \frac{z}{2\pi i n} \right)^k e^{2\pi i nx} \\
    &= 1 + a(x)z - \lim_{N \to \infty} \sum_{-N \leq n \leq N, n \neq 0} \sum_{k=1}^{\infty} \left( \frac{z}{2\pi i n} \right)^k e^{2\pi i nx} \\
    \text{(5.7)} &= 1 + a(x)z - \lim_{N \to \infty} \sum_{-N \leq n \leq N, n \neq 0} \sum_{n \neq 0} \left( \frac{z}{2\pi i n} \right)^k e^{2\pi i nx}
\end{align*}
\]

The inner sum is now a geometric series that can be evaluated immediately:

\[
\begin{align*}
    u(x, z) &= 1 + a(x)z - \lim_{N \to \infty} \sum_{-N \leq n \leq N, n \neq 0} \sum_{n \neq 0} \left( \frac{z}{2\pi i n} \right)^k e^{2\pi i nx} \\
    &= 1 + a(x)z - \lim_{N \to \infty} \sum_{-N \leq n \leq N, n \neq 0} \sum_{n \neq 0} \left( \frac{z}{2\pi i n} \right)^k e^{2\pi i nx} \\
    &= a(x)z - \lim_{N \to \infty} \sum_{-N \leq n \leq N, n \neq 0} \frac{z}{z - 2\pi i n} e^{2\pi i nx}
\end{align*}
\]

Recalling the definition of \( u(x, z) \), we obtain that:

\[
\begin{align*}
    \text{(5.8)} &= \frac{e^{zx}}{e^z - 1} = a(x) - \lim_{N \to \infty} \sum_{-N \leq n \leq N, n \neq 0} \frac{e^{2\pi inx}}{z - 2\pi i n}
\end{align*}
\]

Because we began from Equation (5.2) (Taylor expansion of \( u \)), we have only proved the Equation (5.8) for \(|z| < 2\pi\). Since both sides are analytic for \( z \neq 2\pi i n \), we can use the Rigidity Principle to remove this restriction. By letting \( x = 0, x = \frac{1}{2} \) and \( x = 1 \), we acquire the following:

\[
\begin{align*}
    \text{(5.9)} &= \frac{e^{-z/2}}{e^{z/2} - e^{-z/2}} = \frac{1}{2} + \lim_{N \to \infty} \sum_{-N \leq n \leq N} \frac{1}{z - 2\pi i n} \\
    \text{(5.10)} &= \frac{1}{e^{z/2} - e^{-z/2}} = \lim_{N \to \infty} \sum_{-N \leq n \leq N} \frac{(-1)^n}{z - 2\pi i n} \\
    \text{(5.11)} &= \frac{e^{z/2}}{e^{z/2} - e^{-z/2}} = \frac{1}{2} + \lim_{N \to \infty} \sum_{-N \leq n \leq N} \frac{1}{z - 2\pi i n}
\end{align*}
\]

Adding Equations (5.9) and (5.11) yields

\[
\begin{align*}
    \text{(5.12)} &= \frac{e^{z/2} + e^{-z/2}}{e^{z/2} - e^{-z/2}} = 2 \lim_{N \to \infty} \sum_{-N \leq n \leq N} \frac{1}{z - 2\pi i n}
\end{align*}
\]

Setting \( z = \pi ix \), we get the following from Equations (5.10) and (5.12):
By integrating Equation (5.14), we see that
\[ \log \sin \pi x = \log x + \sum_{n=1}^{\infty} \left[ \log(n^2 - x^2) - \log(n^2) \right] + c \]
Here \( c \) is a constant of integration. Furthermore, by exponentiating we find
\[ \sin(\pi x) = e^{c}x \prod_{n=1}^{\infty} \left(1 - \frac{x^2}{n^2}\right) \]
Taking the derivative of both sides, we obtain:
\[ \pi \cos(\pi x) = e^{c} \prod_{n=1}^{\infty} \left(1 - \frac{x^2}{n^2}\right) + xe^{c} \left[ \prod_{n=1}^{\infty} \left(1 - \frac{x^2}{n^2}\right) \right]' \]
Setting \( x = 0 \), we obtain that
\[ \pi = e^{c} \]
and so
\[ \sin(\pi x) = \pi x \prod_{n=1}^{\infty} \left(1 - \frac{x^2}{n^2}\right) \]
Now, by setting \( x = \frac{1}{2} \) we get that
\[ 1 = \frac{\pi}{2} \prod_{n=1}^{\infty} \left(1 - \frac{1}{4n^2}\right) = \frac{\pi}{2} \prod_{n=1}^{\infty} \left(\frac{4n^2 - 1}{4n^2}\right) \]
This gives us that
\[ \frac{\pi}{2} = \prod_{n=1}^{\infty} \left(\frac{4n^2}{4n^2 - 1}\right) = \prod_{n=1}^{\infty} \left(\frac{2n}{2n - 1}\right) \frac{2n}{2n + 1} \]
\[ = \frac{2 \cdot 2 \cdot 4 \cdot 4 \cdot 6 \cdot 6 \cdot \ldots}{1 \cdot 3 \cdot 3 \cdot 5 \cdot 5 \cdot 7 \cdot \ldots} \]
which was first given by Wallis in 1655.

**Acknowledgments**

I would like to thank my mentor, Oishee Banerjee, for her help and guidance (and patience) with the writing of this paper. In addition, I would like to thank Daniil Rudenko, for his excellent mentorship in the apprenticeship program, and Peter May, for making the REU possible. It was a truly wonderful summer, one that I will cherish for years to come.
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