ANOSOV DIFFEOMORPHISMS AND MANIFOLDS ADMITTING THEM

MARLIN FIGGINS

Abstract. Anosov diffeomorphisms are dynamical systems that naturally arise as toral automorphisms and as an attempt to locally consolidate a system's dynamics into directions of expansion and contraction. Despite their 'apparent' simplicity, Anosov diffeomorphisms satisfy surprisingly strict dynamical and topological properties and are, historically speaking, difficult to find and construct. Attempts to classify these maps have led to more and more general constructions, but as of now, the problem of finding all Anosov diffeomorphisms on compact manifolds remains open. Here, we present some of the known constructions and modern tools created for and used towards classification results of Anosov diffeomorphisms and the manifolds that admit them.

Contents

1. Introduction to Differentiable Dynamics

Our subject of interest is understanding the global behavior and structure of differentiable dynamical systems. In the most general sense, we want to consider the action of a Lie group G on a smooth manifold M such the group action $G \times M \to M$ is C^r. That is, we want to define some smooth homomorphism between our Lie

Date: September 27, 2018.

group G and $\mathrm{Diff}(M)$, the set of all C^r diffeomorphisms of our manifold M to itself, $1 \leq r \leq \infty$.

Letting $G = \mathbb{R}$ and considering $\mathbb{Z} \subset \mathbb{R}$, given some $f \in \mathrm{Diff}(M)$, we see that our interest is in understanding the iterates of f. Conceptually, this asks: *what can happen to a point in our manifold if we continue apply our map f to it over and over again?*

1.1. **Basic Terminology.**

Definition 1.1. Given $f \in \mathrm{Diff}(M)$ and $x \in M$, the *orbit* of x (under f) is the set

$$O_f(x) = \{f^n(x) \mid n \in \mathbb{Z}\},$$

where $f^n \in \mathrm{Diff}(M)$ is the n-fold composition of f i.e.

$$f^n = \underbrace{f \circ f \circ \ldots \circ f}_{n \text{ times}}.$$

If f is understood, we will just use $O(x)$.

Our problem, then, is asking what properties the orbit of x has for an arbitrary point x in our manifold M. There are several possibilities. One of these possibilities is that the orbit is periodic.

Definition 1.2. An orbit is called *periodic* if $|O_f(x)| = m$ for some $m \in \mathbb{N}$, meaning

$$O_f(x) = \{x, f(x), \ldots, f^{m-1}(x)\} \tag{1.3}$$

and $x = f^m(x)$. We call m the (minimal) period of x. If $O_f(x)$ is periodic, then x is called a *periodic point* (of f). In particular, if x has period 1, we say that x is a *fixed point* (of f). We denote the set of periodic points of f as $\mathrm{Per}(f)$.

Remark 1.4. The set of periodic points $\mathrm{Per}(f)$ is f-invariant. Likewise, the set of fixed points $\mathrm{Fix}(f)$ is f-invariant.

Given that there can be different types of behavior for points under diffeomorphisms in $\mathrm{Diff}(M)$, a natural question is how we can define two diffeomorphisms to be close and establish the topology on $\mathrm{Diff}(M)$.

1.2. **Classifying Diffeomorphisms.** If we assume that M is a compact manifold, we can endow $\mathrm{Diff}(M)$ with a topology of uniform C^r convergence. Though we have a notion of closeness, we still lack a notion of equivalence. We can ask what should it mean for two diffeomorphisms in $\mathrm{Diff}(M)$ to be 'equivalent', what properties of orbit structure we want to be preserved between 'equivalent maps', as well as adapt a notion of stability to our equivalence relation. The first of these notions is *topological conjugacy*.

Definition 1.5. We say that $f \in \mathrm{Diff}(M)$ and $g \in \mathrm{Diff}(M')$ are *topologically conjugate* if there is a homeomorphism $h\colon M' \to M$ such that $f \circ h = h \circ g$. Such a map h is called a conjugacy.

One can easily check that topological conjugacy forms an equivalence relation and that conjugacies preserve the sets fixed and periodic points. Associated with topological conjugacy, there is the notion of *structural stability*, i.e. we say f is structurally stable if for g sufficiently 'close' to f in $\mathrm{Diff}(M)$, we can find a conjugacy that is arbitrarily close to the identity. As it turns out, even this notion is too strict. It was conjectured, and shown to be false, that structurally stable diffeomorphisms were dense in $\mathrm{Diff}(M)$ with respect to the C^r topology.[8, Sec. 1.1.]

In order to introduce the next notion of equivalence, we must define some new objects.

Definition 1.6. Given $f \in \mathrm{Diff}(M)$, $x \in M$ is said to be *non-wandering* if for every neighborhood U of x, we can find some $n \in \mathbb{N}$ such that $f^n(U) \cap U \neq \emptyset$.

The set of non-wandering points of f is denoted by $\Omega = \Omega(f)$.We can then define a weaker notion of equivalence by saying that f, f' are topologically conjugate on Ω if there is a homeomorphism $h\colon \Omega(f) \to \Omega(f')$ such that $h \circ f = f' \circ h$. Instead of structural stability, we call the analogous property for Ω-conjugacy Ω*-stability*.

Proposition 1.7. *If two diffeomorphisms f and g are topologically conjugate, they are topologically conjugate on Ω.*

Proof. Restricting the domain of the conjugacy h to $\Omega(f)$ gives a homeomorphism between $\Omega(f)$ and $h(\Omega(f))$ with respect to the subspace topologies. As h is a bijective, open map, we see that when $y \in \Omega(g)$, $h(U)$ is a neighborhood of y for any neighborhood U of $h^{-1}(y)$. Since $h^{-1} \circ g^n \circ h = f^n$, a point in $g^n(h(U)) \cap h(U)$ maps to a point in $f^n(U) \cap U$. Therefore since y is non-wandering, $h^{-1}(y)$ is non-wandering. The other direction follows similarly. $\square$

This justifies the statement that this is a weaker notion of equivalence. With this in mind, we want to find an example of a class of Ω-stable diffeomorphisms.

1.3. **Hyperbolicity.**

Definition 1.8. Suppose $A \in \mathbf{GL}_n(\mathbb{R})$, where $\mathbf{GL}_n(\mathbb{R})$ is the set of all $n \times n$ invertible matrices with entries in $\mathbb{R}$. We say that A is a *hyperbolic* matrix if each of its eigenvalues $\lambda_i \in \mathbb{C}$ satisfy $|\lambda_i| \neq 1$. We call a eigenvalue λ_i *contracting* if $|\lambda_i| < 1$ or *expanding* if $|\lambda_i| > 1$. Similiarly, a matrix A is called contracting (expanding) if its eigenvalues are contacting (expanding).

Remark 1.9.
1. Every $A \in \mathbf{GL}_n(\mathbb{R})$ is a smooth diffeomorphism of $\mathbb{R}^n$
2. The subset of hyperbolic matrices in $\mathbf{GL}_n(\mathbb{R})$ is open and dense.
3. If two hyperbolic matrices are conjugate, then they have the same eigenvalues.

Given a hyperbolic $A \in \mathbf{GL}_n(\mathbb{R})$, we can split the domain of A into the direct sum of A-invariant subspaces E^s and E^u i.e. $\mathbb{R}^n = E^s \oplus E^u$, where E^s and E^u are the generalized eigenspaces corresponding to the contracting and expanding eigenvalues of A respectively. It follows that A restricted to one of these subspaces is contracting (E^s) or expanding (E^u). This gives us a direction in which A is expanding and another in which it's contracting, giving us the picture below (Fig. 1).

Consider a hyperbolic $A \in \mathbf{GL}_n(\mathbb{Z})$, the set of invertible $n \times n$ matrices with entries in $\mathbb{Z}$ and determinant ± 1. Since $\mathbf{GL}_n(\mathbb{Z}) \subset \mathbf{GL}_n(\mathbb{R})$, A is still a diffeomorphism from $\mathbb{R}^n$ to itself. Most importantly in this case, $A(\mathbb{Z}^n) = \mathbb{Z}^n$. Therefore by quotienting $\mathbb{R}^n$ by $\mathbb{Z}^n$, A induces a map

$$\tilde{A}\colon x + \mathbb{Z}^n \mapsto A(x) + \mathbb{Z}^n \tag{1.10}$$

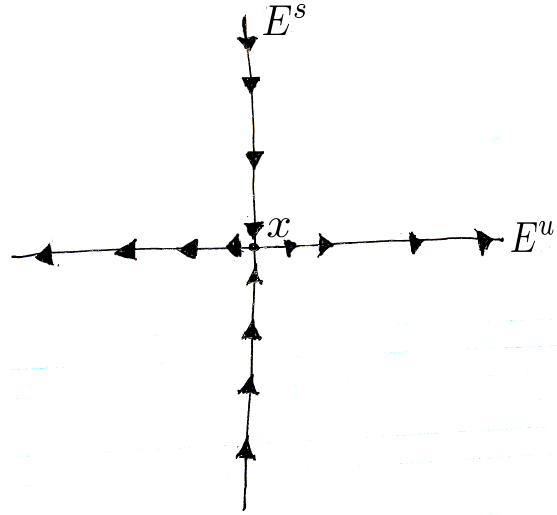


FIGURE 1. Example of a splitting $\mathbb{R}^n = E^s \oplus E^u$

on $\mathbb{T}^n = \mathbb{R}^n/\mathbb{Z}^n$, the n-torus to itself. One expects that the induced map $\tilde{A}$ has some of the properties of A, in particular that it is 'hyperbolic' in a sense. First, we check that $\tilde{A}$ a 'reasonable' map on the n-torus.

Proposition 1.11. *If $A \in \mathbf{GL}_n(\mathbb{Z})$ is hyperbolic, then $\tilde{A}$ is a diffeomorphism of $\mathbb{T}^n$.*

Proof. We need only show that $\tilde{A}$ and its inverse are well-defined. Since $|\det A| = 1$, we know that $A^{-1} \in \mathbf{GL}_n(\mathbb{Z})$ and that $\widetilde{A^{-1}}$ is well-defined.Since $\widetilde{A^{-1} \circ A} = \widetilde{A^{-1}} \circ \tilde{A} = \mathrm{id}_{\mathbb{T}^2}$, we see that $\widetilde{A^{-1}} = \tilde{A}^{-1}$. $\square$

Since $\mathbb{T}^n$ is a compact manifold, we can begin to generalize the notion of hyperbolicity to diffeomorphisms on compact manifolds, i.e. we can attempt to form an idea of what it means for $\tilde{A}$ to be 'hyperbolic' as an object in its own right.

Most importantly, we want our definition of hyperbolic maps to have some concept of an 'expanding direction' and a 'contracting' direction for at least some collection of points in its domain. We first need to give some idea of what this could mean mathematically. Luckily, differential topology has some terminology that naturally lends itself to this problem.

Definition 1.12. Given a smooth compact Riemannian manifold M, we define the tangent space at $x \in M$, T_xM to be the real vector space comprising the collection of tangent vectors to M at x. If M is embedded in $\mathbb{R}^n$, this can be pictured as the hyperplane tangent to x. Further, for a diffeomorphism $f \colon M \to M$, we define the differential of f at x to be the linear map $df_x \colon T_xM \to T_{f(x)}M$, which takes tangent vectors in T_xM to the corresponding tangent vector in $T_{f(x)}M$. The exact formulation of this correspondence can be found in a textbook on differential topology.

Remark 1.13. The collection of tangent spaces of M can be 'glued' together to form TM, which is called the tangent bundle of M. TM is also a smooth manifold with twice the dimension of M.

Adapting the situation described above to this new terminology leads us to the following definition of hyperbolicity.

Definition 1.14. [1, Chap. 3] Given a diffeomorphism f of a compact, Riemannian manifold, we say that a closed, invariant subset $\Lambda \subset M$ is *hyperbolic* if for every $x \in \Lambda$, T_xM, the tangent space of M at x, can be written as the direct sum of subspaces E^s_x, E^u_x such that $df_x(E^s_x) = E^s_{f(x)}$ and $df_x(E^u_x) = E^u_{f(x)}$, and there exists $c > 0$, $\lambda \in (0,1)$ such that for every $n \geq 0$,

$$\|df^n_x(v)\| \leq c\lambda^n \|v\| \text{,when } v \in E^s_x \tag{1.15}$$

$$\|df^n_x(v)\| \leq c\lambda^{-n} \|v\| \text{,when } v \in E^u_x, \tag{1.16}$$

where $\|\cdot\|$ is given by the Riemannian metric on M. The subspaces E^s_x and E^u_x are called the stable and unstable subspaces at x respectively.

Remark 1.17. This definition can also stated in terms of a splitting of the tangent manifold TM into invariant subbundles E^s and E^u[7].

This is intuitively similar to the hyperbolic maps defined in Definition 1.8. In fact, noticing this leads us to our first class of examples of hyperbolic diffeomorphisms.

1.4. **Hyperbolic Toral Automorphisms.**

Theorem 1.18. *If $A \in \mathbf{GL}_n(\mathbb{Z})$ is a hyperbolic matrix, then $\mathbb{T}^n$ is hyperbolic with respect to $\tilde{A}\colon \mathbb{T}^n \to \mathbb{T}^n$ in the sense of Definition 1.14.*

Proof. The space $\mathbb{R}^n$ is hyperbolic with respect to A in the sense of Definition 1.14. Therefore, we can consider a splitting of $\mathbb{R}^n$ into $E^s_x \oplus E^u_x$. Since the tangent space of $\mathbb{R}^n$ is naturally identified with $\mathbb{R}^n$ for every $x \in \mathbb{R}^n$, we can pass this splitting to the tangent space of the coset of x in $\mathbb{T}^n$, giving a new splitting under $\tilde{A}$. □

The diffeomorphisms induced by the hyperbolic matrices satisfying the conditions of Theorem 1.18 are called *hyperbolic toral automorphisms.* Hyperbolic toral automorphisms are an interesting class of objects, and have some pretty neat dynamical behavior described by the following proposition.

Proposition 1.19. *Properties of hyperbolic toral automorphisms. If $\tilde{A}$ is a hyperbolic toral automorphism of the n-torus, $n \geq 2$, then*

1. *$Per(\tilde{A})$ is dense in $\mathbb{T}^n$.*
2. *The number of fixed points of $\tilde{A}$ is equal to* $\det(A - I)$
3. *The number of points with period n is given by* $\det(A^n - I)$.

Proof of (1). Suppose that we have p an arbitrary rational point of the n-torus. We can represent p as $(\frac{p_1}{q}, \frac{p_2}{q}, \ldots, \frac{p_n}{q}) \in \mathbb{Q}^n/\mathbb{Z}^n \subset \mathbb{T}^n$, where $q \in \mathbb{Z}$ and each p_i is non-negative and less than q. Since the action of $\tilde{A}$ returns the fractional part of each entry in $A(p)$, we see that the possible set of values is $\{\big(\frac{q_1}{q}, \frac{q_2}{q}, \ldots, \frac{q_n}{q}\big) \mid 0 \leq q_i < q\}$ which is finite, meaning if we continue to apply A to p we will eventually have a repeat. Therefore, all rational points are periodic. Since the rational points of $\mathbb{T}^n$ are dense in $\mathbb{T}^n$, $Per(\tilde{A})$ is dense in $\mathbb{T}^n$. □

Proof of (2) and (3). We see that a point x is a fixed point if $\tilde{A}(x) = x$ that is if

$$(A - I)(x) = N \in \mathbb{Z}^n.$$

We need to check how many points in $(A - I)(\mathbb{T}^n)$ lie on the lattice $\mathbb{Z}^n$. Since there is exactly one in the fundamental domain $[0, 1)^n$ of the n-torus, we can see that this just corresponds to the volume of the parallelepiped $(A - I)([0, 1)^n)$, i.e. $\left|\mathrm{Fix}(\tilde{A})\right| = |\det(A - I)|$. This proves (2). Since the fixed points of $\tilde{A}^k$ are the periodic points of $\tilde{A}$ with period k, we get (3). $\square$

2. Anosov Diffeomorphisms

In truth, hyperbolic toral automorphisms fall into a more general class of objects called Anosov diffeomorphisms, but starting with the example and construction of hyperbolic toral automorphisms, will prove useful in providing intuition for later constructions. We now begin to consider more general Anosov diffeomorphisms. We start with the precise definition of an Anosov diffeomorphism.

Definition 2.1. A diffeomorphism f of a smooth, compact, Riemannian manifold is called *Anosov* if M is hyperbolic.

It should be noted that since $\mathrm{Per}(f) \subset \Omega(f)$ for any diffeomorphism f, this definition entails that all periodic and fixed points of f are hyperbolic. We begin by presenting a specific example of an Anosov diffeomorphism.

2.1. A First Class of Examples.

Example 2.2. Arnold's Cat Map. Consider the following matrix in $\mathbf{GL}_2(\mathbb{R})$:

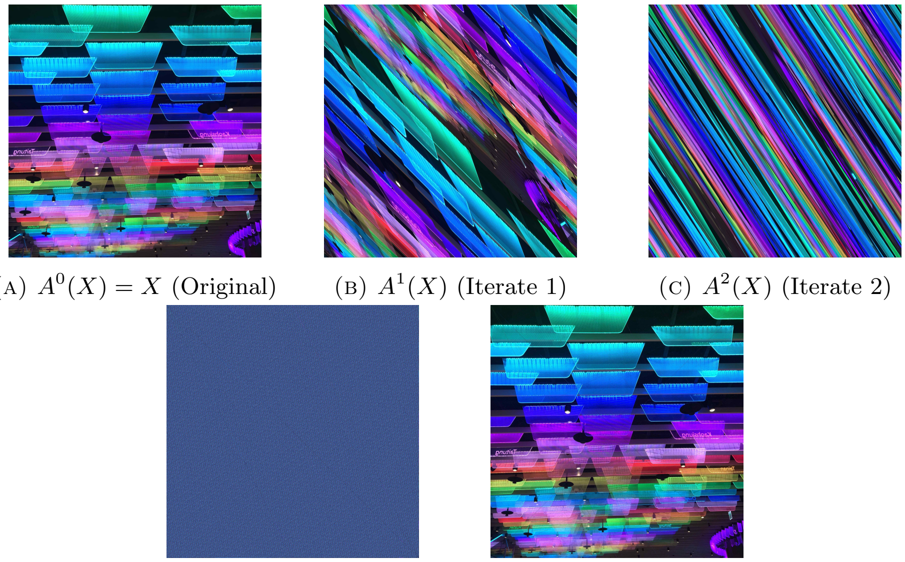
$$A = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}. \tag{2.3}$$

One can easily check that $A \in \mathbf{GL}_2(\mathbb{Z})$, $\det A = 1$, and that A is hyperbolic. Theorem 1.18 shows that A induces a hyperbolic toral automorphism, and in particular that $\tilde{A}$ is an Anosov diffeomorphism. A routine calculation shows that its eigenvalues are λ and λ^{-1}, where $\lambda = \frac{3+\sqrt{5}}{2} > 1$. Hence, A is diagonalizable and for any point $x \in \mathbb{T}^n$, the unstable and stable subspaces of x correspond to the eigenspaces for λ and λ^{-1} respectively.

Remark 2.4. If we take a square picture, we can reorder its pixels according to the action of A. An example of this is shown in Fig. 2. The computer code to do this was written by the author in the R programming language. Because of the limitations of computers, this only shows what is happening at rational points.

Theorem 2.5. *Every hyperbolic toral automorphism is an Anosov diffeomorphism.*

Hyperbolic toral automorphisms are really the poster children of Anosov diffeomorphisms. In fact, a lot of the study of classifying Anosov diffeomorphisms has been dedicated to finding Anosov diffeomorphisms which were *not* topologically conjugate to a hyperbolic toral automorphism. We will begin to tackle questions of classification a bit later, but first, we will tackle some of the general properties of Anosov diffeomorphisms, starting with a lemma about the metric used in Definition 1.14.



(A) $A^0(X) = X$ (Original) (B) $A^1(X)$ (Iterate 1) (C) $A^2(X)$ (Iterate 2)

(D) $A^{10}(X)$ (Iterate 10) (E) $A^{36}(X)$ (Iterate 36)

FIGURE 2. Arnold's Cat Map on a picture. Notice that it has a period of 36.

2.2. **Dynamics of Anosov Diffeomorphisms.**

Lemma 2.6. *Every Anosov diffeomorphism has an* adapted *metric* $\|\cdot\|$ *such that* $c = 1$ *in Definition 1.14.*

Proof. See [6]. $\square$

For simplicity, we will use an adapted metric going forward. Given that for Anosov systems our dynamics consist of contraction and expansion, one might, reasonably, attempt to classify the behavior of orbits of f. We start by describing the sets of points whose orbits stay close to the orbit of a given point x.

Definition 2.7. [1] Given $x \in M$ and an Anosov diffeomorphism f, we define the following:

$$W^s_\epsilon(x) = \{y \in M \mid d(f^n(x), f^n(y)) \leq \epsilon, \text{ for every } n \geq 0\}, \tag{2.8}$$

$$W^u_\epsilon(x) = \{y \in M \mid d(f^{-n}(x), f^{-n}(y)) \leq \epsilon, \text{ for every } n \geq 0\}, \tag{2.9}$$

$$W^s(x) = \{y \in M \mid \lim_{n\to\infty} d(f^n(x), f^n(y)) = 0\}, \tag{2.10}$$

$$W^u(x) = \{y \in M \mid \lim_{n\to\infty} d(f^{-n}(x), f^{-n}(y)) = 0\}. \tag{2.11}$$

We call $W^s_\epsilon(x)$ and $W^u_\epsilon(x)$ the ϵ*-stable* and ϵ*-unstable manifolds* of x respectively. Similarly, $W^s(x)$ and $W^u(x)$ are called the *stable* and *unstable manifolds* respectively.

In order to understand the behavior of an Anosov diffeomorphism f on these sets and justify calling them manifolds, we rely on the following theorem.

Theorem 2.12 (Stable Manifold Theorem)**.** [1, 6] *Suppose f is a C^r diffeomorphism and Λ is a hyperbolic set for f. For $x \in \Lambda$ and sufficiently small positive ϵ,*

a) $d(f^n(x), f^n(y)) \leq \lambda^n d(x,y)$ when $y \in W^s_\epsilon(x)$, and $d(f^{-n}(x), f^{-n}(y)) \leq \lambda^n d(x,y)$ when $y \in W^u_\epsilon(x)$ for all $n \geq 0$;
b) $W^\sigma_\epsilon(x)$ is an embedded C^r submanifold for $x \in \Lambda$ such that $T_x W^\sigma_\epsilon(x) = E^\sigma_x$, and
c) $W^\sigma_\epsilon(x)$ varies continuously with x, where ($\sigma = s, u$).

Proof. See [6, Sec. 2, Theorem 1]. □

Remark 2.13. $W^s(x)$ and $W^u(x)$ are submanifolds of M.

The following corollary establishes the relationship between our ϵ-(un)stable manifolds and our (un)stable manifolds.

Corollary 2.14. *For a diffeomorphism f, hyperbolic set Λ, and $x \in \Lambda$ as in the statement of Theorem 2.12,*

$$W^s(x) = \bigcup_{n\geq 0} f^{-n}(W^s_\epsilon(f^n(x))), \text{ and} \tag{2.15}$$

$$W^u(x) = \bigcup_{n\geq 0} f^{-n}(W^u_\epsilon(f^{-n}(x))). \tag{2.16}$$

Proof. We see $W^s_\epsilon(x) \subset W^s(x)$ by Theorem 2.12. Next, we notice that if $y \in W^s(x)$, then there is N such that

$$y \in \{z \mid d(f^n(x), f^n(z)) < \epsilon \text{ for } n \geq N\} = \{z \mid f^N(z) \in W^s_\epsilon(f^N(x))\}.$$

Therefore $y \in f^{-N}(W^s_\epsilon(f^N(x)))$. The unstable case follows similarly. □

The stable and unstable manifolds are incredible tools which are essential in the proofs of many of properties of Anosov diffeomorphisms. Not only this, but these manifolds allow us to establish some ideas about the local product structure of a diffeomorphism on our manifold M.

Lemma 2.17 (Local Product Structure)**.** *Suppose f is Anosov. For sufficiently small positive ϵ, there is some positive δ such that $W^s_\epsilon(x) \cap W^u_\epsilon(y)$ contains a unique point $[x,y] \in M$ for any two $x, y \in M$ that are δ-close. Furthermore the map $[\cdot,\cdot] : \Omega(f) \times \Omega(f) \to M$ is continuous.*

Proof. Since the submanifolds $W^s_\epsilon(x)$,$W^u_\epsilon(x)$ are transversal, the point in their intersection x varies continuously with x. We see that $[x,y]$ exists, is unique, and varies continuously. □

Theorem 2.18 (Spectral Decomposition)**.** [1] *Given f Anosov, we can write the non-wandering set of f as $\Omega(f) = \Lambda_1 \cup \Lambda_2 \cup \ldots \cup \Lambda_s$, where each Λ_i is a closed, and f-invariant set such that*

a) $f_{|\Lambda_i}$ is topologically transitive
b) $\Lambda_i = X_{i,1} \cup \ldots \cup X_{i,n_i}$ where $X_{i,j}$ are pairwise disjoint and $f(X_{i,j}) = X_{i,(j+1)}$ (i is taken mod n_i) and $f_{|X_{i,j}}$ is topologically mixing.

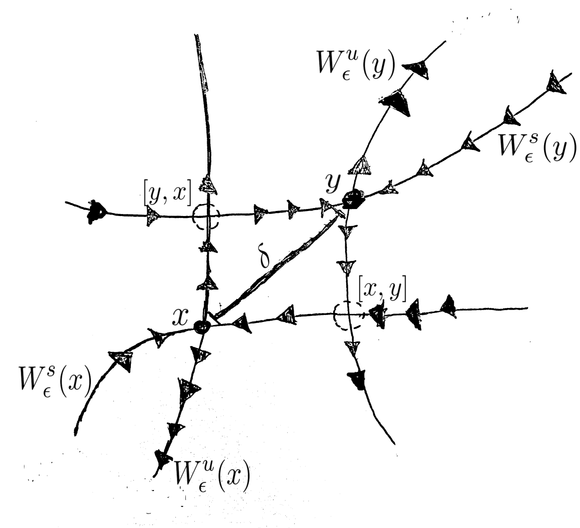


FIGURE 3. Finding $[x, y]$ according to Lemma 2.17

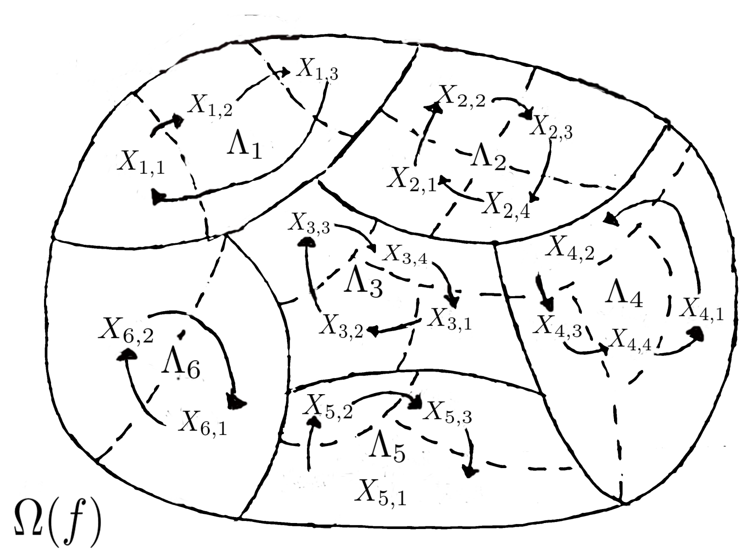


FIGURE 4. Partitioning $\Omega(f)$ according to Theorem 2.18

A proof of this can be found in [1, 3, 8] and most any introductory text on dynamical systems. Instead of proving this, we present Fig. 4 in order to visualize what the statement Theorem 2.18 tells us about the dynamics of f and its set of non-wandering points.

In words, the Spectral Decomposition theorem says that if f is an Anosov diffeomorphism of a manifold M, then M is in some sense *decomposed* by the dynamics of

f. This motivates the question of what kinds of decompositions can exist by Theorem 2.18 as well as the question of which manifolds allow for such a decomposition i.e. *What manifolds admit Anosov diffeomorphisms?*. This question is especially non-obvious, but we have some examples of manifolds which cannot admit Anosov diffeomorphisms.

Proposition 2.19. *There are no Anosov diffeomorphisms on the n-sphere.*

In addition to the question of manifolds admitting Anosov diffeomorphisms, we can ask about the stability properties of these Anosov diffeomorphisms i.e. whether Anosov diffeomorphisms are topologically conjugate to their perturbations? There are several theorems that are useful in answering these questions. In particular, we have the Anosov Shadowing Lemma which describes the orbits of Anosov diffeomorphisms.

Definition 2.20. Given a finite sequence $\bar{x} = \{x_i\}_{i=a}^{b}$ where a, b are integers. We say that $\bar{x}$ is a *δ-pseudo orbit of f* if

$$d(f(x_i), x_{i+1}) < \delta, \text{ for every } i \in [a, b-1). \tag{2.21}$$

A point $x \in M$ is said to *ϵ-shadow* $\bar{x}$ if

$$d(f^i(x), x_i) \leq \epsilon, \text{ for every } i \in [a, b]. \tag{2.22}$$

Theorem 2.23 (Anosov Shadowing Lemma)**.** [1] *Suppose f is Anosov. Then for every $\epsilon > 0$, there is some $\delta > 0$ such that every delta-pseudo orbit of f in Ω is ϵ-shadowed by some $x \in \Omega$ i.e. every sufficiently closely pseudo orbit of f in Ω can be approximated arbitrarily close by an actual orbit of f.*

Proof. Pick $\alpha > 0$ and choose $\beta < \alpha$ to satisfy Lemma 2.17. Take N such that $\lambda^N \alpha < \frac{\beta}{2}$, where λ is as in Definition 1.14. Let $\delta > 0$ such that for any δ-pseudo orbit $\{y_i\}_{i=0}^{N}$ in Ω, $d(f^j(y_0), y_j) < \frac{\beta}{2}$ for each $j \in [0, N]$.

Suppose $\{x_i\}_{i=0}^{rN}$ is a δ-pseudo orbit. For $k \in [0, r]$, define a new sequence x'_{kN} by

$$x'_{(k+1)N} = W^u_\alpha(f^N(x'_{kN})) \cap W^s_\alpha(x_{(k+1)N}) \in \Omega, \tag{2.24}$$

where $x'_0 = x_0$. According to Lemma 2.17, the point in this intersection exists and is unique by our choice of β and δ. Let $x = f^{-rN}(x'_{rN})$. Let $i \in [0, rN]$ and s with $i \in [sN, (s+1)N)$. Since each $x'_{tN} \in W^u_\alpha(f^N(x')_{(t-1)N})$ according to (2.24), we have the following inequalities:

$$d(f^i(x), f^{i-sN}(x'_{sN})) \leq \sum_{t=s+1}^{r} d(f^{i-tN}(x'_{tN}), f^{i-(t-1)N}(x'_{(t-1)N})) \tag{2.25}$$

$$\leq \alpha \sum_{t=s+1}^{r} \lambda^{tN-i} \leq \frac{\alpha\lambda}{1-\lambda}. \tag{2.26}$$

Similarly, we see that

$$d(f^{i-sN}(x'_{sN}), f^{i-sN}(x_{sN})) \leq \alpha, \tag{2.27}$$

since $x'_{sN} \in W^s_\alpha(x_{sN})$. Due to our choice of β, we have that

$$d(f^{i-sN}(x_{sN}), x_i) < \frac{\beta}{2} < \alpha. \tag{2.28}$$

Combining (2.25), (2.27), and (2.28) with the triangle inequality, we see that

$$d(f^i(x), x_i) \leq \alpha(\frac{\lambda}{1-\lambda} + 2), \tag{2.29}$$

which can be made less than ϵ for sufficiently small α. □

Remark 2.30. This theorem also holds in the case of infinite pseudo-orbits ($a = -\infty, b = \infty$).

An immediate corollary of this tells us about the periodic points in Ω.

Corollary 2.31. *Given $\epsilon > 0$, then there is $\delta > 0$ such that if $x \in \Omega$ with $d(f^n(x), x) < \delta$ for some $n \in \mathbb{N}$, then there is some n-periodic $x^* \in \Omega$ such that*

$$d(f^k(x), f^k(x^*)) \leq \epsilon, \text{ for every } k \in \{0, 1, \ldots, n\}. \tag{2.32}$$

This tells us that we can realize almost periodic orbits of f with nearby actual periodic orbits.

This imposes what seems to be a pretty strict condition on the dynamics of Anosov diffeomorphisms, which once again alludes to the question of classifying them.

3. Classifying Anosov Diffeomorphisms & Manifolds Admitting Them

We start by stating a result.

Theorem 3.1. *Anosov diffeomorphisms of a compact manifold are structurally stable.*

Despite all the properties of Anosov diffeomorphisms discussed in the previous section, we've as of yet only see one class of them (those representing the hyperbolic toral automorphisms). Because of the difficulties in constructing examples of Anosov diffeomorphisms as well as their 'generic' behavior, Smale poses the question of finding all examples of Anosov diffeomorphisms on a compact manifold in [8]. Indeed this question remains unanswered, but it is at least known that there exist non-toral Anosov diffeomorphisms. Though in order to see an example of this, we'll need to make room for more general algebraic construction relying on Lie groups.

3.1. Lie Groups and Nilmanifolds.

First, we present some basic ideas of Lie groups and their structure as manifolds.

Definition 3.2. A group G is a Lie group if it is a smooth manifold and if its group operation ($(g, h) \mapsto gh$) and inverse function ($g \mapsto g^{-1}$) are smooth maps.

Example 3.3. The following are examples of Lie groups:

1. $S^1 = \mathbb{R}/\mathbb{Z}$,
2. $\mathbb{R}^n$, $\mathbb{C}^n$,
3. $\mathbf{GL}_n(\mathbb{R})$ and $\mathbf{SL}_n(\mathbb{R})$,
4. $\mathbf{GL}_n(\mathbb{C})$ and $\mathbf{SL}_n(\mathbb{C})$,
5. $\mathbf{O}_n(\mathbb{R})$ and $\mathbf{O}_n(\mathbb{C})$.

The following proposition simplifies the problem of determining if a diffeomorphism of a Lie group is Anosov.

Proposition 3.4. *The tangent spaces of any $g, g' \in G$ are isomorphic to one another as vector spaces.*

Proof. We show that any tangent space of G is isomorphic to T_eG.
Given $g \in G$, we define $L^g(x) = gx$ where $x \in G$. This is a group homomorphism, and we see that dL^g_e is a vector space isomorphism between T_eG and T_gG. □

Lie groups behave essentially in the same way as regular groups but with an added differentiable structure. Our interest here is in constructing new examples of Anosov diffeomorphisms and manifolds admitting them, and we'll be taking an algebraic approach, attempting to construct such manifolds from Lie groups by considering cosets of certain subgroups and the orbit spaces of actions on them. These constructions will be generalizations of our earlier construction of the n-torus and hyperbolic toral automorphisms, so it will be useful to consider how they (again) come about as examples of what follows.

The setting we're interested in is a Lie group G and $\alpha \in \mathrm{Aut}(G)$, where α is a group automorphism of G. We want to pick this α such that there is a Γ that is a co-compact (meaning the set of right cosets $\Gamma\backslash G$ is compact), discrete subgroup of G with $\alpha(\Gamma) = \Gamma$. That is, $\Gamma\backslash G$ is a compact manifold. In particular, we're interested in the case where G is nilpotent.

Definition 3.5. Suppose we have a group G. Define a sequence of normal subgroups by $G_{i+1} = [G, G_i] = \{ghg^{-1}h^{-1} \mid g \in G, h \in G_i\}$ with $G = G_0$. G is said to be *nilpotent* if there is some n such that this sequence terminates in the trivial group i.e.

$$G = G_0 \supset G_1 \supset G_2 \supset \ldots \supset G_n = \{e_G\}.$$

Remark 3.6. Every abelian group is nilpotent.

If we require that G be a nilpotent Lie group and Γ be a compact lattice in the construction above, the resulting manifold $\Gamma\backslash G$ is called a *nilmanifold* and Γ is called a *co-compact lattice*.

Example 3.7. Consider the following groups:

$$G = \left\{ \begin{pmatrix} 1 & x & y \\ 0 & 1 & z \\ 0 & 0 & 1 \end{pmatrix} \mid (x, y, z) \in \mathbb{R}^3 \right\} \tag{3.8}$$

$$\Gamma = \left\{ \begin{pmatrix} 1 & x & y \\ 0 & 1 & z \\ 0 & 0 & 1 \end{pmatrix} \mid (x, y, z) \in \mathbb{Z}^3 \right\}. \tag{3.9}$$

The group G is known as the *Heisenberg group* and Γ is its restriction to integer matrices. The Heisenberg group is an example of a nilpotent Lie group. Since the multiplication of G acts as follows:

$$\begin{pmatrix} 1 & x_1 + x_2 & y_1 + y_2 + x_1 y z_2 \\ 0 & 1 & z_1 + z_2 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & x_1 & y_1 \\ 0 & 1 & z_1 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & x_2 & y_2 \\ 0 & 1 & z_2 \\ 0 & 0 & 1 \end{pmatrix}, \tag{3.10}$$

we see that G is a group, and restricting to integer matrices shows Γ is a subgroup of G. Furthermore, Γ is a discrete, co-compact subgroup, and so $\Gamma\backslash G$ is an example of a nilmanifold.

Following this same construction, one can also see that the torus is an example of a nilmanifold, but the question at hand is what happens to an automorphism α when it is 'passed' to a nilmanifold. Similar to the construction of the toral automorphisms, α induces a diffeomorphism $\tilde{\alpha}\colon \Gamma\backslash G \to \Gamma\backslash G$ on $\Gamma\backslash G$ defined by

$$\tilde{\alpha}(\Gamma g) = \Gamma\alpha(g). \tag{3.11}$$

We call the induced map $\tilde{\alpha}$ a *nilmanifold automorphism*. In the case where α is hyperbolic, $\tilde{\alpha}$ is a hyperbolic nilmanifold automorphism and is an Anosov diffeomorphism. In fact, this goes both ways for Anosov diffeomorphisms on nilmanifolds.

Theorem 3.12. [4, Theorem 5.2.] *An Anosov diffeomorphism of a nilmanifold is topologically conjugate to a hyperbolic nilmanifold automorphism.*

It is important to note that this is not true for Anosov diffeomorphisms on a more general class of spaces called *infra-nilmanifolds*. To see why this is, we need to introduce the concept of an infra-nilmanifold, which is a generalization of our previous notion of a nilmanifold.

3.2. **Constructing Infra-nilmanifolds.**

Definition 3.13. Suppose G is a simply connected, connected nilpotent Lie group. We define the affine group of G as $\mathrm{Aff}(G)$=$G\rtimes\mathrm{Aut}(G)$. Given $(g, \alpha) \in \mathrm{Aff}(G)$ acts on $x \in G$ by transformation by α followed by left translation by g i.e.

$$(g, \alpha) \cdot x = g\alpha(x). \tag{3.14}$$

Remark 3.15. In this way, we can see that $\mathrm{Aff}(G)$ is a subgroup of $\mathrm{Diff}(G)$.

Suppose we have some compact subgroup K of $\mathrm{Aut}(G)$, and consider a discrete subgroup of $G \rtimes K$ with no elements of finite order, Γ. This Γ acts as a subgroup of $\mathrm{Aff}(G)$. We then require that the quotient of G under the action of $\Gamma\backslash G$ is compact. This action is free and properly discontinuous, and it follows that $\Gamma\backslash G$ is a manifold. We call manifolds constructed in this way *infra-nilmanifolds*.

We see the reason for this name and the prefix *infra* in the following theorem.

Proposition 3.16. *Every infra-nilmanifold* $\Gamma\backslash G$ *is finitely covered by a nilmanifold.*

The proof of this relies on the following result from L. Auslander (as repaired in [4]).

Lemma 3.17. *If* G *is a lie group and* Γ *a co-compact, torsion-free, discrete subgroup of* $G \rtimes K$ *for some compact subgroup* K *of Aut(*G*), then* $\Gamma \cap G$ *is a uniform lattice of* G *and the holonomy group of* Γ *i.e.* $F = \Gamma\backslash(\Gamma \cap G)$ *is a finite group.*

Now onto the proof of Proposition 3.16.

Proof. Letting p be the canonical projection $G \rtimes K$ to K, we see that $p(\Gamma) \simeq F$. Therefore $\Gamma \subset G \rtimes F$. In particular if we take $\Gamma' \subset \Gamma$ such that $\Gamma' \subset (G \rtimes \{e_F\})$, then $(\Gamma' \cap G)\backslash G$ is a nilmanifold which finitely covers $\Gamma\backslash G$. This gives us that every infra-nilmanifold is finitely covered by a nilmanifold. $\square$

3.3. **Infra-nilmanifold Automorphisms & Anosov Diffeomorphisms.** Now that we have a notion of a infra-nilmanifold, it is historically necessary to mention the following conjecture which has motivated a lot of the ideas that follow.

Conjecture 3.18. *Every Anosov diffeomorphism is topologically conjugate to an infra-nilmanifold automorphism.*

In order for us to begin to check the validity of this conjecture, we proceed by constructing and defining the notion of an infra-nilmanifold automorphism.

Fixing G and F as before and letting Γ be a discrete, co-compact, torsion-free subgroup of $G \rtimes F$, we can write $(g, \alpha) \in \Gamma$ as $g\alpha$ and $(g, \varphi) \in \mathrm{Aff}(G)$ as $g\varphi$. Furthermore, we can define multiplication for $(g_1, \psi_1), (g_2, \psi_2) \in \mathrm{Aff}(G)$:

$$g_1\psi_1 g_2 \psi_2 = g_1 \psi_1(n_2)\psi_1\psi_2. \tag{3.19}$$

Lemma 3.20. [4] *Suppose we have G a connected, simply connected nilpotent Lie group and F a subgroup of Aut(G). Given an automorphism φ of $G \rtimes F$ such that $\varphi(F) = F$, then for any $(g, \alpha) \in G \rtimes F$*

$$\varphi(x) = \psi(g, \alpha)\psi^{-1}$$

where ψ is the first projection of $G \rtimes F$. This tells us that φ is given by a conjugation in $\mathrm{Aff}(G)$.

Proof. Identifying G with its inclusion into $G \rtimes \{e_{\mathrm{Aut}(G)}\}$ and likewise $\mathrm{Aut}(G)$ with $\{e_G\} \rtimes \mathrm{Aut}(G)$, we see that

$$\varphi(g) = \psi(g) = \psi g \psi^{-1}, \tag{3.21}$$

where $g \in G$ and $\psi g \psi^{-1}$ is a conjugation. Likewise for any $\mu \in F$, $g \in G$, we have that

$$\mu(g) = \mu g \mu^{-1} \tag{3.22}$$

in $G \rtimes F$. Applying our homomorphism φ to (3.22) gives us that

$$\varphi(\mu(g)) = \varphi(\mu)\psi(g)\varphi(\mu)^{-1} = \varphi(\mu)(\psi(g)).$$

Therefore, we know that $\varphi \circ \mu = \varphi(\mu) \circ \psi$ meaning

$$\varphi(\mu) = \psi \mu \psi^{-1}. \tag{3.23}$$

We can present $x \in G \rtimes F$ as $x = g\mu$ which gives us that

$$\varphi(x) = \varphi(g)\varphi(\mu) = \psi g \psi^{-1} \psi \mu \psi^{-1} = \psi x \psi^{-1} \tag{3.24}$$

using (3.21) and (3.23). □

In the case where φ is an automorphism of $G \rtimes F$ with $\varphi(F) = F$ as above and $\varphi(\Gamma) \subset \Gamma$. For $\gamma = h\mu$, we can define the action of Γ on $g \in G$ by $\gamma g = h\mu(g)$. A computation similar to the one in (3.23) allows us to see that $\varphi(\gamma g) = \varphi(\gamma)\varphi(g)$. Therefore $\varphi(\Gamma g) = \Gamma\varphi(g)$, meaning that φ induces a map of the left orbit space $\Gamma\backslash G$. This enables us to define an analogue of our toral automorphisms for more general infra-nilmanifolds.

Definition 3.25. [4] Suppose G is a connected, simply connected, nilpotent Lie group and $F \subset \mathrm{Aut}(G)$ a finite group, and Γ a discrete, co-compact, torsion-free subgroup of $G \rtimes F$. Given φ an automorphism of $G \rtimes F$ such that $\varphi(F) = F$ and $\varphi(\Gamma) \subset \Gamma$, we define

$$\tilde{\varphi} \colon \Gamma g \mapsto \Gamma\varphi(g) \tag{3.26}$$

to be the *infra-nilamanifold endomorphism* induced by φ. In the case $\varphi(\Gamma) = \Gamma$, $\tilde{\varphi}$ is called an infra-nilmanifold automorphism.

This gives us a more general algebraic approach to constructing Anosov diffeomorphisms since all hyperbolic infra-nilmanifold automorphisms are Anosov diffeomorphisms though this doesn't completely solve the classification problem posed by Smale.

As they are constructed above, all hyperbolic infra-nilmanifold automorphisms are Anosov diffeomorphism. The converse of this turns out to be false; it is known that there are Anosov diffeomorphisms which are *not* conjugate to one of these infra-nilmanifold automorphisms.

Despite this complication, we still have a result about the existence of such maps on infra-nilmanifolds.

Theorem 3.27. *A infra-nilmanifold M admits an Anosov diffeomorphism if and only if it admits a hyperbolic infra-nilmanifold automorphism.*

Proof. Suppose $\tilde{f}$ is Anosov on an infra-nilmanifold $M = \Gamma \backslash G$. We can see that $\tilde{f}$ induces a hyperbolic automorphism f of $\pi_1(M, y) \to \pi_1(M, f(y))$ which are both isomorphic to Γ. Since $f\colon \Gamma \to \Gamma$ is an automorphism of Γ, we can find ψ such that f is just a conjugation in $\mathrm{Aff}(G)$, i.e.

$$f(\gamma) = (1, \psi)\gamma(1, \psi)^{-1}. \tag{3.28}$$

From this, we get a conjugation automorphism $\Phi\colon G \rtimes F \to G \rtimes F$ defined by $\Phi\colon x \mapsto (1, \psi)x(1, \psi)^{-1}$. One can check that $\Phi(\Gamma) = \Gamma$ and $\Phi(F) = F$, giving us that Φ induces a hyperbolic infra-nilmanifold automorphism of $\Gamma \backslash G$. □

Next, we construct an Anosov diffeomorphism which is *not* topologically conjugate to an infra-nilmanifold automorphism.

Example 3.29. [4] Let the holonomy group F be $\mathbb{Z}_2$, represented as subgroup $\{I_4, L_f\}$ where I_4 is the 4×4 identity matrix and

$$L_f = \begin{pmatrix} -I_2 & 0 \\ 0 & I_2 \end{pmatrix}.$$

We let Γ be the torsion-free, discrete and uniform subgroup of $\mathbb{R}^4 \rtimes \mathbb{Z}_2$ generated by the following elements:

$$a = (e_1, I_4), b = (e_2, I_4), c = (e_3, I_4), d = (e_4, I_4), f = \left(\frac{e_3 + e_4}{2}, L_f\right)$$

where the e_i are the standard basis elements of $\mathbb{R}^4$. Letting $\alpha\colon \mathbb{R}^4 \to \mathbb{R}^4$ be defined by

$$\alpha = \left(\frac{e_3 + e_4}{2}, \begin{pmatrix} 13 & 8 & 0 & 0 \\ 8 & 5 & 0 & 0 \\ 0 & 0 & 13 & 8 \\ 0 & 0 & 8 & 5 \end{pmatrix}\right) \in \mathrm{Aff}(\mathbb{R}^4),$$

we can compute the following relations:

$$\alpha a \alpha^{-1} = a^{13}b^8, \alpha b \alpha^{-1} = a^8 b^5, \alpha c \alpha^{-1} = c13d^5, \alpha d \alpha^{-1} = c^8 d^5, \alpha f \alpha^{-1} = abc^{10}d^6 f.$$

This allows us to see that $\alpha\Gamma\alpha^{-1} = \Gamma$ and therefore α induces a diffeomorphism $\tilde{\alpha}$ on $\Gamma\backslash\mathbb{R}^4$. We will give a name to diffeomorphisms constructed in this way in Definition 3.31. Since none of the eigenvalues of the linear part of α have eigenvalues with modulus one, we see that $\tilde{\alpha}$ is indeed an Anosov diffeomorphism. We claim that this $\tilde{\alpha}$ is not topologically conjugate to an infra-nilmanifold automorphism of $\Gamma\backslash\mathbb{R}^4$.

Suppose there was such a $\tilde{\varphi}$ (induced by $\varphi\colon \mathbb{R}^4 \to \mathbb{R}^4$). It follows that $\varphi\Gamma\varphi^{-1} = \Gamma$, $\varphi\mathbb{Z}_2\varphi^{-1} = \mathbb{Z}_2$, $\varphi\mathbb{Z}^4\varphi^{-1} = \mathbb{Z}^4$, so we can write

$$\varphi = \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}, \text{where } A, B \in \mathbf{GL}_2(\mathbb{Z}). \tag{3.30}$$

This form implies that $\varphi f \varphi^{-1} = c^k d^l f$ for some $k, l \in \mathbb{Z}$. Likewise, the existence of a conjugacy h leads to the following statement about induced maps of the fundamental group Γ of $\Gamma\backslash\mathbb{R}^4$ i.e. $\alpha_\sharp = h_\sharp^{-1} \circ \varphi_\sharp \circ h_\sharp$. Modding out by the normal subgroup $Z(\Gamma)[\Gamma,\Gamma]$ of Γ, we see that $\alpha_\sharp, h_\sharp, \varphi_\sharp$ induce maps on $\mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2$. Considering theses as matrices M_α, M_φ, M_h, leads to a contradiction since the equality $M_\alpha = M_h^{-1} M_\varphi M)h$ demands that M_α be the identity.

This shows that Conjecture 3.18 is in fact false, but there is hope if one considers the more general affine endomorphisms.

Definition 3.31. [4] If $\alpha \in \mathrm{Aff}(G)$ is such that $\alpha\Gamma\alpha^{-1} = \Gamma$, then α induces a map

$$\tilde{\alpha}\colon \Gamma g \mapsto \Gamma\alpha(g) \tag{3.32}$$

called the *affine automorphism induced by* $\tilde{\alpha}$.

Remark 3.33. The $\tilde{\alpha}$ constructed in Example 3.29 is an example of an affine infra-nilmanifold automorphism. Furthermore, Theorem 3.27 also holds for hyperbolic affine infra-nilmanifolds automorphisms. See [4] for the revised statement and proof of this.

With this idea of affine infra-nilmanifold automorphisms in mind, we have a 'revised' version of Conjecture 3.18.

Conjecture 3.34. *Every Anosov diffeomorphism is conjugate to an affine infra-nilmanifold automorphism.*

Instead of approaching the still open question of whether all Anosov diffeomorphisms on an infra-nilmanifold are topologically conjugate to some affine automorphism of that infra-nilmanifold (as posed in [4]), we seek to answer the following:

Question 3.35. *When is an Anosov diffeomorphism topologically conjugate to an infra-nilmanifold automorphism?*

3.4. **Polynomial Global Product Structure.** One approach to answering Question 3.35 will involve establishing *Global Product Structure*, which has been shown to be a key step in proving some of the results towards answering Conjecture 3.34.

First, we need to define the notion of a foliation.

Definition 3.36. Suppose M is a smooth manifold. A *foliation* $\mathcal{F}$ of M is given by an open cover $\{\mathcal{F}(x)\}_{x\in M}$ such that each $\mathcal{F}(x)$ is a smooth, immersed submanifold of M with a fixed dimension. We call the elements of a foliation $\mathcal{F}$ *leaves*.

For an Anosov diffeomorphism, we can define the following sets

$$\mathcal{F}^s = \{W^s(x) \mid x \in M\}, \text{ and } \mathcal{F}^u = \{W^u(x) \mid x \in M\}. \tag{3.37}$$

By Theorem 2.12, the splitting of the tangent bundle is continuous. Likewise, we know each (un)stable manifold $W^s(x)(W^u(x))$ is an immersed submanifold of M meaning that $\mathcal{F}^s$ and $\mathcal{F}^u$ are foliations called the *stable* and *unstable* foliations respectively. Likewise, we will refer to elements of $\mathcal{F}^s$ and $\mathcal{F}^u$ as stable and unstable leaves.

We now move on to the definition of global product structure.

Definition 3.38 (Global Product Structure)**.** Foliations $\mathcal{F}$ and $\mathcal{G}$ have *Global Product Structure* if the intersection of each pair of leaves $(\mathcal{F}(x), \mathcal{G}(y)) \in \mathcal{F} \times \mathcal{G}$ consists of a unique point which we call $[x, y]$. We say that an Anosov diffeomorphism $f\colon M \to M$ has global product structure if its unstable and stable foliations have global product structure on $\widetilde{M}$, the universal cover of M.

It's currently unknown whether or not Global Product Structure is a strong enough condition to give a classification for Anosov diffeomorphisms. Instead, we restrict our attention to a stronger condition, *polynomial Global Product Structure.*

Definition 3.39. Foliations $\mathcal{F}$ and $\mathcal{G}$ have *polynomial Global Product Structure* if there is a polynomial P such that every $x, y \in M$,

$$d_{\mathcal{F}}(x, [x, y]) + d_{\mathcal{G}}(y, [x, y]) < P(d(x, y)), \tag{3.40}$$

where $d_{\mathcal{F}}$ and $d_{\mathcal{G}}$ are distances across leaves of the respective foliation and d is the metric on M.

Our goal is to provide a pathway to understanding the following result of Hammerlindl.

Theorem 3.41. [5] *An Anosov diffeomorphism is topologically conjugate to an infra-nilmanifold automorphism if and only if it has polynomial growth structure.*

This result gives us an answer to Question 3.35.

In order to understand where this result comes from, we'll need to define two more notions related to Anosov diffeomorphisms and their foliations: *polynomial bounds on rectangles* and *polynomial growth of volume.*

Definition 3.42 (Polynomial Bounds on Rectangles)**.** Given a pair of foliations $(\mathcal{F}, \mathcal{G})$, a R-rectangle is a continuous map $\varphi\colon [0,1]^2 \to M$ such that $\varphi(s,t)$ is in some leaf of $\mathcal{F}$ for every t and is in some leaf of $\mathcal{G}$ for every s and

$$\sup \{d_{\mathcal{G}}(\varphi(s, t_1), \varphi(s, t_2))\}, \tag{3.43}$$

i.e. the maximum possible distance between points in your rectangle across all leaves of $\mathcal{G}$ is bounded by R. We say that $(\mathcal{F}, \mathcal{G})$ has *polynomial bounds on rectangles* if there is P such that for every R-rectangle φ,

$$d_{\mathcal{F}}(\varphi(0,0), \varphi(1,0)) < 1 \implies d_{\mathcal{F}}(\varphi(0,1), \varphi(1,1)) < P(R) \tag{3.44}$$

i.e. the side lengths along $\mathcal{F}$ of your rectangle grow at a rate that is bounded by a polynomial when moving along leaves in $\mathcal{G}$. Notice that this definition is *dependent* on the order of the foliations. We write that $\mathcal{F}$ and $\mathcal{G}$ have polynomial bounds on rectangles when both ordered pairs $(\mathcal{F}, \mathcal{G})$ and $(\mathcal{G}, \mathcal{F})$ do.

Definition 3.45 (Polynomial Growth of Volume)**.** A Riemannian manifold M has *polynomial growth of volume* if there is some polynomial P such that the volume of every open ball in M is bounded by that polynomial evaluated at its radius, that is

$$\mathrm{Vol}(B_r(x)) < P(r) \text{ for every } x \in M \text{ and } r > 0. \tag{3.46}$$

We also say that a foliation $\mathcal{F}$ has polynomial growth of volume if there is a polynomial P such that

$$\mathrm{Vol}(\mathcal{F}_r(x)) < P(r) \text{ for every } x \in M \text{ and } r > 0, \tag{3.47}$$

where $\mathcal{F}_r(x)$ denotes the set of points reachable by a path in $\mathcal{F}(x)$ with length less than r.

With the previous definitions in mind, the following lemma is known for Anosov diffeomorphisms.

Lemma 3.48. [5, Lemmas 2.1, 2.2] *The stable and unstable foliations of an Anosov diffeomorphism have polynomial bounds on rectangles and polynomial growth of volume when lifted to the universal cover.*

We also have a proposition relating the previous polynomial notions of foliations with the polynomial growth of universal covers.

Proposition 3.49. [5, Proposition 2.3] *Let $\mathcal{F}, \mathcal{G}$ be continuous foliations of a compact Riemannian manifold M with C^1 leaves such that $T\mathcal{F}$ and $T\mathcal{G}$ are continuous subbundles of TM. If the foliations have polynomial global product structure, polynomial bounds on rectangles, and polynomial growth of volume when lifted to the universal cover $\tilde{M}$, then the universal cover has polynomial growth of volume.*

Together, the previous two results give us the following:

Corollary 3.50. *All compact Riemannian manifolds admitting Anosov diffeomorphisms with polynomial global product structure have polynomial growth of volume on their universal covers.*

Theorem 3.51. [2] *If an Anosov diffeomorphism of M has Global Product Structure and the universal cover of M has polynomial growth of volume, then f is topologically conjugate to an infra-nilmanifold automorphism.*

From these two results, we have one direction of Theorem 3.41. The other direction is a matter of Lie group theory and will not be discussed here. The important takeaway of this is that we have another method of exhibiting that an Anosov diffeomorphism is on an infra-nilmanifold by the means of establishing polynomial Global Product Structure. Furthermore, foliations and product structure are indispensable tools in the *classification* of Anosov diffeomorphism and manifolds admitting them as we will see in the next section.

3.5. **Franks-Newhouse Theorem.**

Using some of the concepts shown earlier, we provide a proof of a classification result for Anosov diffeomorphisms of *co-dimension one*, that is for Anosov diffeomorphisms such that either E^s or E^u has dimension 1.

Theorem 3.52 (Franks)**.** [7] *If f is an Anosov diffeomorphism of a compact Riemannian manifold M with co-dimension one and $\Omega(f) = M$, then f is topologically conjugate to a hyperbolic toral automorphism.*

This result forms a natural pair with the following theorem of Newhouse.

Theorem 3.53 (Newhouse)**.** [7] *If an Anosov diffeomorphism $f\colon M \to M$ is of co-dimension one, then $\Omega(f) = M$.*

Together these result gives a partial answer to Smale's question by saying that *Anosov diffeomorphisms of co-dimension one are precisely hyperbolic toral automorphisms.*

Theorem 3.54 (Franks-Newhouse Theorem)**.** [7] *If f is an Anosov diffeomorphism of co-dimension one on a compact Riemannian manifold M, then f is topologically conjugate to a hyperbolic toral automorphism.*

The proof of this relies of the following lemma.

Lemma 3.55. [7] *If $\dim \mathcal{F}^u = 1$ and $\Omega(f) = M$, then there is a collection of measures $\mu = \{\mu_I \mid I \text{ is an arc in } \mathcal{F}^u\}$ such that*

(1) $\mu_{J|I} = \mu_I$ if $I \subset J$,
(2) $\mu_J \circ \varphi = \mu_I$ if $\varphi : I \to J$ is a projection along $\mathcal{F}^s$,
(3) each μ_I is non-atomic and positive on non-empty open sets,
(4) and $\mu_I(I) = \infty$ if and only if I is unbounded.

The collection of measures μ is called a transverse invariant measure for $\mathcal{F}^s$

Proof of Theorem 3.52. [7] Suppose $\mathcal{F}^u$ is orientable and $\dim \mathcal{F}^u = 1$. Otherwise, if $\dim \mathcal{F}^s = 1$, take f^{-1} and the same argument holds. We know that f has a periodic point for some n, therefore f^n has a fixed point and has some non-zero index at that point. If we notice that this index does not vanish under homotopy, then we have that f has a fixed point p.

By Lemma 3.55, we have a transverse invariant measure $\mu = \{\mu_I\}$ for $\mathcal{F}^s$. Consider $(x, y) \in W^u(p) \times W^s(p)$ where p is a fixed point as above. If we use $[s, t]$, to denote the arc in $W^u(s)$ with end points s and t, we see that there is exactly one point $z \in W^u(y)$ such that

$$\mu([p, x]) = \mu([y, z]) \tag{3.56}$$

and the orientation from p to x is the same as that from y to z. This idea is pictured in Fig. 5. Due to the uniqueness of z, we can define a map $\pi_p\colon W^u(p)\times W^s(p) \to M$ by $\pi_p\colon (x, y) \mapsto z$. We see that since $[y, \pi_p(x, y)]$ is completely contained in the unstable manifold of $W^u(y))$ and μ is preserved by projecting along stable leaves, the map π_p is continuous. Also, $\pi_p(x, y) \in W^s(x) \cap W^u(y)$ for every $(x, y) \in W^u(p) \times W^s(p)$.

We define $\tilde{M} \coloneqq W^u(p) \times W^s(p)$ which is isomorphic to $\mathbb{R} \times \mathbb{R}^{\dim M-1}$. Taking a product neighborhood N,that is a neighborhood that is foliated by unstable and stable manifolds, we see that N must contain a point in M that is evenly covered by π_p. This allows us to see that $\pi_p\colon \tilde{M} \to M$ is the universal covering of M.

Consider the lift of f by π_p to the universal cover $\tilde{M}$ defined by $F = f_{|W^u(p)} \times f_{|W^s(p)}$. In this case, we want to think of the fundamental group of M, $\pi_1(M)$, as the group of deck transformations of M. Given such a transformation $\alpha \in \pi_1(M)$ and $x \in W^u(p)$, let $\overline{\alpha(x)} \in W^u(p)$ be the first component of $\alpha(x, p)$. We can think of this as projecting arcs in $W^u(p)$ along $\mathcal{F}^s$. Therefore, for any arc I in $W^u(p)$,

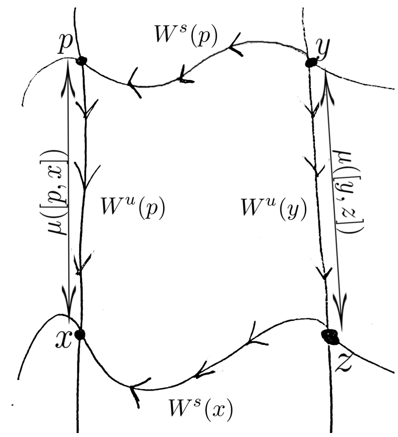


FIGURE 5. Illustrating the definition of $z = \pi_p(x, y)$.

$\mu(I) = \mu(\bar{\alpha}(I))$. This allows us to think of the set of maps $\bar{\alpha}$ induced by $\alpha \in \pi_1(M)$ as a free abelian group, G.

With a little bit of work (restricting π_p to some stable manifold), one can see that the identity of the group G corresponds to the identity transformation in $\pi_1(M)$, and we get the following are isomorphic:

$$\pi_1(M) \cong \{\bar{\alpha} \mid \alpha \in \pi_1(M)\} \cong \mathbb{Z}^m, \text{ for some } m. \tag{3.57}$$

We know that F induces an isomorphism F_* on the fundamental group of M and that $F \circ \alpha = F_*(\alpha) \circ F$ for each $\alpha \in \pi_1(M)$. This gives us that $f \circ \bar{\alpha} = \overline{F_*(\alpha)} \circ f$. Using the fact that the set of $\overline{\alpha(p)}$ is dense in $W^u(p)$, we can think of $f \colon W^u(p) \to W^u(p)$ as a linear map.

Let $\varphi \colon \mathbb{Z}^m \to \pi_1(M)$ be an isomorphism. We can then define a map A from the lattice $\mathbb{Z}^m$ to itself by $A = \varphi^{-1} \circ F_* \circ \varphi$, which induces a toral automorphism $\tilde{A}$. This $\tilde{A}$ is a hyperbolic toral automorphism that is conjugate to f.

To construct conjugacy, we first construct a bijection $h \colon \alpha \mapsto \varphi(\alpha)(p, p)$ from $\mathbb{Z}^m \cong \pi_1(M)$ to $\pi_p^{-1}(p) \subset \tilde{M}$, giving us that

$$F_{|\pi_p^{-1}(p)} \circ h = h \circ A_{|\mathbb{Z}^m}. \tag{3.58}$$

This map h can be extended continuously to a map $H \colon \mathbb{R}^m \to \tilde{M}$ such that $F \circ H = H \circ A$. Applying the above argument and replacing A with F gives us that H is invertible. Therefore we get that H is a homeomorphism and moreover, it projects to a topological conjugacy between f and $\tilde{A}$.

In the case that $\mathcal{F}^u$ is not orientable, choose a orientation covering $f^* \colon M^* \to M^*$ such that $\mathcal{F}^*$ of f^* is orientable. We see that f^* is topologically conjugate to a hyperbolic toral automorphism by the above proof. Therefore, we can find a deck transformation α such that $\bar{\alpha}$ reverses orientation of a leaf W^u. This is an orientation-reversing transformation of $\mathbb{R}$ and has a fixed point. This is a contradiction, meaning $\mathcal{F}^u$ is orientable, and f is topologically conjugate to a hyperbolic toral automorphism. $\square$

The classification of Anosov diffeomorphisms and compact manifolds admitting them is still an open question, but the affine infra-nilmanifold automorphisms defined in Definition 3.31 alongside techniques related to foliations and Global Product Structure are important methods to approaching this question.

Acknowledgements

There are several people to be thanked here. To start, I'd like to thank Peter May for organizing the 2018 REU, my mentor Jonathan DeWitt for his constant support as well as for introducing me to the classification problem discussed here, and my fellow participants for allowing me to learn, grow, and encounter new and awe-inducing mathematics alongside them.

References

[1] R. E. Bowen, J. R. Chazottes, and D. Ruelle, *Equilibrium States and the Ergodic Theory of Anosov Diffeomorphisms*, Lecture Notes in Mathematics, Springer Berlin Heidelberg, 2008.
[2] M Brin and A Manning, *Anosov diffeomorphisms with pinched spectrum*, Dynamical systems and turbulence, Warwick 1980, 1981, pp. 48–53.
[3] M. Brin and G. Stuck, *Introduction to Dynamical Systems*, Cambridge University Press, Cambridge, 2002.
[4] K. Dekimpe, *What an infra-nilmanifold endomorphism really should be...*, Topological Methods in Nonlinear Analysis **40** (2012), no. 1, 111–136.
[5] A. Hammerlindl, *Polynomial global product structure*, arXiv preprint arXiv:1203.5319 (2012).
[6] M. W Hirsch and C. C Pugh, *Stable manifolds and hyperbolic sets*, Global Analysis (Proc. Sympos. Pure Math., Vol. XIV, Berkeley, Calif., 1968) (1970), 133–163.
[7] K. Hiraide, *A simple proof of the Franks–Newhouse theorem on codimension-one Anosov diffeomorphisms*, Ergodic Theory and Dynamical Systems **21** (2001), no. 3, 801–806.
[8] S. Smale, *Differentiable dynamical systems*, Bulletin of the American mathematical Society **73** (1967), no. 6, 747–817.