DIFFERENTIAL FORMS AND THEIR APPLICATION TO MAXWELL'S EQUATIONS

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Abstract. This paper begins with a brief review of the Maxwell equations in their "differential form" (not to be confused with the Maxwell equations written using the language of differential forms, which we will derive in this paper). The reader is not expected to have any prior knowledge of the Maxwell equations as the purpose of this paper is not to understand the equations (one can take a physics course if he/she is interested in that), but to express them in a different sort of language than they are commonly seen in. It is expected that the reader already be familiar with the basics of manifolds, vector fields and differential forms but there will be a brief section of this paper dedicated to more precisely enumerating what ideas we will be taking as a given. Then we will cover how the basic operations in vector calculus can be expressed using differential forms. Then expand upon more in depth theory regarding differential forms such as the Hodge star and metrics. Finally we will have developed all the tools necessary to rewrite the Maxwell equations over a well known manifold called "spacetime".

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1. Maxwell's Equations

Maxwell's equations are a description of two vector fields, the electric field $\vec{E}$ and the magnetic field $\vec{B}$. These fields are defined throughout both space, normally taken to be $\mathbb{R}^3$ and are also a function of time. These fields depend on both the electric charge density ρ which is a real valued function on $\mathbb{R}^3$ and upon the current density j which is again a vector field on $\mathbb{R}^3$ that is time dependent. The equations are, in units where the speed of light is 1, given by

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$$\nabla \cdot \vec{B} = 0 \tag{1.1}$$

$$\nabla \times \vec{E} + \frac{\partial \vec{B}}{\partial \vec{t}} = 0 \tag{1.2}$$

$$\nabla \cdot \vec{E} = \rho \tag{1.3}$$

$$\nabla \times \vec{B} - \frac{\partial \vec{E}}{\partial \vec{t}} = \vec{j} \tag{1.4}$$

The first thing to notice about the equations is that they seem to come in two pairs. Equations (1.1) and (1.2), which from now will be referred to as the first pair, are not dependent on either the charge density ρ or the current density j. Additionally when we compare the first and second pair of equations the operations of curl, divergence and time derivative have been swapped around. More precisely, in the absence of ρ and j the first and second pair differ by the transformation

$$\vec{B} \to \vec{E} \tag{1.5}$$

$$\vec{E} \to -\vec{B} \tag{1.6}$$

which takes the first pair to the second pair and the second pair to the first pair and is referred to the duality of the Maxwell equations. However, as nice as this transformation is, it does not work when ρ and j are not both 0. The "niceness" of this duality is one of the primary motivating reasons for the rewriting that we will do in later sections as it will allow us to recover the duality between the first and second pair even in the presence of the charge and current density.

2. Basic Differential forms

There are multiple ways to define differential forms. One way which most of the audience is familiar with or will be familiar with is given in Rudin [2]. This method says that every k-form ω is of the form

$$\omega = \sum_I a_I dx_I \qquad \text{where } I \text{ is a k-index and } a_I \text{ is a smooth function} \tag{2.1}$$

and defines the meaning of the expression by giving the formula for how to integrate the expression. From this all the basic formulas relating the sum of forms, product etc are derived from this formula. Such an approach is perhaps the most simple and will be sufficient for 90% of the results in this paper. however there is a second definition of 1-forms as linear maps from vector fields to smooth functions. With K forms being elements of the exterior algebra. The only reason we will need this definition is that this allows us to turn a 1 form into a cotangent vector. In other words, a 1-form ω gives a cotangent vector ω_p which maps any vector field v on M to R via the formula

$$\omega_p(v_p) = w(v)(p) \tag{2.2}$$

However, no matter what way we choose to define differential forms there are several basic results that are the same no matter the approach. We will not prove these basic results as they appear in every text covering basic differential geometry. For example both [1] and [2] have sections devoted to proving these statements. So here is the list of these basic results.

Lemma 2.3. *all forms can be written in what is called an increasing k index (if ω is a k-form)*

$$\omega = \sum_I a_I dx_I \qquad \textit{where } I \textit{ is an increasing k-index}$$

and

$$dx_I = dx_{i_1} \wedge \cdots \wedge dx_{i_k}$$

Lemma 2.4. *the wedge product is anti-commutative*

$$dx \wedge dy = -dy \wedge dx$$

Definition 2.5. The exterior derivative d of a 0-form or smooth function is

$$df = \sum_{i=1}^{n} \partial_i f dx_i$$

Definition 2.6. the exterior derivative d of a k-form ω is

$$d\omega = \sum_I da_I \wedge dx_I$$

Lemma 2.7. *Taking the exterior of a differential form twice will always give 0*

$$d(d\omega) = d^2\omega = 0$$

Definition 2.8. the wedge product between a k-form ω and an l-form λ is a (k+l)-form given by

$$\omega \wedge \lambda = \sum_I \sum_J a_I b_J dx_I \wedge dx_J$$

Lemma 2.9. *The "product rule" for differential forms is given by*

$$d(\omega \wedge \lambda) = d\omega \wedge \lambda + (-1)^l \omega \wedge d\lambda$$

where ω is an l-form

3. Divergence, curl, and gradient

This section covers how the divergence, curl, and gradient are each a form of the exterior derivative.

In $\mathbb{R}^n$ there is a natural correspondence between 1 forms and vector fields since they both have the same number of components. There exists a canonical mapping taking

$$v = \sum_{\mu=1}^{n} v^\mu \partial_\mu \longrightarrow \omega = \sum_{\mu=1}^{n} v^\mu dx_\mu \tag{3.1}$$

Using this mapping and its inverse we can treat vector fields as 1-forms and vice-versa. We then see that the formula for the exterior derivative of a 0-form f (smooth function) is really just the gradient ∇f after we convert the 1-form df into a vector field using the inverse of (3.1).

$$df = \sum_{i=1}^{n} \partial_i f dx_i \longrightarrow \sum_{i=1}^{n} \partial_i f \partial_i = \nabla f$$

While the gradient takes a smooth function and returns a vector field or a 1-form, the divergence and curl each take a vector field and return either a smooth function or another vector field respectively. This poses a problem for us as the exterior derivative of a 1-form is a 2-form and not a smooth function or another vector field. To remedy this we note that an n-1-form also has n components each corresponding to the exclusion of a dx_i from the wedge product. We denote an operation $\star$ which exchanges between 1-forms and (n-1)-forms given by

Definition 3.2.

$$\star a_i dx_i = \text{sign}(i, I) a_i dx_I \qquad \text{with } I = 1, 2, \ldots, i-1, i+1, \ldots n$$

and

$$\star a_i dx_I = \text{sign}(I, i) a_i dx_i \qquad \text{when } I = 1, 2, \ldots, i-1, i+1, \ldots n$$

Where $\text{sign}(i, I)$ is the sign of the permutation starting with i then taking the elements of I in order.

We will cover the $\star$ operator in more detail in a later section as this definition only holds for $\mathbb{R}^n$ using the standard metric but for now we will be explicit and take $n = 3$ so that the curl can be defined as usual.

Definition 3.3. The $\star$ operator on $\mathbb{R}^3$ under the standard metric is given by

$$\begin{aligned} \star dx &= dy \wedge dz & \star\, dy \wedge dz &= dx \\ \star dy &= dz \wedge dx & \star\, dz \wedge dx &= dy \\ \star dz &= dx \wedge dy & \star\, dx \wedge dy &= dz \end{aligned}$$

We should note now that the choice of sign on the right hand side is really arbitrary, and the choice of sign in (3.3) corresponds to what is commonly called the right hand rule. We could just as easily have the $\star$ operator given by

Definition 3.4.

$$\begin{aligned} \star dx &= dz \wedge dy & \star\, dy \wedge dz &= -dx \\ \star dy &= dx \wedge dz & \star\, dz \wedge dx &= -dy \\ \star dz &= dy \wedge dx & \star\, dx \wedge dy &= -dz \end{aligned}$$

Which gives a left handed star operator, but we will stick to convention and use the right handed $\star$ in (3.2)

Finally we will extend this operator to also switch between the 0-forms and 3-forms as each has only 1 component.

Definition 3.5. In the case of $n = 3$ the right handed star operator is

$$\star f = f dx \wedge dy \wedge dz \qquad \star\, f dx \wedge dy \wedge dz = f$$

Now we can guess at what the formulas for the curl and divergence are and then verify them by explicit calculation. Like the $\star$ operator, the curl requires a choice of a righthand rule but the divergence does not. This implies that the formula for the divergence should have an even number of $\star$ operators so that the sign difference given by $\star$ will cancel out. After fiddling around a bit one can see that the simplest expression returning a 1-form is $\star d \star w$. We claim that this expression is indeed the divergence.

Theorem 3.6.

$$\star d \star \omega \iff \nabla \cdot \vec{\omega}$$

Proof. We will verify this for $\mathbb{R}^n$ by applying definition (3.2). Let ω be a 1-form given by

$$\omega = \sum_{i=1}^{n} a_i dx_i$$

Then we have

$$\star\omega = \sum_{i=1}^{n} a_i dx_I \text{sign}(i, I) \qquad \text{with } I = 1, \dots, i-1, i+1, \dots n$$

Taking the derivative gives

$$d\star\omega = d\sum_{i=1}^{n} a_i dx_I \text{sign}(i, I) = \sum_{i=1}^{n} \partial_i a_i dx_1 \wedge dx_2 \wedge \cdots \wedge dx_n \text{sign}(i, I)$$

and finally

$$\star d\star\omega = \sum_{i=1}^{n} \partial_i a_i \text{sign}(i, I)^2 = \sum_{i=1}^{n} \partial_i a_i$$

Which is the formula for the divergence. As expected the signs from each $\star$ operation did indeed end up cancelling out. □

Following a similar reasoning as before we conclude that we want an odd number of $\star$ operators for the curl in $\mathbb{R}^3$. The simplest expression returning a 1-form is $\star d\omega$. We claim that this expression is equivalent to the right handed curl..

Theorem 3.7.

$$\star d\omega \iff \nabla \times \vec{\omega}$$

Proof. We begin by computing $d\omega$

$$d\omega = \sum_{i=1}^{3} da_i \wedge dx_i$$

Expanding this out gives

$$d\omega = [(\partial_y a_3 - \partial_z a_2) dy \wedge dz + (\partial_z a_1 - \partial_x a_3) dz \wedge dx + (\partial_x a_y - \partial_y a_1) dx \wedge dy]$$

then applying $\star$

$$\star d\omega = (\partial_y a_3 - \partial_z a_2) dx + (\partial_z a_1 - \partial_x a_3) dy + (\partial_x a_y - \partial_y a_1) dz$$

Which is exactly the formula for the right handed curl. □

This unification of the divergence, gradient, and curl into the language of differential forms simplifies the proofs of several notable theorems in vector calculus. Notably

Theorem 3.8.

$$\nabla \times (\nabla f) = 0$$

Theorem 3.9.

$$\nabla \cdot (\nabla \times \vec{F}) = 0$$

Proof. For theorem 3.8 we can write the LHS as

$$\star ddf$$

Then $ddf = 0$ by lemma 2.7 and $\star 0 = 0$

For theorem 3.9 we have the LHS

$$\star d \star \star d\omega = \star dd\omega = 0$$

So we are done. Which greatly simplifies two proofs which would have other wise required more tedious computations. □

Previously in this section we have treated vector fields as 1-forms in $\mathbb{R}^3$ but we could just as well have taken them to be 2-forms via the mapping

$$v = \sum_{\mu=1}^{3} v^{\mu}\partial_{\mu} \longrightarrow \omega = v^1 dy \wedge dz + v^2 dz \wedge dx + v^3 dx \wedge dy \tag{3.10}$$

We will save the details as to why one might choose one representation or the other for the next section and quickly explain how to take the divergence and curl of this two form.

First we note when we take ω to be a 2-form ω is identical to $\star\omega$ when ω is a1-form. So we can quickly deduce each formula by replacing ω with $\star\omega$.

The divergence of a 2-form is.

$$\star d \star \star\omega = \star d\omega \tag{3.11}$$

The curl of a two form is

$$\star d \star \omega \tag{3.12}$$

One can immediately notice that there is a symmetry present in these formulas. The formula for the curl of a 1-form is the same as the formula for the divergence of a 2-form, and the formula for the divergence of a 1-form is the formula for the curl of a 2-form. This is an important symmetry which we will exploit more fully in the following sections.

4. Rewriting the First Pair of Equations

Our next step is to rewrite the first two Maxwell equations in the language of differential forms. We will first consider the static case so that we do not have to deal with the time derivative at first and can remain safely in $\mathbb{R}^3$. Our first step is to decide whether or not the $\vec{E}$ field and $\vec{B}$ field should be 1-forms or 2-forms. To do this we will use what is known as a parity check.

Definition 4.1. A parity check asks what happens to a vector field when all the coordinate axes are flipped to their negatives. We will denote a parity check on a vector field $\vec{F}$ by

$$P(\vec{F}) = \pm\vec{F}$$

if

$$P(\vec{F}) = -\vec{F}$$

We say that $\vec{F}$ has odd parity. If

$$P(\vec{F}) = \vec{F}$$

We say that $\vec{F}$ has even parity.

We will now perform parity checks on the $\vec{E}$ field and the $\vec{B}$ field. In order to perform the parity check on the $\vec{E}$ field and $\vec{B}$ field we consider the Lorentz equation giving the Force on charged particle

$$\vec{F} = q(\vec{E} + \vec{v} \times \vec{B}) \tag{4.2}$$

We then apply a principle of physics which states that the laws of physics, including (4.2), ought to hold independently of our choice of coordinate basis. In other words if we apply a parity check to both sides of (4.2) the equation ought to still hold.

The force and velocity are both vector fields where flipping the coordinate bases does not change the vector fields themselves. This means that both $\vec{F}$ and $\vec{v}$ have odd parity.

$$P(\vec{F}) = -\vec{F} \tag{4.3}$$

$$P(\vec{v}) = -\vec{v} \tag{4.4}$$

We can then apply P to both sides of (4.2) to get.

$$P(\vec{F}) = P(q(\vec{E} + \vec{v} \times \vec{B})) \tag{4.5}$$

$$-\vec{F} = qP(\vec{E}) + qP(\vec{v} \times \vec{B}) \tag{4.6}$$

$$-\vec{F} = q(P(\vec{E}) - \vec{v}P(\vec{B})) \tag{4.7}$$

Now, in order for (4.2) to hold we must have

$$P(\vec{E}) = -\vec{E} \tag{4.8}$$

$$P(\vec{B}) = \vec{B} \tag{4.9}$$

Thus the $\vec{E}$ field has odd parity and the $\vec{B}$ field has even parity. now we will examine the parity of 1 and 2-forms.

Theorem 4.10. *1-forms in $\mathbb{R}^n$ have odd parity and 2-forms have even parity*

Proof. Under a parity switch each dx_i is sent to $-dx_i$. This makes it clear then that under a parity switch a 1-form is sent to its negative and a 2-form is sent to itself as the negatives will cancel out. $\square$

With this in mind we will treat the $\vec{E}$ field as a 1-form and the $\vec{B}$ field as a 2-form. Then the first two equations in the static case become

$$dB = 0 \tag{4.11}$$

$$dE = 0 \tag{4.12}$$

This still leaves the problem of the non static case to deal with. In order to throw time derivatives into the mix we will have to leave the familiar comfort of $\mathbb{R}^3$ and consider our differential forms over spacetime.

Definition 4.13. Spacetime is a manifold that in many ways will look like $\mathbb{R}^4$. We will denote the time component as dt if we are using dx, dy, dz to denote the spatial coordinates or by dx_0 if we are using dx_1, dx_2, dx_3

Over spacetime we will still take the $\vec{E}$ field and $\vec{B}$ field to be 1 and 2 forms with the same components as before, and with zeroes in the components with dt. We will now introduce a unification of the $\vec{E}$ and $\vec{B}$ fields called the electromagnetic field denoted by F

Definition 4.14.

$$F = B + E \wedge dt$$

The advantage to doing this is that we can now write the first two equations as just one.

Theorem 4.15. *The first two Maxwell equations are equivalent to* $dF = 0$

Proof. First we use the property that the derivative is linear under addition

$$dF = dB + d(E \wedge dt)$$

Then we apply lemma 2.9 to the second term to get

$$dF = dB + dE \wedge dt$$

Then we can split up the exterior derivative into a "spacelike" and "timelike" component via

$$d\omega = d_s\omega + \partial_t a_I dt \wedge dx_I$$

where

$$d_s\omega = \sum_{i=1}^{3} \partial_i a_I \wedge dx_I$$

Then (4.15) becomes

$$\begin{aligned} dF &= d_sB + dt \wedge \partial_t B + (dsE + dt \wedge \partial_t E) \wedge dt \\ &= d_sB + (dsE + \partial_t B) \wedge dt \end{aligned}$$

Since the first term has no dt term in it and the second term does, their sum is 0 iff each term is 0 individually. and the first second term being 0 are exactly the first and second Maxwell equations respectively.

□

5. Metrics and The Hodge star operator

In this section we will expand definition of the $\star$ operator given in (4.2) and (4.3) to Spacetime. The main difference between $\mathbb{R}^4$ and spacetime is the metric on spacetime is the Minkowski metric.

Definition 5.1. The Minkowski metric η measuring the distance between two vectors in units where the speed of light is 1 is

$$\eta(v, w) = -v_0w_0 + v_1w_1 + v_2w_2 + v_3w_3$$

Definition 5.2. In general we denote a metric in a vector space by g and require the following conditions

$$\begin{aligned} g &: V \times V \to R \\ g(cv + v', w) &= cg(v, w) + g(v', w) \\ g(v, cw + w') &= cg(v, w) + g(v, w') \\ g(v, w) &= g(w, v) \end{aligned}$$

These conditions also known as Bilinear and symmetric. Additionally a metric must satisfy a non-degeneracy condition which says that if $g(v, w) = 0$ for all $w \in M$ then $v = 0$.

This gives a new way of converting between 1-forms and vector fields on a Manifold called raising and lowering indices. To do this we assign a matrix $g_{\mu\nu}$ to each metric g via

Definition 5.3.

$$g_{\mu\nu} = g(e_\mu, e_\nu)$$

Where each e_i is the i-th unit basis vector

Likewise there exists a similar matrix $g^{\mu\nu}$ given by

Definition 5.4.

$$g^{\mu\nu} = g^{-1}_{\mu\nu}$$

Which we know exists due to the non-degeneracy condition of the metric

We can do the calculation for η explicitly to get

$$\eta_{\mu\nu} = \eta^{\mu\nu} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{5.5}$$

Now in order to find a way to convert a vector field into a 1-form we can recall the definition of 1-forms as elements of the cotangent space. That is, if f is a smooth function we can define the differential df as the map

$$df(v) = v(f) \tag{5.6}$$

Then for a given metric g and vector field v there is a corresponding 1-form $g(v, \cdot)$ where the $\cdot$ is the function input. However, this is not the way of writing 1-forms that we have become familiar with, which is where the matrices $g_{\mu\nu}$ and $g^{\mu\nu}$ come in.

Theorem 5.7. *we can compute the components of $g(v, \cdot)$ via*

$$g(v, \cdot) = \sum_{\mu=1}^{n} \sum_{\nu=1}^{n} g_{\mu\nu} v^\mu dx_\nu$$

Proof. We will prove this theorem holds by showing that it is true for the basis vectors and then applying linearity. Since each dx_ν is the dual to the basis vector e_ν if we feed the right hand side a basis vector e_i it spits out the coefficient

$$\sum_{\mu=1}^{n} g(e_\mu, e_i) v^\mu$$

Which is equal to $g(v, e_i)$ □

To find a vector field from a 1-form we use the inverse

Theorem 5.8. *the vector field corresponding to a 1-form ω is given by*

$$v = \sum_{\mu=1}^{n} \sum_{\nu=1}^{n} g^{\mu\nu} \omega_\mu e_\nu$$

Proof. we will omit the proof of this theorem as it is identical in every way to theorem 5.7 ☐

Now we will show how to turn this metric, which is much like an inner product between vector fields, into something we can regard as an inner product between 1-forms. For this definition note that we can rewrite $g(v, w)$ as

$$g(v,w) = \sum_{\mu=1}^{n} \sum_{\nu=1}^{n} g_{\mu\nu} v^{\mu} w^{\nu} \tag{5.9}$$

With this in mind we can make the definition

Definition 5.10. The inner product between two 1-forms ω and λ denoted by $\langle \omega, \lambda \rangle$ is defined by

$$\langle \omega, \lambda \rangle = \sum_{\mu=1}^{n} \sum_{\nu=1}^{n} g^{\mu\nu} \omega_{\mu} \lambda_{\nu}$$

Which takes the components of the 1-forms and converts them into vector fields before using the metric g to take the inner product. The benefit of this approach is that this inner product inherits all the properties laid out in definition 5.2

We now have all the tools we need to give a proper definition of the $\star$ operator

Definition 5.11. The hodge star operator $\star$ is a map from p-forms to n-p forms on an n-dimensional manifold. Assume that we have an orthonormal basis of 1-forms. Meaning

$$\langle dx_i, dx_j \rangle = 0$$

if $i \neq j$, and

$$\langle dx_i, dx_i \rangle = \epsilon(i)$$

with $\epsilon = \pm 1$. Then for distinct $1 \leq i_1, \ldots, i_p \leq n$,

$$\star(dx_{i_1} \wedge dx_{i_2}, \ldots, \wedge dx_{i_p}) = \pm dx_{i_{p+1}} \wedge, \ldots, \wedge dx_{i_n}$$

Where $i_{p+1}, \ldots, i_n$ are the integers 1 to n not included in $i_1, \ldots, i_p$ and the $\pm$ is given by

$$\mathrm{sign}(i_1, \ldots, i_n) \epsilon(i_1) \ldots \epsilon(i_p)$$

6. Rewriting the Second Pair of Equations

We are now ready to rewrite the second pair of equations. Like before we will consider a simplified case which is static in time and is taken over $\mathbb{R}^3$. Then the equations become

$$\star d \star E = \rho \tag{6.1}$$

$$\star d \star B = \vec{j} \tag{6.2}$$

With the $\star$ operator defined explicitly over $\mathbb{R}^3$ in (3.3). Also, in the second equation we are abusing notation by having the LHS be a 1 form and the RHS a vector in $\mathbb{R}^3$ but we will remedy this soon. We now have 2 steps Remaining in fully rewriting these 2 equations. First we notice that like the first pair of equations, the LHS of the second pair has the same general form. This suggests that once again we can unify the pair and that if there is justice in the world the LHS of the equation ought to $\star d \star F$ for the same F we defined in section 4 and with the $\star$ operator we defined in section 5. Second as $\star d \star F$ gives a 1-form over spacetime, we will need to find a way to unify the scalar ρ with the vector $\vec{j}$ into a 1-form.

Definition 6.3. To unify ρ with $\vec{j}$ into a 1-form we will first define a vector field in $\mathbb{R}^4$ combining them by

$$\vec{J} = \rho \partial_t + j^1 \partial_1 + j^2 \partial_2 + j^3 \partial_3$$

Using the Minkowski metric and $\eta_{\mu\nu}$ calculated in 6.5 give a 1-form gives

$$J = -\rho dt + j^1 dx + j^2 dy + j^3 dz \tag{6.4}$$

Usually called the Current.

The last step is to perform the calculation of $\star d \star F$ and verify the equivalence of the two formulations

Theorem 6.5.

$$\star d \star F = J \iff \textit{equations 1.3 and 1.4}$$

Proof. Just as in (4.15) we split the exterior derivative into spacelike and timelike components, we do the same here for the $\star$ operator, splitting it into a spacelike operator $\star_s$ as well as a timelike component. In the language of this operator we are trying to extract the 2 equations

$$\star_s d_s \star_s E = \rho \tag{6.6}$$

$$-\partial_t E + \star_s d_s \star_s B = j \tag{6.7}$$

With this in mind let's compute $\star F$ in terms of the operator$\star_s$. First we use the linearity property to give

$$\star F = \star B + \star (E \wedge dt) \tag{6.8}$$

The key thing to note here is that every component of $E \wedge dt$ contains a dt so when we apply the $\star$ operator no component will have a dt. The converse is true of B, since no terms contain a dt all terms in $\star B$ will. If we now apply definition 5.11 and compute each term then factor each dt from $\star B$ we get

$$\star F = \star_s B \wedge dt + \star_s E \tag{6.9}$$

This is quite a fascinating result as we can now see that taking $\star F$ more or less has switched the roles of E and B, sending

$$B \longrightarrow \star_s E$$
$$E \longrightarrow - \star_s B$$

The computation of $d \star F$ goes identically to the calculation of dF in theorem 4.15 with the above changing of variables to yield

$$d \star F = \star_s \partial_t E \wedge dt + d_s \star_s E - d_s \star_s B \wedge dt \tag{6.10}$$

We then group the terms with a dt and without

$$d \star F = d_s \star_s E + (\star_s \partial_t E - d_s \star_s B) \wedge dt \tag{6.11}$$

Then once again when we take the $\star$ of both sides all terms with a dt will lose it and all terms without will gain it so the grouping stays identical and we get

$$\star d \star F = - \star_s d_s \star_s E \wedge dt + (\partial_t E - \star_s d_s \star_s B) \tag{6.12}$$

Then when we set $\star d \star F = J$ and combine like terms we miraculously recover equations 6.6 and 6.7. Thus the theorem is finished. $\square$

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