SOME TRANSFINITE INDUCTION DEDUCTIONS

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Abstract. This paper develops the ordinal numbers and transfinite induction, then demonstrates some interesting applications of transfinite induction. One such application is the proof that there is a set in $\mathbb{R}^2$ that intersects every line in exactly two places. We also prove that $\mathbb{R}^3$ can be covered by a disjoint union of circles of radius 1. Finally, we introduce the Kirby-Paris Hydra game and prove that every Hydra eventually dies, which is equivalent to proving that Peano arithmetic is incomplete.

Contents

1. Introduction

We assume the reader is familiar with the principle of mathematical induction. Having learned mathematical induction, it is natural to question the limitations of it; why must induction be restricted to the natural numbers? If induction can be expanded beyond the natural numbers, does this allow us to explore interesting and useful mathematical ideas?

Transfinite induction is the extension of mathematical induction to ordinal numbers. While proofs via mathematical induction are often a tedious exercise in algebra, the statements of the theorems we will prove are far from intuitive. Transfinite induction is particularly useful in the area of analytic geometry, since it allows us to prove results which are nearly impossible to picture – this will be demonstrated in the first two proofs of this paper. Also unexpected is the statement that every Hydra eventually dies. Although this finding may seem an unnecessary piece of mathematical trivia, its proof is quite significant for Peano arithmetic.

At the end of the 19th century, Giuseppe Peano formulated Peano arithmetic as a set of axioms for the natural numbers [4]. While the axioms are obvious, they are not sufficient to prove every true statement about the natural numbers – this is called *incompleteness*. One such statement is that all Kirby-Paris Hydras die in finitely many steps [6].

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These fascinating theorems constitute only a few applications of transfinite induction; some other theorems that can be proved via transfinite induction are [3]:

- $\mathbb{R}^2$ cannot be covered by a disjoint union of circles.
- Suppose we call a subset $S \subset \mathbb{R}^2$ a "circle" if there exists a point s, called the *center*, such that every half-line beginning from s intersects S in a single point. Then $\mathbb{R}^2$ can be covered with countably many circles.
- There is a partition of the plane into countably many pieces such that the distance between any two points in the same piece is irrational.

2. Preliminaries

Here are some standard definitions and propositions that will be useful for the theorems to be proven in later sections.

Definition 2.1. A *relation* [2] between two sets is a collection of ordered pairs containing one object from each set. Inductively, we say that an *n-ary relation* R over a set X is a relation between X^{n-1} and X.

We sometimes write $R(x_1, ..., x_n)$ for $(x_1, ..., x_n) \in R$, and if R is binary, we may write xRy for $(x, y) \in R$.

$<, \leq$, and $\in$ are all examples of relations.

Definition 2.2. A set X is *strictly well ordered* [2] by the binary relation R if the following hold:

i. For all $x, y \in X$, either xRy or yRx;

ii. Every nonempty subset Y of X has a least element (*i.e.*, an element $y \in Y$ such that yRz for all $z \in Y$);

iii. $x \in X$ implies not xRx, and the relation $R \cup \{(x, x) : x \in X\}$ satisfies (i) and (ii).

Definition 2.3. An *ordinal* [2] is a set α such that $\bigcup \alpha \subset \alpha$ and α is strictly well ordered by the $\in$ relation.

The notation $\bigcup \alpha$ is somewhat ambiguous; we simply mean that for an element x, $x \in \bigcup \alpha$ if and only if there exists an element β such that $\beta \in \alpha$ and $x \in \beta$.

We can intuitively think of an ordinal as a type of well ordering, but the trick of treating it as a set whose elements are sets makes for simpler notation.

Definition 2.4. The *successor* of an ordinal α [2] is the ordinal $\alpha + 1 = \alpha \bigcup \{\alpha\}$, which is the least ordinal greater than α.

For example, the ordinal 0 is defined to be the empty set $\emptyset$, so the ordinal 1 is the set $\{0\}$ and the ordinal 2 is the set $\{0, \{0\}\} = \{0, 1\}$. The successor ordinal n is the set $\{0, 1, ...n - 1\}$.

Definition 2.5. An ordinal α is said to be a *limit ordinal* [2] if and only if it is not a successor ordinal.

Note that by this definition, 0 is a limit ordinal, and we define ω to be the smallest limit ordinal other than 0.

Now we will develop some basic properties of ordinals.

Proposition 2.6. *Every element of an ordinal is an ordinal* [2].

Proof. Let α be an ordinal and $x \in \alpha$.

If $y \in x$ then $y \in \bigcup \alpha$. Since $\bigcup \alpha \subset \alpha$, $y \in \alpha$ which means that all elements of x are also elements of α. Hence $x \subset \alpha$ and since α is strictly well ordered by $\in$, it follows that x is strictly well ordered by $\in$.

We will now show that $\bigcup x \subset x$. In other words, if $z \in y$, $y \in x$ then $z \in x$.

Let $z \in y$, $y \in x$. Since $\bigcup \alpha \subset \alpha$, we have $y \in \alpha$ and hence $z \in \alpha$. Now, since α is strictly well ordered by the $\in$ relation, $z \in x$.

Then by definition, x is an ordinal. □

Proposition 2.7. *If α, β are ordinals, then $\alpha \subset \beta$ if and only if $\alpha \in \beta$ or $\alpha = \beta$* [2].

Proof. Note that if $\alpha = \beta$ then it is clear that $\alpha \subset \beta$, and if α is not a subset of β then not all the elements of α are also elements of β, so there is no way for α to equal β.

Therefore, we must prove that if $\alpha \neq \beta$, then $\alpha \subset \beta$ if and only if $\alpha \in \beta$.

First suppose that $\alpha \in \beta$ and $\alpha \neq \beta$. If $\gamma \in \alpha$ then $\gamma \in \bigcup \beta$, and since $\bigcup \beta \subset \beta$, we have $\alpha \subset \beta$ (this is the same argument we used at the beginning of proposition 2.6).

Now suppose $\alpha \subset \beta$ and $\alpha \neq \beta$. Let γ be the least element of the nonempty set $\beta \setminus \alpha$. To show that $\alpha \in \beta$ we will show that $\alpha = \gamma$.

If $\delta \in \gamma$, then $\delta \in \beta$ and since γ is the least element of $\beta \setminus \alpha$, $\delta \in \alpha$. Therefore all elements of γ are also elements of α, so $\gamma \subset \alpha$.

Now since $\delta \in \alpha$ and $\alpha \subset \beta$, $\delta \in \beta$. β is strictly well ordered by $\in$, so we have either $\gamma \in \delta$, $\gamma = \delta$, or $\delta \in \gamma$. We know that $\delta \in \alpha$ and $\gamma \in \beta \setminus \alpha$, so $\gamma \notin \alpha$. Therefore, we can conclude that $\gamma \neq \delta$ and $\gamma \notin \delta$. Hence $\delta \in \gamma$, which shows that $\alpha \subset \gamma$.

Recall that $\gamma \subset \alpha$, so α must be equal to γ. Since $\gamma \in \beta$, $\alpha \in \beta$. □

Proposition 2.8. *All the natural numbers are ordinals* [2].

Proof. Consider the set of of all the natural numbers.

For every element in this set, every smaller element is also in the set, which means it is strictly well ordered by the $\in$ relation. Moreover, it is easy to see that an element x is in the set of natural numbers if and only if x is in some subset of the natural numbers.

Then by definition, the set of natural numbers is an ordinal (it is in fact the smallest infinite ordinal ω). Hence by proposition 2.6, every natural number is an ordinal. □

The following proposition will be needed later.

Proposition 2.9. *There is no infinite decreasing sequence of ordinals* [3].

Proof. Suppose for contradiction that we have an infinite sequence of ordinals $\{\alpha\}$ such that $\alpha_0 > \alpha_1 > \alpha_2 > ...$

By our definition of ordinals, this means that $\alpha_{i+1} \in \alpha_i$. Since $\alpha_1 \in \alpha_0$, $\alpha_2 \in \alpha_1$, and so on, it is clear that α_0 is an ordinal that contains the ordinals $\alpha_1, \alpha_2, ...$. Consider the sequence $\{\alpha_1, \alpha_2, ...\}$ where $\alpha_1 > \alpha_2 > ...$. Since this sequence is infinite, it can have no least element.

Therefore, α_0 has no least element, which means it is not an ordinal, a contradiction. □

Definition 2.10. Let α and β be ordinal numbers. Then *ordinal multiplication* is defined so that:

i. $\alpha * 0 = 0$;

ii. If β is a successor ordinal then $\alpha*$ (successor of β) $= \alpha * \beta + \alpha$;

iii. If β is a limit ordinal then $\alpha * \beta$ is the least ordinal greater than any ordinal in the set $\{\alpha * \gamma : \gamma < \beta\}$.

Note that multiplication is not commutative but is associative.

Definition 2.11. Let α and β be ordinal numbers. Then *ordinal exponentiation* [3] is defined so that:

i. If $\beta = 0$ then $\alpha^{\beta} = 1$;

ii. If β is a successor ordinal, then $\alpha^{\beta+1} = \alpha^{\beta} * \alpha$;

iii. If β is a limit ordinal, then $\alpha = 0$ implies $\alpha^{\beta} = 0$. If $\alpha \neq 0$ then α^{β} is the least ordinal greater than every ordinal in the set $\{\alpha^{\gamma} : \gamma < \beta\}$.

Note that any ordinal αcan be written uniquely as $\omega^{\beta_1} + \omega^{\beta_2} + \omega^{\beta_k}$, where k is a natural number and $\beta_1 \geq \beta_2 \geq ... \geq \beta_k \geq 0$ are ordinals.

Definition 2.12. (Transfinite Induction) [2]. Let $P(\alpha)$ be a property of ordinals. Suppose that for all ordinals β, if $P(\gamma)$ holds for $\gamma < \beta$, then $P(\beta)$ holds. Then we have $P(\alpha)$ for all ordinals α.

Proof. Suppose for contradiction that for all ordinals β, if $P(\gamma)$ holds for $\gamma < \beta$, then $P(\beta)$ holds, but there is some ordinal α for which $P(\alpha)$ does not hold.

Let $X = \{\gamma \leq \alpha : P(\gamma) \text{ fails}\}$. X is not empty because α is in X. Therefore, X has a least element which we will call β. Then any $\gamma < \beta$ is not in X, or in other words $P(\gamma)$ holds. But by hypothesis, if $P(\gamma)$ holds for $\gamma < \beta$, then $P(\beta)$ holds. Contradiction. □

Definition 2.13. A function f whose domain is an ordinal α is called an *α-termed sequence* [2].

Definition 2.14. An *enumeration* [2] of a set X is a sequence whose range is X.

Definition 2.15. The *cardinality* [2] of a set X, denoted $|X|$ is the least ordinal α such that X is enumerated by an α-termed sequence.

Intuitively, we can think of the cardinality of X as the size of X, or the number of elements in X.

Definition 2.16. An ordinal α is said to be a *cardinal* [2] if $\alpha = |\alpha|$.

The ξ-th infinite cardinal is denoted $\aleph_{\xi}$. For example, $\aleph_0$ is the cardinality of ω, the first limit ordinal, or equivalently the set of all natural numbers.

$\aleph_1$ is the second smallest infinite cardinal number, which is the cardinality of the set of real numbers. In fact, it can be shown that $|\mathbb{R}| = \aleph_1 = 2^{\aleph_0}$.

Definition 2.17. A map f is said to be *injective* [3] if $f(a) = f(b)$ implies $a = b$.

Definition 2.18. A map f from A to B is said to be *surjective* [3] if for every element $b \in B$ there exists some $a \in A$ such that $f(a) = b$.

Definition 2.19. A map f is *bijective* [3] if it is both injective and surjective. Sets A and B have the same cardinality if and only if there is a bijection from A to B.

Definition 2.20. A set A is *countable* [3] if and only if there is a bijection from A to a subset of the natural numbers. In symbols, $|A| \leq |\mathbb{N}|$.

Definition 2.21. (Peano arithmetic) [8].

Axiom 1. 0 is a natural number.

Axiom 2. If α is a natural number, then the successor of α is also a natural number.

Axiom 3. 0 is not the successor of a natural number.

Axiom 4. For all natural numbers α and β, $\alpha = \beta$ if and only if the successor of α is equal to the successor of β.

Axiom 5. (Induction axiom.) If a set S of natural numbers contains 0 and also the successor of every number in S, then every natural number is in S.

Finally, we introduce some basic concepts used in logic.

Definition 2.22. A set of axioms is *consistent* [2] if there is no statement such that both the statement and its negation are true according to the axioms. In other words, the set of axioms does not contain any contradictions.

Definition 2.23. A set of axioms is *complete* [2] if for any statement, either the statement or its negation can be proved from the axioms.

3. Two point sets exist

A two point set is a subset of the plane which intersects every line in exactly two points. There are many interesting things to prove about the symmetries of two point sets [1], but here we simply prove that such sets exist.

Theorem 3.1. *There exists a set $A \subset \mathbb{R}^2$ which intersects every line in exactly two points.*

Proof. Let $\{L_\alpha\}$ be a labeling of all the lines in $\mathbb{R}^2$ using ordinals α. We will use transfinite induction on α to construct a sequence $\{A_\alpha\}$ of subsets of $\mathbb{R}^2$ such that for every α, the following three properties hold:

i. A_α has at most two points;

ii. $\bigcup\limits_{\beta\leq\alpha} A_\beta$ does not have any three points collinear;

iii. $\bigcup\limits_{\beta\leq\alpha} A_\beta$ contains exactly two points of L_β.

Then the set $A = \bigcup A_\alpha$ of all ordinals α will have the required property.

Base case. $\alpha = 0$. L_0 is our first line in $\mathbb{R}^2$, so we choose $A_0 \subset L_0$ to have exactly two points. Then properties (i)-(iii) are satisfied.

Successor case. Suppose for successor ordinal α the sequence $\{A_\beta\}_{\beta\leq\alpha}$ satisfies properties (i)-(iii). We will prove that then these properties are satisfied by $\{A_\beta\}_{\beta\leq\alpha+1}$ as well.

Let $B = \bigcup\limits_{\beta\leq\alpha} A_\beta$ and note that $B \bigcup A_{\alpha+1} = \bigcup\limits_{\beta\leq\alpha+1} A_\beta$, the set we are interested in. Let C be the set of all lines containing two points from B.

Since property (ii) holds for $\{A_\beta\}_{\beta\leq\alpha}$ and $L_{\alpha+1}$ is a straight line, the set $B \bigcap L_{\alpha+1}$ has at most two points.

If $B \bigcap L_{\alpha+1}$has exactly two points, then we choose $A_{\alpha+1} = \emptyset$, which satisfies properties (i)-(iii).

If $B \bigcap L_{\alpha+1}$ has less than two points, then $L_{\alpha+1}$ intersects every line from C in at most one point, *i.e.*, for all lines $L \in C$, $|L_{\alpha+1} \bigcap L| \leq 1$. Clearly we want to choose $A_{\alpha+1}$ to be a subset of $L_{\alpha+1} \setminus \bigcup C$ in order to satisfy properties (ii) and (iii), however there is a danger that $L_{\alpha+1} \setminus \bigcup C = \emptyset$. We will show this is not the case:

Consider the line along the x-axis in the Cartesian coordinate system − the real number line, which we will denote by the set X. Note that its elements are all the real numbers, so the number of elements in X is exactly the cardinality of the reals; $|X| = |\mathbb{R}| = 2^{\aleph_0}$. Now, every line in $\mathbb{R}^2$ is simply a transformation of X, which implies that $|L_{\alpha+1}| = 2^{\aleph_0}$.
Consider a fixed point (a, b) and the line $y = ax + b$. Since both this point and line are uniquely determined by this notation, it is clear that there is a bijective map between (a, b) and $y = ax + b$. Therefore, there must be exactly as many lines as there are points in $\mathbb{R}^2$, *i.e.* the set of all lines has cardinality $2^{\aleph_0}$.
C is by definition the set of all lines containing two points from B, which we know does not yet include all the lines in $\mathbb{R}^2$. Hence $|C| < 2^{\aleph_0}$.
Recalling that for all lines $L \in C$, $|L_{\alpha+1} \bigcap L| \leq 1$, we have

$$|\bigcup_{L\in C}(L_{\alpha+1} \bigcap L)| = |L_{\alpha+1} \bigcap (\bigcup C)| < 2^{\aleph_0}$$

which means $L_{\alpha+1} \setminus \bigcup C$ is not empty.

Therefore, choose $A_{\alpha+1} \subset L_{\alpha+1} \setminus \bigcup C$ to have one element if $B \bigcap L_{\alpha+1}$ has one element, and to have two elements if $B \bigcap L_{\alpha+1} = \emptyset$. It is clear that either choice of $A_{\alpha+1}$ satisfies properties (i)-(iii).
Limit case. Suppose α is a limit ordinal and for all $\beta < \alpha$, $\{A_\gamma\}_{\gamma\leq\beta}$ satisfies properties (i)-(iii). We will prove that then these properties are satisfied by $\{A_\gamma\}_{\gamma\leq\alpha}$, using a similar argument to the successor case.

Consider the set $(\bigcup_{\beta<\alpha} A_\beta) \bigcap L_\alpha$, which has at most two points since $\{A_\gamma\}_{\gamma\leq\beta}$ satisfies property (ii).
If $(\bigcup_{\beta<\alpha} A_\beta) \bigcap L_\alpha$ has exactly two points, we choose $A_\alpha = \emptyset$.
If $(\bigcup_{\beta<\alpha} A_\beta) \bigcap L_\alpha$ has less than two points, then L_α intersects at most once with every line from the set, D, of all lines containing two points from $\bigcup_{\beta<\alpha} A_\beta$. We write, for all lines $L \in D$, $|L_\alpha \bigcap L| \leq 1$.
Again, we must check that $L_\alpha \setminus \bigcup D$ is not empty:

By the same reasoning as above, $|L_\alpha| = 2^{\aleph_0}$ and $|D| < 2^{\aleph_0}$. Hence

$$|\bigcup_{L\in D}(L_\alpha \bigcap L)| = |L_\alpha \bigcap (\bigcup D)| < 2^{\aleph_0}$$

which means $L_\alpha \setminus \bigcup D$ is not empty. Therefore, choose $A_\alpha \in L_\alpha \setminus \bigcup D$ to have one element if $(\bigcup_{\beta<\alpha} A_\beta) \bigcap L_\alpha$ has one element, and to have two elements if $(\bigcup_{\beta<\alpha} (A_\beta)) \bigcap L_\alpha$ is empty.

Taking $A = \bigcup A_\alpha$ for all ordinals α, we have constructed a set that intersects every line in exactly two points. □

4. Covering $\mathbb{R}^3$

In the introduction, we mentioned that one interesting proof via transfinite induction is that it is not possible to cover $\mathbb{R}^2$ with a disjoint union of circles. This makes the following theorem all the more unexpected, for it is hard to imagine covering $\mathbb{R}^3$ without first covering $\mathbb{R}^2$.

Theorem 4.1. $\mathbb{R}^3$ *can be covered by a disjoint union of circles of radius 1.*

Proof. Let $\{p_\alpha\}$ be a labeling of all the points in $\mathbb{R}^3$ by ordinals α. We want to construct a collection, A, of subsets of $\mathbb{R}^3$ such that each $C \in A$ is a circle of radius 1, different elements of A are disjoint, and A covers $\mathbb{R}^3$. We will use transfinite induction on α to construct A as $\{C_\alpha\}$, requiring that for every α,

i. $p_\alpha \in \bigcup_{\beta \leq \alpha} C_\beta$;

ii. $C_\alpha \bigcap (\bigcup_{\beta < \alpha} C_\beta) = \emptyset$.

Then $A = \bigcup C_\alpha$ will have the required property.

Base case. $\alpha = 0$. p_0 is our first point in $\mathbb{R}^3$, so we choose C_0 to be a circle of radius 1 in $\mathbb{R}^3$ such that $p_0 \in C_0$. Then properties (i) and (ii) are satisfied.

Successor case. Suppose for successor ordinal α the sequence $\{C_\beta\}_{\beta \leq \alpha}$ satisfies properties (i) and (ii). We will prove that then these properties are satisfied by $\{C_\beta\}_{\beta \leq \alpha+1}$ as well.

If $p_{\alpha+1} \notin \bigcup_{\beta < \alpha+1} C_\beta$, we will define $p = p_{\alpha+1}$. Otherwise, we choose an arbitrary point $p \in \mathbb{R}^3 \setminus \bigcup_{\beta < \alpha+1} C_\beta$. It is possible to choose such a point because $\alpha + 1$ is a successor ordinal, so we clearly have not constructed enough circles yet to cover $\mathbb{R}^3$; in other words, $\mathbb{R}^3 \setminus \bigcup_{\beta < \alpha+1} C_\beta$ is not empty.

Now we will choose $C_{\alpha+1}$ so that it contains p and satisfies property (ii). Consider a plane, P, in $\mathbb{R}^3$ which contains p and does not contain any of the circles C_β for $\beta < \alpha + 1$. There is a danger that such a plane does not always exist, but we will prove that this is not so:

A plane in $\mathbb{R}^3$ is uniquely determined by three points – since p is already fixed, we see that there must be exactly $2^{\aleph_0}$ many planes passing through p. However, there are at most $|\alpha+1|$ many planes which contain circles from $\{C_\beta\}_{\beta<\alpha+1}$, where $|\alpha + 1| < 2^{\aleph_0}$. Hence P exists.

Note that by our construction of P, it can intersect each C_β in at most two points. Therefore the set

$$B = P \bigcap (\bigcup_{\beta<\alpha+1} C_\beta) = \bigcup_{\beta<\alpha+1} (P \bigcap C_\beta)$$

is the union of $\beta < 2^{\aleph_0}$ many finite sets, so $|B| < 2^{\aleph_0}$.

Fix a line L in P which contains p. Call A_0 the set of all circles in P which contain p and are tangent to L. Then different circles in A_0 can only intersect at p. Hence there is a circle $C_{\alpha+1} \in A_0$ which is disjoint from B, and$\{C_\beta\}_{\beta\leq\alpha+1}$ satisfies properties (i) and (ii).

Limit case. Suppose α is a limit ordinal and for all $\beta < \alpha$, $\{C_\gamma\}_{\gamma\leq\beta}$ satisfies properties (i)-(iii). We will prove that then these properties are satisfied by $\{C_\gamma\}_{\gamma\leq\alpha}$, using a similar argument to the successor case.

If $p_\alpha \notin \bigcup_{\beta<\alpha} C_\beta$, define $p = p_\alpha$. Otherwise, choose $p \in \mathbb{R}^3 \setminus \bigcup_{\beta<\alpha} C_\beta$. It is clear that it is possible to choose such a point if we consider the set

$$L \bigcap (\bigcup_{\beta<\alpha} C_\beta) = \bigcup_{\beta<\alpha} (L \bigcap C_\beta)$$

where L is any straight line in $\mathbb{R}^3$. This set is the union of β many finite sets $L \bigcap C_\beta$, where $\beta < 2^{\aleph_0}$. Hence $\bigcup_{\beta<\alpha}(L \bigcap C_\beta)$ has cardinality less than $2^{\aleph_0}$, which means $\mathbb{R}^3 \setminus \bigcup_{\beta<\alpha} C_\beta$ cannot be empty.

Consider a plane, P in $\mathbb{R}^3$ which contains p and does not contain any of the circles C_β for $\beta < \alpha$. It is possible to construct such a plane because there are $2^{\aleph_0}$ many planes passing through p, but there are at most $|\alpha| < 2^{\aleph_0}$ many planes which contain circles from $\{C_\beta\}_{\beta<\alpha}$.

Note that by our construction of P, it can intersect each C_β in at most two points. Therefore the set

$$B = P \bigcap (\bigcup_{\beta<\alpha} C_\beta) = \bigcup_{\beta<\alpha}(P \bigcap C_\beta)$$

is the union of $\beta < 2^{\aleph_0}$ many finite sets, so $|B| < 2^{\aleph_0}$.

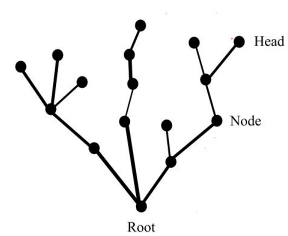
Fix a line L in P which contains p. Call A_0 the set of all circles in P which contain p and are tangent to L. Then different circles in A_0 can only intersect at p. Hence there is a circle $C_\alpha \in A_0$ which is disjoint from B, so $\{C_\beta\}_{\beta\leq\alpha}$ satisfies properties (i) and (ii).

Taking $A = \bigcup C_\alpha$, for all ordinals α, we have covered $\mathbb{R}^3$ with a disjoint union of circles of radius 1. □

5. The Hydra Game

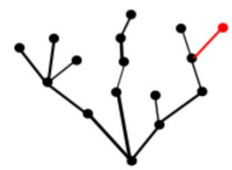
The Kirby-Paris Hydra game takes its name from the Greek myth – the Hydra is a monster with multiple heads that either grow straight out of its body or are connected to the body by necks. Every time we chop off a head growing directly out of the body, that head dies. However, if we chop off a head connected to a neck, the Hydra grows more heads, according to any set of rules we wish.

We will represent the Hydra as a rooted tree, *i.e.*, the root is the body of the Hydra, the leaves are its heads, and the nodes are the necks. This is one possible Hydra [5]:

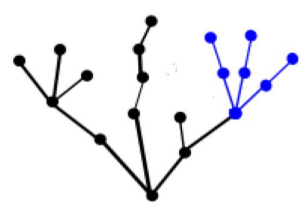


We start with the simple rule that when we chop off a head connected to a neck, we descend a node and from there, grow two subtrees identical to the subtree that was attacked.

Suppose we decide to cut off the head shown in red.



Then according to our rules, this is the resultant Hydra:



Instead of playing with the rule that when we chop off a head connected to a neck, the Hydra grows two subtrees identical to the subtree that was attacked, we could also play that the Hydra grows *three* new copies of the attacked subtree, or even n new copies at the *nth* step of the game; any rules are valid as long as the Hydra never grows infinitely many heads at once.

Theorem 5.1. *All Hydras eventually die, regardless of what rules we play with.*

Remark. In 1982, Kirby and Paris proved that any proof technique which proves that every Hydra eventually dies must be strong enough to prove that Peano arithmetic is consistent [6]. Godel's first Incompleteness Theorem, which we will not prove, states that all consistent axiomatic formulations of number theory include propositions that can neither be proved nor disproved by those axioms, *i.e.* any consistent system is incomplete [7]. Hence Peano arithmetic is incomplete.

Proof. First we will prove that all Hydras eventually die when we adopt the rule that when we chop off a head connected to a neck, the Hydra grows two subtrees identical to the subtree that was attacked. Then we will extend this to prove that it does not matter how many subtrees grow back, as long as the number is finite.

Let $\{H_\alpha\}$ be a labeling of Hydras by ordinals α such that each H_α has one more head than $H_{\alpha-1}$. We will use transfinite induction on α, with the hypothesis that for every α, H_α can be killed in finitely many steps.

Base case. $\alpha = 0$. H_0 is our first Hydra; a rooted tree with a single head. Then we cut off the head, which kills that head and hence the whole Hydra has been killed in one step.

Successor case. Suppose for successor ordinal α, H_α eventually dies. We will prove that then $H_{\alpha+1}$ also eventually dies.

$H_{\alpha+1}$ has one more head than H_α. Suppose this head grows directly out of the Hydra's body. Then it clearly only takes one more step to kill $H_{\alpha+1}$ than H_α, so $H_{\alpha+1}$ must die in finitely many steps.

Suppose this head does not grow directly out of the body. We begin by chopping off that head, which will produce two more copies of the subtree that was attacked – we will denote this subtree by S. Note that $S \subset H_\alpha$. Since H_α dies in finitely many steps, certainly S must die in finitely many steps. This implies that three copies of S must also die in finitely many steps, hence $H_{\alpha+1}$ dies in finitely many steps.

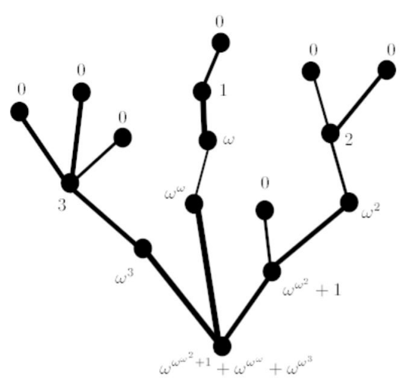
Limit case. Suppose α is limit ordinal and for all $\beta < \alpha$, H_β eventually dies. We will prove that then H_α also eventually dies.

We will assign an ordinal to H_α like this:

i. Heads are assigned the ordinal 0;

ii. Suppose a node x has sub-Hydras $H_1, ...H_k$ growing from it. We will assign each sub-Hydra its ordinal recursively, and order the ordinals in descending order so that $\alpha_1 \geq \alpha_2 \geq ... \geq \alpha_k$. Let the ordinal assigned to x be $\omega^{\alpha_1} + ... + \omega^{\alpha_k}$.

This is how we would label the Hydra from the example above:



Then it is clear that every move decreases the ordinal assigned to H_α. Since there is no infinite decreasing sequence of ordinals by proposition 2.9, every H_α must die in finitely many steps.

If we play with any other set of rules, the base case and limit case will be exactly the same as above. We would argue the successor case in a similar way, since at any step, the number of subtrees the Hydra grows must be finite. Therefore, all Hydras eventually die. □

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