QUILLEN EQUIVALENCES OF MODEL CATEGORIES

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ABSTRACT. In this paper, we assume no knowledge of model structures on the categories of topological spaces and simplicial sets, and build to Quillen's equivalence between these structures. Along the way, we discuss the geometric realization and total singular complex functors and briefly establish the theory of minimal Kan fibrations.

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1. INTRODUCTION

In the 1960s, Quillen introduced model categories, which provide general contexts for homotopy theory. The category of topological spaces admits a model structure, with a natural corresponding homotopy theory. Simplicial sets can also be thought of as spaces, and the category of simplicial sets also admits a model structure. We can pass between these two categories via the geometric realization and total singular complex functors, and in fact Quillen demonstrated an equivalence between these two categories. This paper details that equivalence.

We will define a model category, and construct model structures on the category of topological spaces and on the category of simplicial sets. We discuss two model structures on topological spaces: the Hurewicz and Quillen model categories. In this paper, we focus on the latter, and prove the equivalence of this category and the category of simplicial sets. In order to do this, we briefly introduce the theory of minimal fibrations. Several existing papers, for example [10], instead prove this equivalence using the theory of anodyne extensions, and in fact J. P. May and K. Ponto's *More Concise* does so with neither anodyne extensions nor minimal fibrations [1].

2. Basic Definitions: Model Categories

Before constructing the model categories of topological spaces and simplicial sets, we provide a brief introduction to model categories in general. These structures are meant to provide a nice context in which to do homotopy theory, so we attach to our category equivalences (maps which will make sense in their corresponding category) and restrictions that make it well-behaved. Formally, we say:

Definition 2.1. A *model category* is a bicomplete category $\mathscr{M}$ equipped with three classes of morphisms $(\mathscr{W}, \mathscr{C}, \mathscr{F})$ satisfying the following axioms:

(i) *$\mathscr{W}$ has 2 out of 3:* Given composable morphisms f, g of $\mathscr{M}$, if any two of f, g, and gf are in $\mathscr{W}$, then so is the third.

(ii) $(\mathscr{C}, \mathscr{F} \cap \mathscr{W})$ is a weak factorization system on $\mathscr{M}$.

(iii) $(\mathscr{C} \cap \mathscr{W}, \mathscr{F})$ is a weak factorization system on $\mathscr{M}$.

We call $\mathscr{W}$ the weak equivalences, $\mathscr{C}$ the cofibrations, and $\mathscr{F}$ the fibrations. Recall the definition of a weak factorization system:

Definition 2.2. A *weak factorization system* $(\mathscr{L}, \mathscr{R})$ is a pair of classes of morphisms in a category $\mathscr{M}$ such that:

(i) Every morphism $f : X \to Y$ of $\mathscr{M}$ factors as the composition of morphisms $\lambda \in \mathscr{L}$ and $\rho \in \mathscr{R}$:

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, "\lambda"'] & & Y \\ & Z \arrow[ur, "\rho"'] & \end{tikzcd}$$

(ii) $\mathscr{L} = {}^{\square}\mathscr{R}$, i.e. $\mathscr{L}$ is exactly the class of morphisms with the left lifting property with respect to $\mathscr{R}$. That is, $\mathscr{L}$ is precisely the class of morphisms $\lambda : A \to X$ such that for $\rho \in \mathscr{R}$ and for all morphisms $f, g \in \mathscr{M}$ such that $\rho \circ g = f \circ \lambda$, there exists a lift h making the following diagram commute:

$$\begin{tikzcd} A \arrow[r, "g"] \arrow[d, "\lambda"'] & E \arrow[d, "\rho"] \\ X \arrow[r, "f"'] \arrow[ur, dashed, "h"] & B \end{tikzcd}$$

(iii) $\mathscr{R} = \mathscr{L}^{\square}$, i.e. $\mathscr{R}$ is exactly the class of morphisms with the right lifting property with respect to $\mathscr{L}$. That is, $\mathscr{R}$ is precisely the class of morphisms $\rho : E \to B$ such that for $\lambda \in \mathscr{L}$ and for all morphisms $f, g \in \mathscr{M}$ such that $\rho \circ g = f \circ \lambda$, there exists a lift h making the diagram in (ii) commute.

We make use of some additional terminology. Call the morphisms in $\mathscr{F} \cap \mathscr{W}$ *acyclic fibrations* and the morphisms in $\mathscr{C} \cap \mathscr{W}$ *acyclic cofibrations*. Now since $\mathscr{M}$ is bicomplete, it has some initial object $\varnothing$ and terminal object $*$. Call an object X *fibrant* if the morphism $X \to *$ is in $\mathscr{F}$ and *cofibrant* if the morphism $\varnothing \to X$ is in $\mathscr{C}$.

We will often use what is called the *small object argument* to show that a category has a model structure (in particular, that it is a compactly generated model category). First, note that we say that a set $\mathcal{I}$ of morphisms in a category $\mathscr{M}$ is *compact* if every object A of the category that is the domain of a map in $\mathcal{I}$ is compact with respect to $\mathcal{I}$, i.e. if for every object A of the category that is the domain of a map in $\mathcal{I}$ and every relative $\mathcal{I}$-cell complex $X \to Z = \mathrm{colim}_n Z_n$, the map

$\mathrm{colim}_n \mathscr{M}(A, Z_n) \to \mathscr{M}(A, Z)$ is a bijection. The small object argument, detailed in [1], says that given a compact set $\mathcal{I}$ of morphisms in a category $\mathscr{M}$, there is a functorial weak factorization system $(\mathscr{C}(\mathcal{I}), \mathcal{I}^\square)$.

Definition 2.3. A model category $\mathscr{M}$ with weak equivalences $\mathscr{W}$, cofibrations $\mathscr{C}$, and fibrations $\mathscr{F}$ is *generated* by sets $\mathcal{I}$ and $\mathcal{J}$ if $\mathscr{F} = \mathcal{J}^\square$ and $\mathscr{F} \cap \mathscr{W} = \mathcal{I}^\square$.

In this case, we call $\mathcal{I}$ the set of *generating cofibrations* and $\mathcal{J}$ the set of *generating acyclic cofibrations*. These terms make sense since the conditions in Definition 2.3 are equivalent to:

$$\mathscr{F} \cap \mathscr{W} = \mathcal{I}^\square = \mathscr{C}(\mathcal{I})^\square \iff \mathscr{C} = \mathscr{C}(\mathcal{I}) \text{ and } \mathscr{F} = \mathcal{J}^\square = \mathscr{C}(\mathcal{J})^\square \iff \mathscr{C} \cap \mathscr{W} = \mathcal{J}$$

The model categories that we will encounter in this paper (and many other interesting model categories) will be generated by compact sets. Showing this (stronger) statement will actually help us avoid proving some of the conditions in Definitions 2.1 and 2.2, due to the following theorem:

Theorem 2.4. *Let $\mathscr{M}$ be a bicomplete category with category of weak equivalences $\mathscr{W}$ and sets of morphisms $\mathcal{I}$ and $\mathcal{J}$. If $\mathcal{I}$ and $\mathcal{J}$ are compact, then $\mathscr{M}$ is a compactly generated model category with generating cofibrations $\mathcal{I}$ and generating acyclic cofibrations $\mathcal{J}$ if and only if the following hold:*
(i) Every relative $\mathcal{J}$-cell complex is a weak equivalence.
(ii) $\mathcal{I}^\square = \mathcal{J}^\square \cap \mathscr{W}$

Proof. The first direction is immediate, and the converse can be verified using the small object argument mentioned above. □

3. Two Model Categories of Topological Spaces

We will actually define two model structures on topological spaces: the Hurewicz and Quillen model structures, denoted $\mathscr{M}_h$ and $\mathscr{M}_q$, respectively. In the Hurewicz model structure, we will have homotopy equivalences $\mathscr{W}_h$ as weak equivalences, Hurewicz cofibrations $\mathscr{C}_h$ as cofibrations, and Hurewicz fibrations $\mathscr{F}_h$ as fibrations. In the Quillen model structure, we will have weak homotopy equivalences $\mathscr{W}_q$ as weak equivalences, retracts of of relative cell complexes $\mathscr{C}_q$ as cofibrations, and Serre fibrations $\mathscr{F}_q$ as fibrations. By a *weak homotopy equivalence*, we mean a map between two spaces that induces isomorphisms on all homotopy groups.

The main focus of this paper will be on the Quillen model structure (and its equivalence to a model category of simplicial sets), so we will show that $\mathscr{M}_q$ is a model structure, and leave the other proof to the reader (although it can be found in [1]). But before we show this, we examine both constructions a bit. First, recall some definitions essential to these structures:

Definition 3.1. A *relative cell complex* is a subspace inclusion $f : X \to Y$ such that Y can be constructed from X by repeatedly attaching cells (possibly infinitely many times).

The cofibrations $\mathscr{C}_q$ in the Quillen model category are retracts of these. The fibrations in the same category will be Serre fibrations:

Definition 3.2. A continuous map $p : E \to B$ is a *Serre fibration* if it has the homotopy lifting property for disks D^n. That is, given any homotopy $f : D^n \times I \to B$ and lift $\widetilde{f}_0 : D^n \to E$ such that $f_0 = p \circ \widetilde{f}_0$, there is a homotopy $\widetilde{f} : D^n \times I \to E$ lifting

f (such that $f = p \circ \widetilde{f}$). Equivalently, p is a Serre fibration if for any D^n we get a lift $\widetilde{f}$ making this diagram commute:

$$\begin{tikzcd} D^n \arrow[r, "\widetilde{f}_0"] \arrow[d] & E \arrow[d, "p"] \\ D^n \times I \arrow[r, "f"] \arrow[ur, dashed, "\widetilde{f}"] & B \end{tikzcd}$$

Meanwhile, the fibrations $\mathscr{F}_h$ in the other category will be Hurewicz fibrations, and the cofibrations $\mathscr{C}_h$ will be the natural dual of this (and the usual definition of a cofibration):

Definition 3.3. A continuous map $p : E \to B$ is a *Hurewicz fibration* if it has the homotopy lifting property for any space Y. That is, given any homotopy $f : Y \times I \to B$ and lift $\widetilde{f}_0 : Y \to E$ such that $f_0 = p \circ \widetilde{f}_0$, there is a homotopy $\widetilde{f} : Y \times I \to E$ lifting f (such that $f = p \circ \widetilde{f}$). Equivalently, p is a Hurewicz fibration if for any Y we get a lift $\widetilde{f}$ making this diagram commute:

$$\begin{tikzcd} Y \arrow[r, "\widetilde{f}_0"] \arrow[d] & E \arrow[d, "p"] \\ Y \times I \arrow[r, "f"] \arrow[ur, dashed, "\widetilde{f}"] & B \end{tikzcd}$$

Whereas Hurewicz fibrations are defined to satisfy the covering homotopy property, cofibrations satisfy the homotopy extension property:

Definition 3.4. A continuous map $i : A \to X$ is a *cofibration* if given the diagram below such that $h \circ i_0 = f \circ i$, there exists $\widetilde{h}$ making it commute:

$$\begin{tikzcd} A \arrow[rr, "i_0"] \arrow[dd, "i"] & & A \times I \arrow[dl, "h"] \arrow[dd, "i \times \mathrm{id}"] \\ & Y & \\ X \arrow[ur, "f"] \arrow[rr, "i_0"] & & X \times I \arrow[ul, dashed, "\widetilde{h}"] \end{tikzcd}$$

Notice that in our definition of a Hurewicz cofibration, Y can be any space, so this is dual to the notion of Hurewicz fibration, not a Serre fibration. However, a precise discussion of this duality is beyond the scope of this paper.

Also, notice that the Serre fibration is a weaker notion of a Hurewicz fibration, in which we may have any space Y instead of a disk D^n. In fact, the following theorem relates the two:

Theorem 3.5. *A Serre fibration between CW complexes is a Hurewicz fibration.*

A proof of this is beyond the scope of this paper, although it is quite interesting and actually suggests a proof of the Quillen equivalence (defined in Section 7) between $\mathscr{M}_q$ and $\mathscr{M}_h$ [7].

Next, we comment on Quillen's original definition of a model category. Defining $\mathscr{C}_q$ as retracts of relative cell complexes is actually a natural way to make $\mathscr{M}_q$ satisfy Definition 2.1. Quillen originally defined "closed model categories," where "closed" meant that the classes of weak equivalences, cofibrations, and fibrations were closed under *retracts*.

Definition 3.6. A class $\mathscr{A}$ of morphisms in $\mathscr{M}$ is *closed under retracts* if given $j : X \to Y$ in $\mathscr{A}$ and the commutative diagram below, where $g \circ f = \mathrm{id}$ and $p \circ q = \mathrm{id}$, the morphism $i : A \to B$ is necessarily in $\mathscr{A}$.

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "i"] & X \arrow[r, "g"] \arrow[d, "j"] & A \arrow[d, "i"] \\
B \arrow[r, "p"] & Y \arrow[r, "q"] & B
\end{tikzcd}$$

For the sake of brevity, we omit the proof that Definition 2.1 suffices to show that $\mathscr{W}, \mathscr{C}$, and $\mathscr{F}$ are closed under retracts, but it can be found in [1]. Now that we have established exactly what the classes in $\mathscr{M}_q$ are, we prove that they induce a model structure, as claimed:

Theorem 3.7. *The category of topological spaces equipped with weak equivalences $\mathscr{W}_q$, cofibrations $\mathscr{C}_q$, and fibrations $\mathscr{F}_q$ is a model category.*

Proof. We see that limits and colimits exist, and thus $\mathscr{M}_q$ is bicomplete. Next we check that $\mathscr{W}_q$ has 2 out of 3. Suppose $f : X \to Y$, $g : Y \to Z$, and $h = g \circ f : X \to Z$ are morphisms in $\mathscr{M}_q$. If f, g are weak homotopy equivalences, clearly h is. If g, h are weak homotopy equivalences, clearly f is. Finally, if f, h are weak homotopy equivalences, we must show that for all $y \in Y$, We have an isomorphism:

$$g_* : \pi_*(Y, y) \to \pi_*(Z, g(y))$$

Note that while we may not have $y \in f(X)$ for every y, there necessarily exists $x_0 \in X$ and a path p from $f(x_0)$ to y. We define $e : \pi_*(Y, y) \to \pi_*(Y, f(x_0))$ via $e(q) = pqp^{-1}$, and $e' : \pi_*(Z, g(y)) \to \pi_*(Z, g(f(x_0)))$ via $e'(q) = (g \circ p)q(g \circ p)^{-1}$. Then the following diagram commutes:

$$\begin{tikzcd}
\pi_*(Y, y) \arrow[r, "g_*"] \arrow[d, "e"] & \pi_*(Z, g(y)) \arrow[d, "e'"] \\
\pi_*(Y, f(x_0)) \arrow[r, "g_*"] & Y\pi_*(Z, g(f(x_0)))
\end{tikzcd}$$

But $g_* : \pi_*(Y, f(x_0)) \to \pi_*(Z, g(f(x_0)))$ is an isomorphism since f_* and $g_* \circ f_*$ are isomorphisms, so $g_* : \pi_*(Y, y) \to \pi_*(Z, g(y))$ is an isomorphism, as well. Thus, $\mathscr{W}_q$ has 2 out of 3.

Now that we have shown $\mathscr{M}_q$ is bicomplete and $\mathscr{W}_q$ is an appropriate category of weak equivalences, we apply Theorem 2.4 to show that $\mathscr{M}_q$ is compactly generated model category by cofibrations $\mathcal{I}$ and acyclic cofibrations $\mathcal{J}$, where $\mathcal{I}$ is the set of inclusions $i : S^{n-1} \to D^n$ for $n \geq 0$ (let $S^{-1} = \varnothing$) and $\mathcal{J}$ is the set of maps $j : D^n \to D^n \times I$ for $n \geq 0$. We assume that $\mathcal{I}$ and $\mathcal{J}$ are compact (this is not hard to check), and we apply the small object argument to get weak factorization systems $(\mathscr{C}(\mathcal{I}), \mathcal{I}^\square)$ and $(\mathscr{C}(\mathcal{J}), \mathcal{J}^\square)$, where $\mathscr{C}(\mathcal{I})$ and $\mathscr{C}(\mathcal{J})$ are retracts of the relative $\mathcal{I}$- and $\mathcal{J}$-cell complexes, repectively.

We check the conditions of Theorem 2.4, beginning with (i). Suppose $f : X_0 \to \operatorname{colim} X_q$ is a relative $\mathcal{J}$-cell complex. Then each $f_i : X_i \to X_{i+1}$ is an inclusion of a deformation retract, so f is a weak homotopy equivalence. Finally, we check that $\mathcal{I}^\square = \mathcal{J}^\square \cap \mathscr{W}_q$ by checking both inclusions.

First, suppose $f : X \to Y$ is a weak homotopy equivalence in $\mathcal{J}^\square$. We want to show that $f \in \mathcal{I}^\square$, i.e. that we have the lift $\widetilde{h} : D^n \to X$ making every diagram of the following form commute:

$$\begin{tikzcd}
S^{n-1} \arrow[r, "g"] \arrow[d, "d"'] & X \arrow[d, "f"] \\
D^n \arrow[r, "h"'] \arrow[ur, dashed, "\widetilde{h}"] & Y
\end{tikzcd}$$

Define $g_0 : S^{n-1} \times I \to X$ via $g_0(s,t) = g(s)$, and since $f \in \mathscr{W}_q$, we get $\psi : D^n \to X$, $\varphi : D^n \times I \to Y$, and the commutative diagrams:

$$\begin{tikzcd}
S^{n-1} \arrow[rr] \arrow[dd, "d"'] \arrow[dr, "g"] & & S^{n-1} \times I \arrow[dl, "g_0"'] \arrow[dd, "d\times\mathrm{id}"] \\
& X & \\
D^n \arrow[rr] \arrow[ur, dashed, "\psi"] & & D^n \times I
\end{tikzcd}
\qquad
\begin{tikzcd}
S^{n-1} \arrow[rr] \arrow[dd, "d"'] & & S^{n-1} \times I \arrow[dl, "f\circ g_0"'] \arrow[dd, "d\times\mathrm{id}"] \\
& Y & \\
D^n \arrow[rr] \arrow[ur, "h"] & & D^n \times I \arrow[ul, dashed, "\varphi"]
\end{tikzcd}$$

Since $f : X \to Y$ is in $\mathcal{J}^{\square}$ and φ and ψ make the diagrams commute, we can now see that f must satisfy the right lifting property with respect to $D^n \to D^n \times I$. Thus, we get a lift $\widetilde{f} : D^n \times I \to E$ such that $\varphi = f \circ \widetilde{f}$, ψ is the composition of $\widetilde{f}$ and the map $D^n \to D^{n-1}$ in the diagram on the left, and $\widetilde{f}$ restricted to $S^{n-1} \times I$ is just g_0. Then $\widetilde{h}$ is just the composition of $\widetilde{f}$ and the map $D^{n-1} \to D^n$ in the diagram on the right. Hence, $f : X \to Y$ is in $\mathcal{I}^{\square}$, and $\mathcal{J}^{\square} \cap \mathscr{W}_q \subseteq \mathcal{I}^{\square}$.

The reverse inclusion is slightly simpler to show, and is left as an exercise to the reader. We apply Theorem 2.4, and thus $\mathscr{M}_q$ equipped with weak homotopy equivalences, retracts of relative cell complexes, and Serre fibrations is a model structure, in fact a compactly generated model structure with generating cofibrations $\mathcal{I}$ and generating acyclic cofibrations $\mathcal{J}$. □

Theorem 3.8. *The category of topological spaces equipped with weak equivalences $\mathscr{W}_h$, cofibrations $\mathscr{C}_h$, and fibrations $\mathscr{F}_h$ is a model category.*

Not only do both of these model structures on topological spaces exist, but they are equivalent (a precise notion of equivalence of model structures is discussed in Section 8). A proof of this equivalence can be found in [1], and while interesting, the equivalence between the model categories of topological spaces and simplicial sets is more surprising. We will now develop this equivalence, beginning with establishing a means of traversing between these model categories (and, of course, we will define the model category of simplicial sets).

4. Functors between *Top* and *sSet*

First, recall the formal definition of a simplicial set:

Definition 4.1. A *simplicial set* S is a sequence of sets S_n (n-simplices) with face maps $d_i : S_n \to S_{n-1}$ and degeneracy maps $s_j : S_n \to S_{n+1}$ satisfying the following identities:

(i) $d_i \circ d_j = d_{j-1} \circ d_i$ if $i < j$
(ii) $s_i \circ s_j = s_j \circ s_{i-1}$ if $i > j$
(iii) $d_i \circ s_j = s_{j-1} \circ d_i$ if $i < j$
(iv) $d_i \circ s_j = \mathrm{id}$ if $i = j$ or $i = j + 1$
(v) $d_i \circ s_j = s_j \circ d_{i-1}$ if $i > j + 1$

Example 4.2. We will make use of a covariant functor sending $[n]$ to the standard topological n-simplex:

$$\Delta_n = \left\{(t_0, ..., t_n) \mid 0 \le t_i \le 1, \sum_{i=0}^{n} t_i = 1\right\}$$

with maps $\delta_i : \Delta_{n-1} \to \Delta_n$:

$$\delta_i(t_0, ..., t_{n-1}) = (t_0, ..., t_{i-1}, 0, t_i, ..., t_{n-1})$$

and maps $\sigma_j : \Delta_{n+1} \to \Delta_n$:

$$\sigma_j(t_0, ..., t_{n+1}) = (t_0, ..., t_{j-1}, 0, t_{j+1}, ..., t_{n+1})$$

Simplicial sets are the objects of a category *sSet* (where the morphisms are natural transformations). We can define adjoint functors between *sSet* and the category *Top* of topological spaces. These will be called the *geometric realization* and *singular simplicial complex* functors.

First, we define the geometric realization of a simplicial set K. We get a functor $T : sSet \to Top$ defined as follows:

Definition 4.3. The *geometric realization* of a simplicial set K, denoted $T(K)$ or $|K|$, is given by

$$T(K) = \coprod_{q \ge 0} \left(K_q \times \Delta_q\right)/\sim$$

where K has the discrete topology and $\sim$ is the equivalence relation generated by:

$$(d_i k_q, x_{q-1}) \sim (k, \delta_i x) \text{ and } (s_i k_q, x_{q+1}) \sim (k, \sigma_i x)$$

for every $q \ge 0$.
(Here, $k_q \in K_q$, $x_{q-1} \in \Delta_{n-1}$, $x_{q+1} \in \Delta_{n+1}$, d_i and s_i are the usual face and degeneracy maps on K, and δ_i and σ_i are the maps defined in Example 4.2).

It is interesting to note that this topological space $T(K)$ that we have constructed is a CW complex, with one n-cell for each non-degenerate simplex of K. (A simplex is non-degenerate so long as it is not in the image of some degeneracy map s_i.) A formal proof of this result can be found in [2].

Now we define an adjoint functor $S : Top \to sSet$. For a topological space X, $S(X)$ is called the total singular complex of X. Again, S will make use of Example 4.2:

Definition 4.4. Given a topological space X, the *total singular complex* $S(X)$ consists of q-simplices:

$$S_q(X) = \{f : \Delta_q \to X \mid f \text{ is continuous}\}$$

Our face maps $d_i : S_q(X) \to S_{q-1}(X)$ and degeneracy maps $s_i : S_q(X) \to S_{q+1}(X)$ are given by:

$$d_i = f \circ \delta_i \text{ and } s_i = f \circ \sigma_i$$

It is left to the reader to check that these maps satisfy the desired identities.

Now we claim that S and T are an *adjoint* pair, i.e. that we have an isomorphism

$$\mathrm{Hom}_{Top}(T(K), X) \cong \mathrm{Hom}_{sSet}(K, S(X))$$

Construction 4.5. We construct a bijection between maps $T(K) \to X$ and maps $K \to S(X)$. Let $\varphi : T(K) \to X$ and $\psi : K \to S(X)$, and

$$\varphi(k_q)(x_q) \longleftrightarrow \psi|k_q, x_q|$$

illustrates this bijection.

We will assume some basic properties of S and T in Section 7, and a more detailed discussion of these functors can be found in [2] and [8], but for the sake of brevity we proceed to constructing a model structure on *sSet*.

5. A Model Category of Simplicial Sets

We now provide constructions of the morphism classes in the model category $\mathscr{M}_s$ of simplicial sets. It is possible to define the weak equivalences topologically, namely as maps whose geometric realizations are weak homotopy equivalences. However, in this paper, we seek to construct the model of category of simplicial sets internally, without trivial equivalence to our model categories of topological spaces, and then show that this equivalence is inherent nonetheless. Our definition of a weak equivalence will however rely on the notion of a *Kan complex*, which we define using our corresponding notion of a fibration.

Our combinatorial analog of a Serre/Hurewicz fibration is a *Kan fibration.* Before we define Kan fibrations, we define the *horn* Λ^i_n as the simplicial set obtained from the boundary of the standard n-simplex Δ_n (defined in Example 4.2), disregarding the ith face.

Definition 5.1. A morphism $p : X \to Y$ of simplicial sets is a *Kan fibration* if it satisfies the lifting property for all horn inclusions $\Lambda^i_n \to \Delta_n$, i.e. if for any horn inclusion there exists a lift $\Delta_n \to X$ making the following diagram commute:

$$\begin{tikzcd} \Lambda^i_n \arrow[r] \arrow[d] & X \arrow[d, "p"] \\ \Delta_n \arrow[r] \arrow[ur, dashed] & Y \end{tikzcd}$$

We can now also define the notion of a Kan complex:

Definition 5.2. A simplicial set X is a *Kan complex* if the map $p : X \to *$ is a Kan fibration. (Note that $*$ represents the one-point simplicial set.)

Equivalently, we may say that a simplicial set X is a Kan complex if and only if all of its horns extend to simplices, as in Definition 5.1. Sometimes, we may say "all horns have *fillers*" to mean exactly this.

We also have an analogous (to Example 3.7) description of a (locally) *trivial* Kan fibration:

Definition 5.3. A Kan fibration $p : X \to Y$ is *locally trivial* if for every simplicial map $f : \Delta_n \to Y$, the pullback fibration $p^*X : \Delta_n \times_Y X \to \Delta_n$ is isomorphic over Δ_n to a product fibration $\Delta_n \times F_y \to \Delta_n$.

Our class of fibrations $\mathscr{F}_s$ will be Kan fibrations. Before we define the weak equivalences, we must introduce the notion of homotopy groups for Kan complexes. Given a Kan complex Z with basepoint $*$, define $\pi_n(Z, *)$ to be the set of homotopy equivalence classes of based maps from $\delta\Delta^{n+1}$ into Z. The basepoint of $\delta\Delta^{n+1}$

is given by the simplicial set of Δ^{n+1} generated by a 1-simplex. Note that the equivalence classes follow from Z being a Kan complex and all homotopies are relative to the basepoint. Now we can define our weak equivalences $\mathscr{W}_s$ to be the class of maps $f : X \to Y$ such that

$$f^* : \pi(X, Z) \to \pi(Y, Z)$$

is a bijection for all Kan complexes Z. Finally, our class of cofibrations $\mathscr{C}_s$ will be the class of all maps $f : X \to Y$ such that all maps $f_q : X_q \to Y_q$ of q-simplices are injective.

We will precisely reconcile these three classes $\mathscr{W}_s, \mathscr{F}_s$, and $\mathscr{C}_s$ with the corresponding classes in a model structure of topological spaces later in the paper, but for now note that these definitions do make sense at first glance. Our notion of a weak equivalence appears analogous to a weak homotopy equivalence and our notion of a cofibration is an injection (and Definition 3.4 does imply inclusion).

But as of now, these correspondences are meaningless in the context of model categories. We first must show that these classes of morphisms actually induce a model structure on *sSet*.

Theorem 5.4. *The category of simplicial sets equipped with weak equivalences $\mathscr{W}_s$, cofibrations $\mathscr{C}_s$, and fibrations $\mathscr{F}_s$ is a model category.*

Proof. Again, we check the model category axioms and appeal to the small object argument (Theorem 2.4). It is left to the reader to verify that $\mathscr{M}_s$ is bicomplete with weak equivalences $\mathscr{W}_s$. Let $\mathcal{I}$ be the set of inclusions $\partial\Delta_n \hookrightarrow \Delta_n$ and let $\mathcal{J}$ be the set of inclusions $\Lambda_n^i \hookrightarrow \Delta_n$. Then $\mathcal{I}$ and $\mathcal{J}$ are compact, and we invoke the following, true by assumption: $\mathcal{J}^\Box = \mathscr{F}_s$ and $\mathcal{I}^\Box \subseteq \mathcal{J}^\Box = \mathscr{F}_s$.

We first prove that condition (i) of Theorem 2.4 is satisfied: every relative $\mathcal{J}$-cell complex is a weak equivalence. The geometric realization T of an element of $\mathcal{J}$ is homeomorphic to some $i : D^n \hookrightarrow D_n \times I$, so we are done by condition (i) on the category of topological spaces.

Finally we show that $\mathcal{I}^\Box = \mathcal{J}^\Box \cap \mathscr{W}_s$. First, suppose $f : X \to Y$ is in $\mathcal{J}^\Box \cap \mathscr{W}_s = \mathscr{F} \cap \mathscr{W}_s$, i.e. f is an acyclic Kan fibration. We use that SX and SY are Kan complexes and that S takes Serre fibrations to Kan fibrations (the latter fact will be implied by Proposition 7.2 and Lemma 7.3). Furthermore, we use the fact that the weak equivalences $\mathscr{W}_s$ are precisely the maps of simplicial sets whose geometric realizations are weak equivalences. Therefore, we can apply ST to f to get a Kan fibration between Kan complexes, say f', that is a weak equivalence.

There is a combinatorial analog of Whitehead's theorem (that weak homotopy equivalences between CW complexes are homotopy equivalences) that says that a weak equivalence between Kan complexes is a homotopy equivalence. A formal proof of this theorem is beyond the scope of this paper, but we assume it in this proof to get that f' is a homotopy equivalence. Thus, the fiber of f' is contractible and the retract r from the fiber of f to $*$ is in $\mathcal{I}^\Box$. Extending this to $f : X \to Y \in \mathcal{I}^\Box$ is left to the reader, but can be found in [1].

Conversely, we show $\mathcal{I}^\Box \subseteq \mathcal{J}^\Box \cap \mathscr{W}_s$. Let $f : X \to Y$ be in $\mathcal{I}^\Box \subseteq \mathscr{F}_s$. Then we have $g : Y \to X$ and:

$$\begin{tikzcd}
\varnothing \arrow[r] \arrow[d] & X \arrow[d, "f"] \\
Y \arrow[r, equal] \arrow[ur, dashed, "g"] & Y
\end{tikzcd}$$

and a lift $h : X \times I \to X$ such that the following diagram commutes:

$$\begin{tikzcd} X \times \delta I \arrow[r, "\mathrm{id}\times gf"] \arrow[d] & X \arrow[d, "f"] \\ X \times I \arrow[r] \arrow[ur, dashed, "h"] & Y \end{tikzcd}$$

This lift h is a homotopy between gf and id, and so f is a homotopy equivalence. Hence, f is in $\mathscr{W}_s$. We also have $f \in \mathcal{I}^{\square} \subseteq \mathscr{F}_s = \mathcal{J}^{\square}$, so f is in $\mathcal{J}^{\square} \cap \mathscr{W}_s$. Thus, by Theorem 2.4, $\mathscr{M}_s$ is a model category of simplicial sets, in fact a compactly generated model category with generating cofibrations $\mathcal{I}$ and generating acyclic cofibrations $\mathcal{J}$. □

6. Minimal Fibrations and Minimal Kan Fibrations

To prove that the model structures of topological spaces and simplicial sets are equivalent, we will need to show that the geometric realization functor T takes the class of Kan fibrations $\mathscr{F}_s$ to the class of Serre fibrations $\mathscr{F}_q$. Our proof of this will rely on the theory of minimal fibrations. This section provides a brief treatment of this theory, and a more thorough discussion can be found in [5] and [9].

Definition 6.1. A Kan fibration $p : X \to Y$ is *minimal* if for all cells $x, y : \Delta_n \to Y$

$$p(x) = p(y) \text{ and } \delta_i x = \delta_i y \implies \delta_k x = \delta_k y \text{ for all } k$$

We use the word minimal because this type of Kan fibration has, within its homotopy class, fibers of minimal size.

Proposition 6.2. *A minimal Kan fibration is locally trivial.*

Proof. Let $p : X \to Y$ be a minimal Kan fibration, and let f be a map from the standard topological q-simplex to Y. Consider the pullback Z, shown below:

$$\begin{tikzcd} Z \arrow[r] \arrow[d, "p^*"] & X \arrow[d, "p"] \\ \Delta_q \arrow[r, "f"] & Y \end{tikzcd}$$

It can be checked that a pullback of a minimal Kan fibration along any morphism is a minimal Kan fibration, so $p^* : Z \to \Delta_q$ is a minimal Kan fibration. Let F be the fiber of p^*. To show that p is trivial, we want to show that $Z \cong \Delta_q \times F$.

Let $H : \Delta_q \times \Delta_1 \to \Delta_q$ be a homotopy. Since p^* is a Kan fibration, we get a deformation $\widetilde{H} : Z \times \Delta_1 \to Z$ from the retraction of $r : Z \to F$ to the identity on Z such that $p^* \circ \widetilde{H} = H \circ (p^* \times i)$, where i is the identity on Δ_1.

We can now define the map $\varphi : Z \to \Delta_q \times F$ by $\varphi(z) = (p^*(z), r(z))$. The minimality of p^* implies that φ is an isomorphism. □

We state the following corollary without proof, but the omitted argument that the geometric realization of a locally trivial simplicial map is a fibration can be found in [5]:

Corollary 6.3. *The geometric realization of a minimal Kan fibration is a Serre fibration.*

We do, however, show two related results (the second relies on Corollary 6.3 and the first lemma):

Lemma 6.4. *The geometric realization of an acyclic Kan fibration is an acyclic Serre fibration.*

Proof. Let $p : X \to Y$ be an acyclic Kan fibration, i.e. a Kan fibration that is also a weak equivalence in the model category of simplicial sets. Let $q : X \times Y \to Y$ be the projection map onto Y, and let $p^* : X \to X \times Y$ consist of the identity on X and p. These maps satisfy the following commutative diagram:

$$\begin{tikzcd} X \arrow[r, "\mathrm{id}_Z"] \arrow[d, "p^*"] & X \arrow[d, "p"] \\ X \times Y \arrow[r, "q"] & Y \end{tikzcd}$$

Since p is an acyclic fibration, we get $h : X \times Y \to X$ such that the diagram below commutes:

$$\begin{tikzcd} X \arrow[r, "\mathrm{id}_Z"] \arrow[d, "p^*"] & X \arrow[d, "p"] \\ X \times Y \arrow[r, "q"] \arrow[ur, dashed, "h"] & Y \end{tikzcd}$$

i.e. such that $h \circ p^* = \mathrm{id}_Z$ and $p \circ h = q$. $T(X)$ is a retract of $T(X \times Y)$ by $T(h)$, so $T(p) = T(q)T(X)$. But it is easy to check that the geometric realization of a projection is a projection, and that a projection is a Serre fibration. Thus, $T(q)$ is a Serre fibration $\implies$ $T(p)$ is an (acyclic) Serre fibration. $\square$

Theorem 6.5. *The geometric realization of a Kan fibration is a Serre fibration.*

Proof. By the above lemma, the geometric realization of an acyclic Kan fibration is a Serre fibration, and by Corollary 6.3, the geometric realization of a minimal Kan fibration is a Serre fibration. Thus, if we can factor any Kan fibration p into the composition of an acyclic Kan fibration r and a minimal Kan fibration q, then we will be done. (It is simple to check that the composition of Serre fibrations is a Serre fibration).

Let $p : X \to Y$ be a Kan fibration. By Proposition 6.5, we can take a strong deformation retract Y' of Y and $X' \subseteq p^{-1}(Y')$ such that the restriction $q : X' \to Y'$ is a minimal Kan fibration. Let $i : X' \to X$ be the inclusion map, and $r : X \to X'$ be the retraction (if r is acyclic, we are done). Let $j : \delta\Delta_q \to \Delta_q$ be the inclusion map, and suppose we are given f, g such that the following solid diagram commutes:

$$\begin{tikzcd} \Delta_q \arrow[r, "f"] & X' \arrow[d, dotted, shift left, "i"] \\ \delta\Delta_q \arrow[u, "j"] \arrow[r, "g"] & X \arrow[u, shift left, "r"] \end{tikzcd}$$

i.e. such that $r \circ g = f \circ j$.

Maps in a category are the objects in another category (with commutative squares for morphisms), so we get a morphism $\varphi : j \to p$ given by $g, p'f$ and a morphism $\psi : j \to p$ given by $irg, p'f$. We have a lift if of ψ, so we must have a lift $\tilde{f} : \Delta_q \to X$:

$$\begin{tikzcd}
\Delta_q \arrow[r, "f"] \arrow[dr, dashed, "\tilde{f}"] & X' \arrow[d, dotted, shift left, "i"] \\
\delta\Delta_q \arrow[u, "j"] \arrow[r, "g"] & X \arrow[u, shift left, "r"]
\end{tikzcd}$$

Let $k_q \in \Delta_n$ be the q-simplex. Since p is minimal, the simplices $(r \circ \tilde{f})(k_q)$ and $f(k_q)$ in X' must be equal. Hence, $r \circ \tilde{f} = f$ and $\tilde{f} \circ j = g \implies r$ is an acyclic fibration. □

Now, in order to apply this (minimal) theory of minimal Kan fibrations, we need to be able to construct such fibrations. The notion below helps us to do this:

Definition 6.6. A deformation retract $p : X' \to Y'$ of a Kan fibration $p : X \to Y$ is *strong* if there is a homotopy H making $p : X' \to Y'$ a deformation retract such that:

$$H(u, v) = u \quad \forall q \geq 0, u \in Y'_q, v \in I_q \quad \text{and} \quad \tilde{H}(u, v) = v \quad \forall q \geq 0, u \in X'_q, v \in I_q$$

It is true that we can restrict Kan fibrations to get minimal Kan fibrations, and in fact that:

Proposition 6.7. *If $p : X \to Y$ is a Kan fibration and Y' is a strong deformation retract of Y, then there exists $X' \subseteq p^{-1}(Y)$ such that $p : X' \to Y'$ is a minimal Kan fibration.*

A proof of this can be found in [2].

7. Quillen's Equivalence

We will demonstrate Quillen's Equivalence between Quillen's model category $\mathscr{M}_q$ of topological spaces $\mathscr{W}_q, \mathscr{C}_q, \mathscr{F}_q$ and the model category $\mathscr{M}_s$ of simplicial sets $\mathscr{W}_s, \mathscr{C}_s, \mathscr{F}_s$. We begin by showing that $T : \mathscr{M}_s \leftrightarrow \mathscr{M}_q : S$ is a *Quillen pair*:

Definition 7.1. An adjoint pair of functors $F : \mathcal{M} \leftrightarrow \mathcal{N} : G$ (where $\mathcal{M}$ and $\mathcal{N}$ are model categories) is a *Quillen pair* if G preserves fibrations and acyclic fibrations.

We note that this condition on G implies a symmetric condition on F. In particular:

Proposition 7.2. *$F : \mathcal{M} \leftrightarrow \mathcal{N} : G$ is a Quillen pair if and only if F preserves cofibrations and acyclic cofibrations.*

Proof. In particular, we claim that the functor F preserves acyclic cofibrations if and only if its right adjoint functor G preserves fibrations (and, symmetrically, the same is true for cofibrations and acyclic fibrations). To see this, let $p : W \to X$ be an acyclic cofibration in $\mathcal{N}$ and $q : Y \to Z$ be a fibration in $\mathcal{M}$. Suppose we have the adjoint commutative diagrams below:

$$\begin{tikzcd}
W \arrow[r] \arrow[d, "p"] & G(Y) \arrow[d, "G(q)"] \\
X \arrow[r] & G(Z)
\end{tikzcd}
\qquad
\begin{tikzcd}
F(W) \arrow[r] \arrow[d, "F(p)"] & Y \arrow[d, "q"] \\
F(X) \arrow[r] & Z
\end{tikzcd}$$

If F preserves acyclic cofibrations, then the diagram on the right has a lift, so the diagram on the left has an adjoint lift. Thus, $G(q)$ has the right lifting property, and is thus a fibration. So G preserves fibrations. If G preserves fibrations, then the diagram on the left has a lift, so the diagram on the right has an adjoint lift. Thus, $F(p)$ is an acyclic cofibration.

Dually, F preserves cofibrations if and only if its right adjoint functor G preserves acyclic fibrations. By assumption, G preserves fibrations and acyclic fibrations, so we conclude that F preserves acyclic cofibrations and cofibrations. Thus, $F : \mathcal{M} \leftrightarrow \mathcal{N} : G$ is a Quillen pair. □

Nevertheless, to prove that T and S form a Quillen pair, we need only show the condition on T. That is, we show that the geometric realization functor takes acyclic Kan cofibrations to acyclic Serre cofibrations and takes Kan cofibrations to Serre cofibrations.

Lemma 7.3. *T preserves cofibrations and acyclic cofibrations.*

Proof. Recall from the proof of Theorem 5.4 the set of generating cofibrations $\mathcal{I}$ and the set of generating acyclic cofibrations $\mathcal{J}$ (both in $\mathscr{M}_s$). The geometric realization of an inclusion $\partial\Delta_n \hookrightarrow \Delta_n$ in $\mathcal{I}$ is a cofibration in $\mathscr{M}_q$ and the geometric realization of an inclusion $\Lambda_n^i \hookrightarrow \Delta_n$ in $\mathcal{J}$ is an acyclic cofibration in $\mathscr{M}_q$. Therefore, T takes Kan cofibrations to cofibrations and acyclic Kan cofibrations to acyclic cofibrations. □

We have shown that T preserves cofibrations and acyclic cofibrations, so by Proposition 7.2, $T : \mathscr{M}_s \leftrightarrow \mathscr{M}_q : S$ is a Quillen pair. Now we demonstrate a correspondence between the weak equivalences in each category, as well, namely that our Quillen pair is in fact a Quillen equivalence:

Definition 7.4. A Quillen pair $F : \mathcal{M} \leftrightarrow \mathcal{N} : G$ is a *Quillen equivalence* if for any cofibrant object X and fibrant object K and for any map $f : FX \to K$ and adjoint map $g : X \to GK$, f is a weak equivalence in $\mathcal{N}$ if and only if g is a weak equivalence in $\mathcal{M}$.

In our proof that T and S form a Quillen equivalence between $\mathscr{M}_q$ and $\mathscr{M}_s$, we appeal to Milnor's Theorem, proved in in Section 4.5 of [10].

Lemma 7.5 (Milnor's Theorem)**.** *For any Kan complex K, the map $\eta_K : K \to S(T(K))$ is a homotopy equivalence.*

Theorem 7.6. *$T : \mathscr{M}_s \leftrightarrow \mathscr{M}_q : S$ is a Quillen equivalence.*

Proof. Let X be a topological space and let K be a simplicial set. With respect to the model structure on simplicial sets, all objects are cofibrant and the Kan complexes are exactly the fibrant objects.

Thus, all that is left to show is that $f : S(X) \to K$ is a weak equivalence in $\mathscr{M}_s$ if and only if $g : X \to T(K)$ is a weak equivalence in $\mathscr{M}_q$. Suppose $f : S(X) \to K$ is in $\mathscr{W}_s$, i.e. that $f^* : \pi(S(X), Z) \to \pi(K, Z)$ is a bijection for all Kan complexes Z. Then

$$g^* : \pi(T(S(X)), U) \to \pi(T(K), U)$$

is a bijection for all spaces U, and thus g is a homotopy equivalence.

Conversely assume $g : X \to T(K)$ is a weak homotopy equivalence. Then for every space U, we have a bijection

$$f_* : \pi(X, U) \to \pi(T(K), U)$$

In particular, for all Kan complexes Z we have a bijection

$$f_* : \pi(S(X), Z) \to \pi(S(T(K)), Z).$$

The fact that this is a bijection follows from the homotopy equivalence of Z and $S(T(Z))$, noted in Lemma 7.5. Given this, $f : S(X) \to K$ is a weak equivalence in $\mathscr{M}_s$ and our proof is complete. □

8. Extensions

This paper served to introduce the theory of model categories, and demonstrate an equivalence between the Quillen model category of topological spaces and the model category of simplicial sets. To conclude, we provide two extensions of this introduction. The first is to explore an alternative method to construct these structures and prove these equivalences, and the second is to discuss some applications of these concepts.

In Section 5, we proposed the geometric realization functor T and the total singular complex functor S, and ultimately showed that these functors make $\mathscr{M}_q$ and $\mathscr{M}_s$ into a Quillen pair. However, Kan developed another approach, using the Ex^∞ functor, also called *Kan's fibrant replacement functor*. Define the (barycentric) subdivision $\mathrm{sd}\,\Delta^n$ as the nerve of the poset of nondegenerate simplicies, and:

$$\mathrm{sd}\ X = \mathrm{colim}_{\Delta^n \to X}\ \ \mathrm{sd}\ \Delta^n$$

This allows us to define a functor $\mathrm{Ex} : sSet \to sSet$ via

$$\mathrm{Ex}(X)_n = \mathrm{Hom}_{sSet}(\mathrm{sd}\,\Delta^n, X),$$

iterate the construction $X \to \mathrm{Ex}(X) \to \mathrm{Ex}^2(X) \to ...$, and define the colimit to be $\mathrm{Ex}^\infty(X)$. This functor Ex^∞ has several relevant properties, including:

Proposition 8.1.
(i) $Ex^\infty(X)$ is a Kan complex.
(ii) $p_X : X \to Ex^\infty(X)$ is an acyclic cofibration.
(iii) Ex^∞ preserves fibrations.
(iv) Ex^∞ is a natural weak equivalence.

In addition to omitting a formal discussion of this functor, we have explored Quillen's equivalence without the use of *anodyne extensions*, which several other sources detail. The theory of anodyne extensions is discussed thoroughly in [5], and can be used to show that the geometric realization of a Kan fibration is a Serre fibration. They are, however, unnecessary given the notion of acyclic cofibrations.

Now that we have introduced model categories, there are beautiful applications. Model categories provide a general space in which to work and prove expansive results, such as:

Theorem 8.2 (Generalized Whitehead's Theorem)**.** *If M is a model category and $X, Y \in M$ are both fibrant and cofibrant, then X and Y are weakly equivalent if and only if they are homotopy equivalent.*

Acknowledgments. It is a pleasure to thank Professor Peter May for organizing an insightful series of lectures in algebraic topology.

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