A BRIEF OVERVIEW OF ALEXANDROV SPACES

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Abstract. This paper will discuss the Alexandrov spaces, an interesting kind of topological space that can be analyzed as a preorder. We will begin by defining what an Alexandrov space *is*, then discuss how we can analyze its topological properties as part of a preorder. Finally, we shall finish up with a quick look at some interesting topological properties of Alexandrov spaces that distinguish them from an arbitrary topological space.

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1. Introduction

In a discussion of an arbitrary topological space, an important axiom for a topology is that a finite intersection of open sets must result in an open set- however, what if we discuss a topology such that an *arbitrary* intersection of open sets is open? Common examples of this kind of space are often far from the mind of the everyday topologist (think about the usual topology on $\mathbb{R}$- it is not Alexandrov as the intersection of all open sets containing some point is the singleton set of that point, which is not open), and the more our discussion develops the more we shall find that these spaces do, indeed, maintain some very interesting properties.

We call such spaces *Alexandrov Spaces*; soon, we shall show that we can view each Alexandrov space as a preorder and that, in fact, Alexandrov spaces and preorders are, essentially, the same. Through this lens, we find a very interesting way to analyze topology: as a preorder. We can even use this to find equivalences between ideas grounded in preorders and ideas in topology, and how an ordering-based concept can imply a topological concept (and vice versa). Simultaneously, we can find Alexandrov spaces to maintain some very interesting properties due to the laxity of the intersection condition of topologies (such as the equivalence of connectedness and path connectedness in these open spaces), which shall be discussed.

Note that a thorough understanding of this paper will necessitate a solid understanding of the basics of order and topology.

2. Preliminaries

In this section, we will briefly go over a few definitions and introduce some notations that shall ease our discussion of Alexandrov spaces (as well as make rigorous some definitions discussed above).

Definition 2.1. A topological space X with a topology τ is an *Alexandrov Space* if, for any $\mathcal{T} \subset \tau$,

$$\bigcap_{T \in \mathcal{T}} T \in \tau.$$

Definition 2.2. In a space X with a topology τ, for all $x \in X$ let

$$U_x = \bigcap_{\substack{U \text{ open} \\ x \in U}} U.$$

Notice that, in a Alexandrov space, all U_x are open, as they are the intersection of open sets. This definition of U_x is vital to our discussion of Alexandrov spaces, as it becomes a very useful vehicle for analyzing the spaces.

Throughout this paper, if A is used without being explicitly defined, regard it as some Alexandrov space.

3. Defining a Preorder on Alexandrov Spaces

As mentioned above, we shall be showing that Alexandrov spaces and preorders are, essentially, equivalent. In the next two definitions, we define what a preorder is and then construct a preorder on any given Alexandrov space.

Definition 3.1. In any set A, a binary relation $\leq$ is a *preorder* on A if $\leq$ is both reflexive and transitive. In other words, if $x \leq x$ for all $x \in A$ and, for all $x, y, z \in A$ such that $x \leq y$ and $y \leq z$, we also have $x \leq z$.

Definition 3.2. In an Alexandrov space X, let $\leq$ be the binary relation such that if $x_1, x_2 \in X$, $x_1 \leq x_2$ if $U_{x_1} \subset U_{x_2}$.

Now that we have our binary relation defined, we demonstrate that it is a preorder.

Proposition 3.3. *The binary relation $\leq$ on an Alexandrov spaces X is a preorder.*

Proof. For all $x \in X$, $U_x \subset U_x$, so $x \leq x$ and thus reflexivity is satisfied.

Now, if $x, y, z \in X$ such that $x \leq y$ and $y \leq z$, then $U_x \subset U_y$ and $U_y \subset U_z$; as subset is a transitive relation, $U_x \subset U_z$, so $x \leq z$, and thus transitivity of $\leq$ is maintained, so $\leq$ as defined is a preorder. □

Through the preceding proposition, we see that every Alexandrov space can be viewed as a preordered set; conversely, every preordered set can be viewed as an Alexandrov space. To do so, take any preordered set X and let $\mathcal{B} = \{\{x \in X : x \leq k\} : k \in X\}$ be a base for a topology τ on X (this is, in fact, an Alexandrov space!). Looking at our defined operations for defining a preorder on an Alexandrov space and constructing an Alexandrov space using a preorder, we see that these are inverse operations- so, we see our equivalence of Alexandrov Spaces and preorders! (In categorical language, we would see that we have an isomorphism of categories.)

4. Separability

We begin a brief discussion of T_0 and T_1 topological spaces (known as two of the "separation axioms"), which will become important distinctions in some of our following results.

Definition 4.1. A topological space X is T_0 if, for all $x_1, x_2 \in X$, there exists an open set containing exactly one of the two points.

Most see the T_0 separation axiom as a relatively weak assumption of a space in a discussion of topology; essentially, in a T_0 space, every point is "topologically distinct", by which we mean that, given two points in the space, there is at least one open set that contains one point and not the other, and so the topology has a way of seeing a difference between the two points. We now see how this idea of topological distinctiveness naturally leads into a discussion of partial orders.

Definition 4.2. A binary relation $\leq$ on a set X is a partial order if it is a preorder and it satisfies the property: if $x, y \in X$ such that $x \leq y$ and $y \leq x$, then $x = y$.

This definition of a partial order says, essentially, that points can be distinguished by the partial order. Thus, with a clear similarity to the T_0 separation axiom, we find the following useful result showing that a T_0 Alexandrov space produces a partial order through our preorder described above.

Proposition 4.3. *An Alexandrov space X is T_0 if and only if $\leq$ is a partial order on X.*

Proof. Let $x, y \in X$ such that $x \neq y$. As X is T_0, there exists some open set $U \subset X$ such that $x \in U$ and $y \notin U$ or $y \in U$ and $x \notin U$. Without loss of generality, $x \in U$ and $y \notin U$. As U is open, by the definition of U_x and the nature of set intersection, $U_x \subset U$. Therefore, as $y \notin U$, we have $y \notin U_x$; as $y \in U_y$ by definition, we see that $U_x \not\subset U_y$, so $x \not\leq y$. Therefore, we see that $x \neq y$ implies that $x \not\leq y$ or $y \not\leq x$. Thus, by contrapositive, $x \leq y$ and $y \leq x$ implies that $x = y$, and so we see that $\leq$ denotes a partial order in a T_0 space.

Now, for the other direction of the proof, assume that $\leq$ is a partial order. Now, fix $x, y \in X$ such that $x \neq y$. So, either $x \not\leq y$ or $y \not\leq x$; without loss of generality, $x \not\leq y$. Thus, $U_x \not\subset U_y$. Similarly to the previous paragraph, we can show that this implies that $x \notin U_y$; so, we see that X is T_0, as $y \in U_y$ is open and $x \notin U_y$. □

We now delve into T_1 spaces, and see why they can be rather uneventful in a discussion of Alexandrov spaces.

Definition 4.4. A topological space X is T_1 if, for all $x_1, x_2 \in X$, there exists an open set $U_1 \subset X$ such that $x_1 \in U_1$ and $x_2 \notin U_1$ as well as an open set $U_2 \subset X$ such that $x_1 \notin U_2$ and $x_2 \in U_2$.

An intuition for T_1 is that any point can be separated from another; essentially, any point can distance itself from another as it is in an open set not containing the other. The following proposition demonstrates why this property, combined with the Alexandrov property, can make a discussion of a topological space somewhat lacking.

Proposition 4.5. *If a topological space X is both T_1 and Alexandrov, then the topology on it is discrete.*

Proof. Fix $x \in U$. Now, fix some $y \in X$ such that $y \neq x$. As X is T_1, there exists some open $U \subset X$ such that $x \in U$ and $y \notin U$. By definition, $U_x \subset U$; thus, $y \notin U_x$. Therefore, as U_x is the intersection of all open sets containing x, we see that $U_x = \{x\}$. As X is Alexandrov, U_x is open; so, $\{x\}$ is open. Thus, we see that all singleton sets are open, and therefore all subsets of X are open (as all sets are unions of singleton sets, which are all open). So, we see that the topology on X is the discrete topology. □

By Proposition 4.5, we see that it is rather trivial to discuss T_1 Alexandrov spaces, as they are all the discrete topology, and thus no pair of points is comparable. Therefore, our discussion will (for the most part) avoid T_1 spaces and center around T_0 spaces. Although many of our theorems capture the general case, it is often useful to discuss only T_0 spaces, which is not asking for too much as every space is homotopy equivalent to some T_0 space.

5. Analyzing Topology as a Preorder

5.1. **Openness/Closedness.** Here we will rigorously define and discuss some concepts in topology and what they mean in the context of Alexandrov Spaces.

Definition 5.1. In a topological space X, a set $V \subset X$ is *closed* if there exists some open $U \subset X$ such that $V = X \setminus U$.

Note that, in all topological spaces, arbitrary intersections and finite unions of closed sets are closed. What follows is a property of the Alexandrov space that follows fairly directly from the definition of Alexandrov spaces.

Proposition 5.2. *Arbitrary unions of closed sets are closed in an Alexandrov space.*

Proof. Let X be an Alexandrov space, and let $\mathcal{V}$ be collection of closed sets in X. For each $V \in \mathcal{V}$, there exists some open $U_V \subset X$ such that $V = X \setminus U$. Let $\mathcal{U} = \{U_V : V \in \mathcal{V}\}$. Observe that

$$\bigcup_{V \in \mathcal{V}} V = \bigcup_{U \in \mathcal{U}} X \setminus U = X \setminus \bigcap_{U \in \mathcal{U}} U.$$

As X is Alexandrov, $\bigcap_{U \in \mathcal{U}} U$ is open (as $\mathcal{U}$ is a collection of open sets), and thus $X \setminus \bigcap_{U \in \mathcal{U}} U = \bigcup_{V \in \mathcal{V}} V$ is closed □

The next theorem gives us an intuition for how open sets "look" in the context of our partial order.

Theorem 5.3. *Let X be an Alexandrov space. A set $U \subset X$ is open if and only if for all $x \in U$, if $y \leq x$, $y \in U$.*

Proof. Let $U \subset X$ be open. Suppose that $x \in U$, and let $y \in X$ such that $y \leq x$. So, $U_y \subset U_x$; as $y \in U_y$ by construction, $y \in U_x$. As $x \in U$, by construction, $U_x \subset U$, so, as $y \in U_x$, we find that $y \in U$.

Let $U \subset X$ be such that it satisfies the property: for all $x \in U$, if $y \leq x$, $y \in U$. We show that $U = \bigcup_{x \in U} U_x$. For all $y \in U$, as $y \in U_y$, $y \in \bigcup_{x \in U} U_x$; so, $U \subset \bigcup_{x \in U} U_x$. Furthermore, if $y \in \bigcup_{x \in U} U_x$, then $y \in U_x$ for some $x \in U$, so $U_y \subset U_x$, and thus $y \leq x$; by hypothesis, that implies that $y \in U$. Therefore, $\bigcup_{x \in U} U_x \subset U$, and thus $\bigcup_{x \in U} U_x = U$. As $\bigcup_{x \in U} U_x$ is open (as it is the union of open sets), U is open. □

Theorem 5.4. *Let X be an Alexandrov space. A set $V \subset X$ is closed if and only if for all $x \in V$, if $x \leq y$, $y \in V$.*

Proof. Using Proposition 5.2, this proof is essentially the same proof as Theorem 5.3. □

5.2. **Limit Points / Sequences.**

Definition 5.5. In a topological space X, a point $x \in X$ is called a *limit point* of a set $B \subset X$ if, for all open $U \subset X$ such that $x \in U$, we have $U \cap (B \setminus \{x\}) \neq \varnothing$. We denote $\mathrm{LP}(B) = \{x \in X : x \text{ is a limit point of } B\}$.

In an Alexandrov space, seeing whether or not a point is a limit point of a set is rather efficient. If $B \subset A$ and $x \in A$, to see if $x \in \mathrm{LP}(B)$, we find that $x \in \mathrm{LP}(B)$ if and only if $U_x \cap (B \setminus \{x\}) \neq \varnothing$, for U_x is open and a subset of all open sets containing x. Now, we look at what a limit point looks like in context of our preorder.

Theorem 5.6. *Let X be an Alexandrov space. If $B \subset X$, $x \in LP(B)$ if and only if there exists some $b \in B$ such that $b \neq x$ and $b \leq x$.*

Proof. Let $x \in \mathrm{LP}(B)$. By the above, $U_x \cap (B \setminus \{x\}) \neq \varnothing$; so, there exists some $b \in U_x \cap (B \setminus \{x\})$. As $b \in U_x$, we see that $U_b \subset U_x$, so $b \leq x$; furthermore, as $b \in B \setminus \{x\}$, $b \in B$ and $b \neq x$.

Let $x \in X$ such that there exists some $b \in B$ for which $b \neq x$ and $b \leq x$. So, $b \in B \setminus \{x\}$, and furthermore $U_b \subset U_x$, so $b \in U_x$; thus, $b \in U_x \cap (B \setminus \{x\})$, and so $U_x \cap (B \setminus \{x\}) \neq \varnothing$. So, $x \in \mathrm{LP}(B)$. □

Notice that this theorem implies that minimal elements of A cannot be limit points of any subset of A (a minimal element being some $a \in A$ such that there exists no $b \in A \setminus \{a\}$ such that $b \leq a$).

Now, we define sequence convergence, which are closely related topics to limit points.

Definition 5.7. We say a sequence a_n in X *converges to* a for some $a \in X$ if, for all open sets $a \in U$, there exists some $N \in \mathbb{N}$ such that $\{a_n, a_{n+1}, \cdots\} \subset U$.

Now, we see what it means to converge in an Alexandrov space.

Theorem 5.8. *Let X be an Alexandrov space. Then a_n is a sequence in X that converges to $a \in X$ if and only if there exists some $N \in \mathbb{N}$ such that, if $n \geq N$, then $a_n \leq a$.*

Proof. Let a_n be a sequence in X that converges to $a \in X$. Thus, as $a \in U_a$, there exists some $N \in \mathbb{N}$ such that $\{a_n : n \in \mathbb{N}, n \geq N\} \subset U_a$. Thus, if $n \in \mathbb{N}$ and $n \geq N$, $a_n \in U_a$, so $a_n \leq a$.

Let a_n be a sequence in X and $a \in X$ such that there exists some $N \in \mathbb{N}$ for which, if $n \in \mathbb{N}$ and $n \geq N$, then $a_n \leq a$. So, if $a' \in \{a_n, a_{n+1}, \cdots\}$, then $a_n \leq a$, and thus $a_n \in U_a$. So, $\{a_n : n \in \mathbb{N}, n \geq N\} \subset U_a$. Thus, if $U \subset X$ is open and $a \in U$, as $U \supset U_a$, we have $\{a_n : n \in \mathbb{N}, n \geq N\} \subset U$. So, a_n converges to a. □

Note that a simple application of Theorem 5.8 is that, if a_n is a sequence in an Alexandrov space that converges to some a, then, for all $a \leq a'$, a_n converges to a'.

5.3. **Continuity.**

Definition 5.9. If X and Y are topological spaces, we say that $F\colon X \to Y$ is *continuous* if, for all open $U \subset Y$, we have that $F^{-1}(U)$ is open in X (where $F^{-1}(U) = \{x \in X\colon F(x) \in U\}$).

Definition 5.10. If A and B are preordered sets, a function $f\colon A \to B$ is *increasing* if, for all $a_1, a_2 \in A$ such that $a_1 \leq a_2$, we have $f(a_1) \leq f(a_2)$.

Theorem 5.11. *If A and B are Alexandrov spaces, a function $f\colon A \to B$ is continuous if and only if it is increasing.*

Proof. Assume that f is continuous. Now, suppose that $a_1, a_2 \in A$ such that $a_1 \leq a_2$. As $U_{f(a_2)}$ is open in B, as $f^{-1}(U_{f(a_2)})$ is open in A (for f is continuous). We know that $f(a_2) \in U_{f(a_2)}$, so $a_2 \in f^{-1}(U_{f(a_2)})$, and thus $U_{a_2} \subset f^{-1}(U_{f(a_2)})$. As $a_1 \leq a_2$, we see that $a_1 \in U_{a_2}$, and thus, $a_1 \in f^{-1}(U_{f(a_2)})$; so, $f(a_1) \in U_{f(a_2)}$. Therefore, $U_{f(a_1)} \subset U_{f(a_2)}$, and so $f(a_1) \leq f(a_2)$. Therefore, f is increasing.

Assume that f is increasing. Now, let $U \subset B$ be open. Suppose that $y \in f^{-1}(U)$ and $x \in A$ such that $x \leq y$. As f is increasing, we see that $f(x) \leq f(y)$, so $U_{f(x)} \subset U_{f(y)}$ (specifically, $f(x) \in U_{f(y)}$). Furthermore, as $y \in f^{-1}(U)$, $f(y) \in U$, so $U_{f(y)} \subset U$, and thus $f(x) \in U$. Therefore, $x \in f^{-1}(U)$. Thus, by Theorem 5.3, $f^{-1}(U)$ is open in A, and therefore f is continuous. □

5.4. **Connectedness.**

Definition 5.12. A topological space A is *connected* if there do not exist non-empty, disjoint, and open $B, B' \subset A$ such that $B \cup B' = A$.

Now, in order to discuss connectedness in Alexandrov spaces, we look at another form of the idea of connectedness

Definition 5.13. Let X be a topological space. We say that $f\colon [0,1] \to X$ *is a path from* x_1, x_2 if f is continuous, $f(0) = x_1$, and $f(1) = x_2$. X is *path connected* if there exists a path between any two points in X.

By looking at the definitions of connectedness and path connectedness, we find that, in an Alexandrov space, they are one and the same.

Theorem 5.14. *A is connected if and only if it is path connected.*

Proof. Let A be a connected Alexandrov space. Now, suppose for contradiction that it is not path connected. Thus, there exists two points $a_1, a_2 \in A$ such that there does not exist a path between a_1 and a_2. Let $A_1 = \{a \in A\colon$ there exists a path from a to $a_1\}$ and $A_2 = \{a \in A\colon$ there does not exist a path from a to $a_1\}$. Clearly, both are non-empty, as $a_1 \in A_1$ and $a_2 \in A_2$. Furthermore, let $\mathcal{A}_1 = \bigcup_{a \in A_1} U_a$ and $\mathcal{A}_2 = \bigcup_{a \in A_2} U_a$. Similarly to A_1 and A_2, $\mathcal{A}_1$ and $\mathcal{A}_2$ are non-empty, and furthermore, they are both open and $\mathcal{A}_1 \cup \mathcal{A}_2 = A$. Now, assume that $\mathcal{A}_1$ and $\mathcal{A}_2$ are not disjoint; thus, there exists some $a \in \mathcal{A}_1 \cap \mathcal{A}_2$. As A_1 and A_2 are clearly disjoint, either $a \in A_1$ or $a \in A_2$. Suppose that $a \in A_1$; so, $a \notin A_2$. Therefore, there must be some $a' \in A_2$ such that $a \in U_{a'}$ (since $a \in \mathcal{A}_2$). As $a \in A_1$, there exists some path f between a_1 and A, and observe that, if $g\colon I \to A$ is such that, for all $t \in I$,

$$g(t) = \begin{cases} a & t \in [0,1) \\ a' & t = 1 \end{cases},$$

then g is a path between a and a'. So, notice we can build a path between a_1 and a' by combining f and g; however, this is a contradiction, as $a' \in A_2$. We can find a similar contradiction if $a \in A_2$. So, $\mathcal{A}_1 \cap \mathcal{A}_2 = \varnothing$. This, however, makes it so that A is disconnected (as $\mathcal{A}_1$ and $\mathcal{A}_2$ are non-empty, open, disjoint, and fill the space A), which is a contradiction. So, A is path connected.

Let A be a path-connected topological space (note that this direction holds even if the space is not Alexandrov!). Now, assume that A is disconnected. Thus, there exist non-empty, disjoint, open sets $B, B' \subset A$ such that $B \cup B' = A$. As neither set is empty, there exists $b \in B$ and $b' \in B'$, and, further, as A is path-connected, there exists a path f from b to b'. Notice $0 \in f^{-1}(\{b\}) \subset f^{-1}(B)$ and $1 \in f^{-1}(\{b'\}) \subset f^{-1}(B')$;

furthermore, as f is continuous and $B \cap B' = \varnothing$, $f^{-1}(B)$ and $f^{-1}(B')$ are disjoint and open. Finally, $f^{-1}(B) \cup f^{-1}(B') = f^{-1}(B \cup B') = f^{-1}(A) = I$. So, we see by $f^{-1}(B)$ and $f^{-1}(B')$ that I is disconnected. However, this is false; so, we have a contradiction, and thus A is connected. □

We come to an interesting result regarding the implications of continuity for maps on A.

Theorem 5.15. *Let X be a T_1 topological space, A be a finite space. Then, $f\colon A \to X$ be a continuous function if and only if $a_1, a_2 \in A$ such that there exists a path F from a_1 to a_2, then $f(a_1) = f(a_2)$ (in other words, f is constant on path components).*

Proof. First of all, notice that all finite spaces are Alexandrov, and thus we have that A is an Alexandrov space.

Assume for the sake of contradiction that $f(a_1) \neq f(a_2)$. Let

$$P = \{a \in A\colon \text{there exists a path from } a_1 \text{ to } a\}.$$

Now, P is clearly path connected, and thus, P is connected. Let g be f restricted to P; observe as f is continuous, g is continuous. Let $Q = g(P)$; as $P \subset A$ and A is finite, P is finite, so Q is finite. Thus, as X is T_1, we can create a pairwise disjoint collection of open sets $\mathcal{U}_q$ indexed by Q such that $q \in \mathcal{U}_q$ for each $q \in P$. Let $R_1 = g^{-1}(\mathcal{U}_{g(a_1)})$ and let

$$R_2 = \bigcup_{\substack{q \in Q \\ q \neq g(a_1)}} g^{-1}(\mathcal{U}_q)$$

Observe that $R_1 \cap R_2 = \varnothing$, $R_1 \cup R_2 = P$, and both R_1 and R_2 are open as g is continuous. Furthermore, neither is empty, as $a_1 \in R_1$ and $a_2 \in R_2$. So, P is disconnected, which contradicts what is stated above; so, it must be true that $f(a_1) = f(a_2)$.

Now, let $f\colon A \to X$ be constant on path components. Let $P \subset A$ be a path component; let $a_1 \in P$ and $a_2 \in A$ such that $a_2 \leq a_1$. We see that a_1 and a_2 have a path between them as the function $P\colon I \to A$ defined such that, for all $t \in I$,

$$P(t) = \begin{cases} a_1 & t \in [0, 1) \\ a_2 & t = 1 \end{cases}$$

is continuous. Therefore, $a_2 \in P$. By Theorem 5.3, this implies that P is open, and so all path components are open. As path components map to singletons, we see that the preimage under f of a singleton in X is either empty or a union of path components; in either case, the preimage of a singleton is open. So, the preimage under f of any subset of X is open, and so trivially f is continuous. Note that this direction holds without necessitating that X is T_1, but just some arbitrary topological space. □

We shall discuss this theorem again later, after looking at finite paths.

5.5. **Countability.** We discuss certain properties of Alexandrov spaces in the context of countability, and introduce some ideas that abstract beyond the discussion of countability. First, we define some metrics for the relative countability of sets.

Definition 5.16. We say that a topological space X is *first countable* if, for any $x \in X$, there exists a countable collection $\mathcal{B}$ of open sets such that, for any open set $x \in U$, there exists some $B \in \mathcal{B}$ such that $x \in B \subset U$ (even if not countable, we call such a $\mathcal{B}$ a *local base about x*).

Furthermore, a topological space X is *second countable* if there exists a countable collection $\mathcal{B}$ of open sets such that for any open set $U \subset X$ there exists some $\mathcal{V} \subset \mathcal{B}$ such that $\bigcup_{V \in \mathcal{V}} V = U$ (even if not countable, we call such a $\mathcal{B}$ a *base* for the topology).

Proposition 5.17. *Every Alexandrov space is first countable.*

Proof. Let X be an Alexandrov space. Now, fix $x \in X$. Let $\mathcal{B} = \{U_x\}$. Let $U \subset X$ be some open set such that $x \in U$; by construction, $x \in U_x \subset U$. Therefore, $\mathcal{B}$ is a finite (and thus countable) local base about x, so we see that X is first countable. □

The following theorem is an interesting result that abstracts well to general discussions of cardinality in Alexandrov spaces.

Theorem 5.18. *In an Alexandrov space X, $\mathcal{U} = \{U_x : x \in X\}$ is a unique minimal base for the topology (as in, if $\mathcal{B}$ is a base for the topology on X, then $\mathcal{U} \subset \mathcal{B}$).*

Proof. By the definition of U_x, $\mathcal{U}$ is a base on the topology on X; if $x \in X$ and $U \subset X$ is open such that $x \in U$, we see that $x \in U_x \subset U$ and $U_x \in \mathcal{U}$.

Now, let $\mathcal{B}$ be a base for the topology on X. Fix $U_x \in \mathcal{U}$; as $x \in U_x$ and $\mathcal{B}$ is a base, then there exists some $B \in \mathcal{B}$ such that $x \in B \subset U_x$. Now, as B is open, $U_x \subset B$; so, $U_x = B$, and thus $U_x \in \mathcal{B}$. So, $\mathcal{U} \subset \mathcal{B}$. □

Corollary 5.19. *If X is a second countable, T_0 Alexandrov space, then X is countable.*

Proof. As X is second countable, there exists some countable base $\mathcal{B}$. By Theorem 5.18, we see that $\mathcal{U} \subset \mathcal{B}$, so $\mathcal{U}$ is countable. Furthermore, as X is T_0, for each $x, y \in X$ such that $x \neq y$, $U_x \neq U_y$; thus, observe that, if $f : X \to \mathcal{U}$ such that $f(x) = U_x$ for all $x \in X$, f is a bijection. Therefore, as $\mathcal{U}$ is countable, X is countable. □

In the other direction, we see that a countable Alexandrov space X is second countable, for we can use $\{U_x : x \in X\}$ as a countable base for the topology.

5.6. **Compactness.**

Definition 5.20. If X is a topological space, set $B \subset X$ is *compact* if for every open cover of B there exists a finite subcover.

We discuss the characteristics of compact sets in Alexandrov spaces. In order to do so, let a *maximal element* of a preordered set A be some $a \in A$ such that if $a' \in A$ is comparable to a, then $a' \leq a$.

Theorem 5.21. *A set $B \subset A$ is compact if and only if B contains only a finite number of topologically distinguishable maximal elements and each element is less than or equal to some maximal element.*

Proof. Let $B \subset A$ be compact. Fix $b \in B$. Let $\mathcal{U} = \{U_x : x \in B\}$, and $\mathcal{U}' \subset \mathcal{U}$ be a finite subcover. Furthermore, observe that there exists some finite set $U \subset B$ such that $\{U_x : x \in U\} = \mathcal{U}'$. Now, there exists some $x_0 \in U$ such that $b \in U_{x_0}$; so, $b \leq x_0$. Now, if x_0 is a maximal element, we are done. If not, then there exists some $x \in B$ such that $x_0 < x$, and, as $\mathcal{U}'$ covers B, there exists some $x_1 \in U$ such that $x \in U_{x_1}$, and thus $x \leq x_1$, and so $x_0 < x_1$. As $\mathcal{U}'$ and U are finite, we can continually use this process to eventually find some maximal element $x_N \in U$ such that $b \leq x_N$ (so, all elements are less than or equal to some maximal element).

We show that B has a finite number of topologically distinguishable maximal elements; we can phrase this as there exists some finite set $M \subset B$ such that, if $b \in B$ is a maximal element, there exists some $m \in M$ such that $U_m = U_b$. Let $\mathcal{U} = \{U_x : x \in B\}$; as B is compact and $\mathcal{U}$ is an open cover of B, there exists some finite open cover $\mathcal{U}' \subset \mathcal{U}$. Observe that there exists a finite set $U \subset B$ such that $\{U_x : x \in U\} = \mathcal{U}'$; let $M = \{m \in U : m \text{ is a maximal element}\}$. Let $m \in B$ be a maximal element. As $\mathcal{U}'$ is an open cover of B, we see by construction of U that there exists some $b \in U$ such that $m \in U_b$; as m is a maximal element, notice that b must also be a maximal element, so $b \in M$ and, furthermore, $U_m = U_b$.

Now, let $B \subset A$ be some set such that for every $b \in B$ there exists some maximal $m \in B$ for which $b \leq m$ and B has a finite number of topologically distinguishable maximal elements. Let $M \subset B$ be a finite set of maximal elements as described in the preceding paragraph. Let $\mathcal{U}$ be an open cover of B. Fix $m \in M$; as $\mathcal{U}$ covers B, there exists some $U \in \mathcal{U}$ such that $m \in U$; for each $m \in M$, let $U^{(m)}$ be such a set. Let $\mathcal{U}' = \{U^{(m)} : m \in M\}$. Observe that $\mathcal{U}' \subset \mathcal{U}$ and that $\mathcal{U}'$ is finite. Now, fix $b \in B$. There exists some maximal element $m' \in B$ such that $b \leq m'$, and furthermore, there exists some $m \in M$ such that $U_m = U_{m'}$,

and so, as $b \in U_{m'}$, $b \in U_m$. As $U^{(m)}$ is open and contains m, $U_m \subset U^{(m)}$, and thus $b \in U^{(m)}$; as $U^{(m)} \in \mathcal{U}'$, we see that $\mathcal{U}'$ is a finite open subcover of $\mathcal{U}$, and so B is compact.

□

6. Analyzing Paths within our Defined Preorder

In this section, we analyze paths within Alexandrov spaces, and notice interesting ways in which they can be analyzed.

Definition 6.1. In a set X with a preorder $\leq$, two points $x_1, x_2 \in X$ are *comparable* if $x_1 \leq x_2$ or $x_2 \leq x_1$; we denote this by $x_1 \gtreqless x_2$.

First, we have a useful construction that can help bring wandering thoughts to paper.

Theorem 6.2. *Let X be a finite topological space with the usual ordering. Then, given a sequence $a_1, \cdots, a_n \in X$ such that $a_1 \gtreqless a_2, a_2 \gtreqless a_3, \cdots, a_{n-1} \gtreqless a_n$, we can construct a path $F\colon [0,1] \to X$ such that for all $k \in \{1, \cdots, n\}$, $F\big((\frac{k-1}{n}, \frac{k}{n})\big) = \{a_k\}$.*

Proof. We proceed by induction for all $n \in \mathbb{N}$. For the base case, it is clear that the constant path $F(t) = a_1$ satisfies the theorem. Now, assume that for a sequence with n terms, it is possible to satisfy the statement. Let $a_1, a_2, \cdots, a_n, a_{n+1} \in X$ such $a_1 \gtreqless a_2, a_2 \gtreqless a_3, \cdots, a_n \gtreqless a_{n+1}$. By the inductive hypothesis, there exists some $f\colon I \to X$ that satisfies the statement for $a_1, \cdots, a_n$. Now, let $F\colon I \to X$ be such that, for $t \in I$,

$$F(t) = \begin{cases} f(t \cdot \frac{n+1}{n}) & t \in [0, \frac{n}{n+1}) \\ \max\{a_n, a_{n+1}\} & t = \frac{n}{n+1} \\ a_{n+1} & t \in (\frac{n}{n+1}, 1] \end{cases}.$$

First, we show that F is continuous. Let $U \subset X$ be open. Fix $t \in F^{-1}(U)$. We have three cases:

(1) $t \in [0, \frac{n}{n+1})$:

As f is continuous as $t \cdot \frac{n+1}{n}$ is continuous, $f(t \cdot \frac{n+1}{n})$ is continuous; thus, observe there exists some open set $R \subset I$ such that $t \in R \subset f^{-1}(U)$.

(2) $t = \frac{n}{n+1}$:

We know that $F(t) = \max\{a_n, a_{n+1}\}$. It is clear that $t \in (\frac{n-1}{n+1}, 1)$. Fix some $t' \in (\frac{n-1}{n+1}, 1)$; observe that $F(t') = a_n$ or $F(t') = a_{n+1}$. In either case, $F(t') \leq F(t)$; so, $U_{F(t')} \subset U_{F(t)}$. As $U_{F(t)} \subset U$, $U_{F(t')} \subset U$, and thus, as $F(t') \in U_{F(t')}$, $F(t') \in U$, so $t' \in F^{-1}(U)$. Therefore, $(\frac{k-1}{n}, \frac{k+1}{n}) \subset F^{-1}(U)$.

(3) $t \in (\frac{n}{n+1}, 1]$:

In this case, we see that $(\frac{n}{n+1}, 1]$ is open in I and $t \in (\frac{n}{n+1}, 1] \subset F^{-1}(U)$, as F is constant on F.

In all cases, there exists an open interval $R \subset I$ such that $t \in R \subset F^{-1}(U)$; so, F is continuous. Furthermore, as f satisfies the statement for $a_1, \cdots, a_n$, observe that $F((\frac{k-1}{n+1}, \frac{k}{n})) = \{a_k\}$ for all $k \in \{1, \cdots, n, n+1\}$. So, the statement holds for $n+1$, and thus, by induction, it holds for all $n \in \mathbb{N}$.

□

Definition 6.3. A *finite path* is a finite sequence a_i of elements in X such that $a_1 \gtreqless a_2 \gtreqless \cdots \gtreqless a_n$ (where n is last index in the sequence). We call $A\colon I \to X$ a *realization of* a_i if A is a continuous and

$$A(t) = \begin{cases} a_1 & t \in [0, \frac{1}{n}) \\ \max\{a_1, a_2\} & t = \frac{1}{n} \\ a_2 & t \in (\frac{1}{n}, \frac{2}{n}) \\ \vdots & \\ a_k & t \in (\frac{k-1}{n}, \frac{k}{n}) \\ \max\{a_k, a_{k+1}\} & t = \frac{k}{n} \\ \vdots & \\ a_n & t \in (\frac{n-1}{n}, 1] \end{cases}.$$

Notice that the function F constructed in Theorem 6.2 is a realization of a finite path.

Now, we come to a very interesting result.

Theorem 6.4. *Let X be an Alexandrov space. Given any path $F\colon I \to X$, it is possible to construct a realization of a finite path $\widetilde{F}\colon I \to X$ such that F and $\widetilde{F}$ are path homotopic.*

Proof. Let $F\colon I \to X$ be a path; further, let $\mathcal{U} = \{F^{-1}(U_x)\colon x \in X\}$. $\mathcal{U}$ is clearly an open cover of I, as each U_x is open and F is continuous. Now, for each $x \in X$, as $F^{-1}(U_x)$ is open in I, there exists some collection of open intervals $\mathcal{R}_x$ such that $\bigcup_{R \in \mathcal{R}_x} R = F^{-1}(U_x)$. Let $\mathcal{R} = \bigcup_{x \in X} \mathcal{R}_x$. Similarly to $\mathcal{U}$, $\mathcal{R}$ is an open cover of I. As I is compact, there exists a finite subcover $\mathcal{R}' \subset \mathcal{R}$. So, $\mathcal{R}'$ is a finite cover of I made up of open intervals.

Let $t_0 = 0$. Since $\mathcal{R}'$ covers I, there exists some $R_0 \in \mathcal{R}'$ such that $t_0 = 0 \in R_0$. As $R_0 \in \mathcal{R}$, there exists some $x_0 \in X$ such that $R_0 \in \mathcal{R}_{x_0}$. Now, let $t_2 = \sup R_0$. We can continue this process inductively such that $t_{k+2} = \sup R_k$ until $t_N = 1$ for some even N (as $\mathcal{R}'$ is finite, covers I, and is a collection of subsets of I, we know that the process will eventually reach this state).

Now, we define the odd t_i. Let $2k \in \{0, \cdots, N-2\}$. We know that $t_{2k+2} = \sup R_{2k}$. Furthermore, as R_{2k+2} is an open interval containing t_{2k+2}, $R_{2k} \cap R_{2k+2} \neq \varnothing$. Now, let $t_{2k+1} \in R_{2k} \cap R_{2k+2} \cap (t_{2k}, t_{2k+2})$ (as $t_{2k+2} > t_{2k}$, $t_{2k} \in R_{2k}$, $t_{2k+2} \in R_{2k+2}$, and $R_{2k} \cap R_{2k+2} \neq \varnothing$, such a t_{2k+1} must exist). Now, let $x_{2k+1} = F(t_{2k+1})$. As $R_{2k}, R_{2k+2}, (t_{2k}, t_{2k+2})$, and $F^{-1}(U_{x_{2k+1}})$ are all open and contain t_{2k+1}, $R_{2k} \cap R_{2k+2} \cap (t_{2k}, t_{2k+2}) \cap F^{-1}(U_{x_{2k+1}})$ is open and contains t_{2k+1}; so, there exists some open interval R_{2k+1} such that $t_{2k+1} \in R_{2k+1} \subset R_{2k} \cap R_{2k+2} \cap (t_{2k}, t_{2k+2}) \cap F^{-1}(U_{x_{2k+1}})$. We finally have all the tools assembled to build our path; let $\widetilde{F}\colon I \to X$ be such that, for all $t \in I$,

$$\widetilde{F}(t) = \begin{cases} x_0 & t \in [0, \inf R_1] \\ x_1 & t \in R_1 \\ x_2 & t \in [\sup R_1, \inf R_3] \\ \vdots & \\ x_{2k} & t \in [\sup R_{2k-1}, \inf R_{2k+1}] \\ x_{2k+1} & t \in R_{2k+1} \\ x_{2k+2} & t \in [\sup R_{2k+1}, \inf R_{2k+3}] \\ \vdots & \\ x_{N-1} & t \in R_{N-1} \\ x_N & t \in [\sup R_{N-1}, 1] \end{cases}$$

Notice that $\widetilde{F}$ is continuous, similarly to the $\widetilde{F}$ in Theorem 6.2, as $x_{2k} \geq x_{2k+1} \leq x_{2k+2}$ for each $2k \in \{0, \cdots, N-2\}$. Now, we must show that F and $\widetilde{F}$ are path homotopic. To do so easily, we introduce a small lemma:

Lemma 6.5. *If F, F' are paths in an Alexandrov space A such that $F(0) = F'(0)$, $F(1) = F'(1)$, and $F(x) \leq F'(x)$ for all $x \in I$, then F, F' are path homotopic.*

Proof. Let $H \colon I \times I \to A$ be such that, for $(x, t) \in I \times I$,

$$H(x,t) = \begin{cases} F(x) & t \in [0,1) \\ F'(x) & t = 1 \end{cases}.$$

Now, we show that H is continuous. Let $U \subset A$ be open. Let $(x, t) \in H^{-1}(U)$. We have two cases:

(1) $t \in [0, 1)$
In this case, observe that $(x, t) \in F^{-1}(U) \times [0, 1) \subset H^{-1}(U)$ and $F^{-1}(U) \times [0, 1)$ is open as F is continuous.

(2) $t = 1$
By definition, $F(x) \leq F'(x)$; so, as $F'(x) \in U$, $F(x) \in U$. Thus, $x \in F^{-1}(U) \cap F'^{-1}(U)$, and so, we see that $(x, t) \in (F^{-1}(U) \cap F'^{-1}(U)) \times I \subset H^{-1}(U)$, and that $(F^{-1}(U) \cap F'^{-1}(U)) \times I$ is clearly open.

Thus, we see that for each $(x, t) \in H^{-1}(U)$, there exists some open $B \subset I \times I$ such that $(x, t) \in B \subset H^{-1}(U)$; so, $H^{-1}(U)$ is open. Thus, H is continuous, and so F and F' are path homotopies. □

Now, let's get back to our main story. Fix $t \in I$. Notice that, by the definition of $\widetilde{F}$, we have that $F(t) \leq \widetilde{F}(t)$; so, by the lemma above, F is path homotopic to $\widetilde{F}$, which is clearly path homotopic to a similar realization of a finite path. Thus, the statement is proven.

□

Now, as was mentioned above, Theorem 6.4 has many implications; in reference to Theorem 5.15 above, we can expand it to include any Alexandrov space A by letting P be the image of the realization of a finite path from a_1 to a_2 found using Theorem 6.4, as this image is a finite, path-connected space. Furthermore, we see that the cardinality of the fundamental group of an Alexandrov space is, at most, the cardinality of the set of all of the space's finite subsets. For instance, we see that the cardinality the fundamental group of any countable Alexandrov space is, at most, countably infinite.

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