NOTES ON THE 2018 REU PROGRAM

Here are some notes about the program. They will be updated throughout the program as more information comes in. Research faculty are often notably late to prepare talks. I myself am an extreme offender, having just written my abstract today, June 11. The program is frontloaded, with the number of talks decreasing as the focus switches to your work on papers.

APPRENTICE PROGRAM
Weeks one through four. Daniiel Rudenko.

What is a space? Every few decades the answer to this question undergoes a substantial modification. One of the biggest breakthroughs of the 19th century was the realization of the essential role played by symmetry groups.

It was an exciting time. Almost simultaneously certain objects whose very existence was speculative if not frightening, turned out to be fundamental in mathematics and physics. Among those objects, complex numbers, higher-dimensional spaces and non-Euclidean geometries were especially important. The goal of this course is to become familiar with these objects and multiple relations existing between them.

We will start our journey in the familiar territory of the Euclidean geometry and will go on to dive into the hyperbolic plane, three-dimensional sphere and higher dimensional Euclidean spaces. Finally, we will find ourselves in the peculiar landscape of projective geometry. Along the way we will develop a major scientific tool provided by linear algebra and group theory. Also, we will touch upon a few related topics in number theory and combinatorics.

It is impossible to achieve comprehension of these concepts without solving plenty of technical exercises as well as challenging problems. Getting this experience will be a large part of the course.

Week 5. TBD.

FINITE SPACES AND LARGER CONTEXTS
Weeks one through eight. Peter May.

What is a topological space? The notion was defined by Hausdorff and, in the still current version, Kuratowski, around a century ago. There is a less general later definition due to Alexandroff. Nearly all mathematicians, regardless of field, know the early definitions. Few know about Alexandroff spaces, but all finite spaces are Alexandroff, and finite spaces are surprisingly interesting. For example there is a space with $2n+2$ points that, to the eyes of algebraic topology, is equivalent to the $n$-sphere. This is a strange, fascinating, and little-known world of mathematics.

It is a striking, if esoteric, fact that in principle one can do all of algebraic topology using Alexandroff spaces. That is even true equivariantly, when one considers
spaces with symmetries given by group actions. Alexandroff spaces in which the topology distinguishes points are the “same thing” as partially ordered sets (posets). This establishes a close connection between algebraic topology and combinatorics. There are many applications, some speculative, others far out. For example, finite spaces have been used to study the role of RNA in evolution.

I am writing a book on this subject, with the same title as the talks, and I will present topics from it and from past REU papers and other papers that have been written over the years. (Maybe some of you might want to contribute.) The intent of the talks will be to introduce both finite spaces and “ordinary” or “usual” spaces, emphasizing how differently the same concepts behave in the two worlds. For example all usual spaces, such as metric spaces, are Hausdorff spaces (as in Hausdoff’s original 1914 definition), but finite Hausdorff spaces are discrete.

**TRAVELS TO EXOTIC SPACES**

Weeks one and two. Shmuel Weinberger.

The space most of us are most familiar with is the Euclidean plane. During these lectures, I’d like to go touring some less familiar places, like high dimensional Euclidean space, or graphs like the one-skeleton of a cube (the phase space of DNA). What do random trees or random graphs look like? Each lecture should contain at least one non-obvious fact, and one application. To quote Lao Tzu, A good traveller has no fixed plans and is not intent on arriving.

**HYPERBOLIC SPACE**

Week 3. Danny Calegari.

**AN INTRODUCTION TO DYNAMICAL SYSTEMS I**

Week 1. Aaron Brown.
I will discuss two dynamical systems on the circle: rotations and multiplication. We will introduce a number of concepts such as minimality, transitivity, ergodicity, and discuss quantitative differences between these two systems.

**AN INTRODUCTION TO DYNAMICAL SYSTEMS II**

Week 2. Howard Masur.
This minicourse will be given in conjunction with that of Aaron Brown the previous week. I will review briefly notions he introduced such as minimality and ergodicity. I will talk about a new example called the Gauss map and its relationship to what is known as the continued fraction expansion of real numbers. Then I will switch to the example of billiards in a polygon in the plane; the connection to rotations of the circle in the case that the polygon is a rectangle and then more general polygons and various new phenomenon that occur.

**NUMBER THEORY**

Weeks 1, 2, 4, 5. Matt Emerton and Frank Calegari.

**STOCHASTIC CALCULUS**

Weeks 2 and 3. Gregory Lawler.
We will discuss stochastic integration, that is, integration with respect to random walk or Brownian motion. This can be viewed as a model of “investing” or “gambling” where one must decide one’s investment (bet) based only on what has happened so far. We will build up to understanding the Black-Scholes formula.
which is used to price options. We will also discuss relations with standard PDEs such as the heat equation.

HOW TO WIN $1,000,000 OR $500 IN A VERY DIFFICULT WAY

Weeks four, five, and six. Will Feldman and Chris Henderson.

Subtitle: Navier-Stokes equations and optimal mixing problems.

Do you know why the US government will not allow more than three ounces of fluid per passenger on a plane? Perhaps it is because it is unknown whether solutions to the Navier-Stokes equation governing fluid velocity will blow-up in a finite time. We discuss this problem (one of the Millennium problems) and how to optimally stir your coffee (Bressan’s conjecture), two active topics in modern mathematical fluid mechanics. We start simple, no background in partial differential equations is required.

AN INVITATION TO CONCENTRATION COMPACTNESS

Week 5. Chenjie Fan.

The Arzela-Ascoli Theorem is one of the fundamental results in analysis. In the language of functional analysis, it says that the embedding $C^1(I) \to C^0(I)$ is compact for any closed interval $I$. Morally, it says that one should expect some such compactness in the space of continuous functions for any family of functions on a closed interval which are smooth in some uniform way. Such compactness will be lost if the closed interval in the statement is replaced by the whole real line. The main reason is that there is a symmetry in the whole line. In this talk, we will see how symmetry could cause the lack of compactness and how “concentration compactness” can be the tool to remedy this.

AN INTRODUCTION TO GEOMETRIC MEASURE THEORY

Week 6. Dali Nimer

What is the dimension of a Cantor set? How to distinguish between sets of the plane of measure zero? How to build a ’tailor made’ measure for a given set? Do we really need differentiability everywhere to describe the geometry of a set? These are some of the questions I will tackle in these two lectures that aim to be an introduction to the kind of questions studied in Geometric measure theory and the tools used to treat them.

SPECIAL TALK: How to solve some inequalities by playing billiards on a curved space (or what happens when number theory meets dynamics).

July 30, Asaf Katz

Abstract - The set of values that a given quadratic form attains over integral vectors is of major interest in number theory. While many techniques to study this set have been developed in the past (most notably, the Hardy-Littlewood circle method), showing the following equality

$$\inf_Q (Z^3 \setminus \{0\}) = 0, \text{ where } Q(x, y, z) = x^2 + y^2 - \sqrt{2}z^2,$$

was out of reach for over 60 years! I will describe the modern approach to this question, involving an ingenious argument transferring the problem from number theory to a question in dynamics over homogeneous spaces, and explain how major theorems in dynamics proven by G. Margulis and M. Ratner help to solve such
a question. Time permitting, I will explain a similar reduction regarding Littlewood's conjecture and the related dynamical theorem, proven in this case by E. Lindenstrauss.

**ULTRAFILTERS AND COMBINATORICS**

Week one. Denis Hirschfeldt.

An ultrafilter on a set $X$ is a subset $U$ of the power set of $X$ with certain properties that allow us to think of $U$ as a notion of largeness for subsets of $X$. We will introduce this concept and discuss a few of its many applications, particularly to combinatorics, for instance in giving a relatively short proof of Hindman’s Theorem that for any coloring of the natural numbers with finitely many colors, there is an infinite set $S$ such that all nonempty sums of distinct elements of $S$ have the same color.

**REGULARITY LEMMAS AND MODEL THEORY.**

Weeks three and four. Maryanthe Malliaris.

The course will explain some recent interactions between Szemeredi’s celebrated regularity lemma for large graphs, and model theory, an area of mathematical logic. No prior knowledge of either will be assumed.

**CATEGORY THEORY AND LIFE**

Special talk, July 19. Eugenia Cheng
(See Eugenia with Stephen Colbert on her web page http://eugeniaccheng.com)

Category theory is often thought of as ”very abstract algebra” with its main applications being in branches of pure mathematics, theoretical physics, and the foundations of computer science. However in this talk I will apply basic concepts of category theory to important questions of life such as prejudice, privilege, blame and responsibility. I will argue that abstraction has a purpose and that broad applicability is one of the powerful consequences. The talk will serve as an both an introduction to the ideas of category theory and a suggestion of how to convince skeptics of the relevance of abstract mathematics. I will introduce the category theory concepts from scratch so no prior knowledge is needed.

Geometry and Topology Summer Workshop at notre Dame
Week 7. See https://www3.nd.edu/ math/rtg/summer.html

**FAIR DIVISION VIA SOCIAL COMPARISON**

Special talk, August 9. Rediet Abebe.

We study cake cutting on a graph, where agents can only evaluate their shares relative to their neighbors. This is an extension of the classical problem of fair division to incorporate the notion of social comparison from the social sciences. We say an allocation is locally envy-free if no agent envies a neighbor’s allocation, and locally proportional if each agent values its own allocation as much as the average value of its neighbors’ allocations. We generalize the classical “Cut and Choose” protocol for two agents to this setting, by fully characterizing the set of graphs for which an oblivious single-cutter protocol can give locally envy-free (thus
also locally-proportional) allocations. We study the price of envy-freeness, which compares the total value of an optimal allocation with that of an optimal, locally envy-free allocation. Surprisingly, a lower bound of $\Omega(\sqrt{n})$ on the price of envy-freeness for global allocations also holds for local envy-freeness in any connected graph, so sparse graphs do not provide more flexibility asymptotically with respect to the quality of envy-free allocations. Along the way, we present a novel graph-theoretic result.

Participant talks
August 8, 9, 10