Exercises

These are meant to solidify definitions in your head. You should do them.

Exercise 1. Prove in detail that $S^2$ is a smooth 2-manifold.

Exercise 2. Let $U \subset \mathbb{R}^n$ be open and let $f : U \rightarrow \mathbb{R}^m$ be a smooth function.
(a) Show that the graph of
$$\Gamma_f := \{(x, y) \in U \times \mathbb{R}^m | y = f(x)\} \subset \mathbb{R}^{n+m}$$
is a smooth $n$-manifold.
(b) Identify the tangent space $T(\alpha, f(\alpha))\Gamma_f$ as a subspace of $\mathbb{R}^{n+m}$.
(c) Show that the projection map $\pi : \Gamma_f \rightarrow U$ is a diffeomorphism.

Exercise 3. Let $M \subset \mathbb{R}^k$ be an $n$-manifold. Show that $TM \subset \mathbb{R}^{2k}$ is a 2$n$-manifold. Hint: If $\phi_\alpha : U_\alpha \rightarrow M$ is a chart for $M$, write down a chart for $TM$ with source $U_\alpha \times \mathbb{R}^n$.

Exercise 4. Let $\text{det} : \text{GL}_n(\mathbb{R}) \rightarrow \mathbb{R}$ denote the determinant. Prove that, under the identification $T_{\text{Id}}\text{GL}_n(\mathbb{R}) = T_{\text{Id}}\text{Mat}_n(\mathbb{R}) = \text{Mat}_n(\mathbb{R})$, we have
$$d(\text{det})_{\text{Id}} = \text{tr}.$$

Note: In case it’s not clear, $\text{Mat}_n$ denotes the vector space of $n \times n$ matrices, and $\text{tr} : \text{Mat}_n \rightarrow \mathbb{R}$ denotes the trace.

Exercise 5. Let $m : \text{GL}_n(\mathbb{R}) \times \mathbb{R}^2 \rightarrow \text{GL}_n(\mathbb{R})$ denote matrix multiplication. Compute
$$dm_{(\text{Id}, \text{Id})} : \text{Mat}_n(\mathbb{R}) \times \text{Mat}_n(\mathbb{R}) \rightarrow \text{Mat}_n(\mathbb{R}).$$

Exercise 6. Let $\mathbb{R}P^n$ denote the set of lines through the origin in $\mathbb{R}^{n+1}$.
(a) Show that there is a canonical bijection between $\mathbb{R}P^n$ and $(n + 1) \times (n + 1)$, symmetric matrices $A$ of trace 1 satisfying $A^2 = A$. So we may identify $\mathbb{R}P^n$ as a subset of $\text{Mat}_{n+1}(\mathbb{R}) \cong \mathbb{R}^{(n+1)^2}$.
(b) Show that $\mathbb{R}P^n$ is a smooth $n$-manifold by using the charts
$$\phi_i : \mathbb{R}^n \rightarrow \mathbb{R}P^n$$
given by sending $(x_1, \ldots, x_n)$ to the line spanned by $(x_1, \ldots, x_{i-1}, 1, x_{i+1}, \ldots, x_n)$.

Exercise 7. Show that $\text{SO}(2)$ (defined below) is diffeomorphic to $S^1$.

Exercise 8. Show that $S^2$ is orientable by writing down an explicit orientation.
Exercise 9. An atlas for a smooth \( n \)-manifold \( M \) is a set \( \{(U_\alpha, \phi_\alpha)\} \) of open subsets \( U_\alpha \subset \mathbb{R}^n \) together with smooth maps \( \phi_\alpha : U_\alpha \to M \) which are diffeomorphisms onto open subsets of \( M \) (i.e. charts). Show that \( M \) is orientable if and only if it is possible to find an atlas \( \{(U_\alpha, \phi_\alpha)\} \) such that the transition functions

\[
\phi_\beta^{-1} \circ \phi_\alpha : U_\alpha \cap \phi_\alpha^{-1}(U_\beta) \to U_\beta \cap \phi_\beta^{-1}(U_\alpha)
\]

are orientation-preserving for the standard orientation on open subsets of \( \mathbb{R}^n \) (i.e. the determinant of the Jacobian is positive). Moreover, a choice of such an atlas determines an orientation.

Exercise 10. Suppose that \( M \subset \mathbb{R}^{n+1} \) is an \( n \)-manifold which admits a smooth function \( \nu : M \to \mathbb{R}^{n+1} \) with the property that, for all \( x \in M \), \( \text{span}(\nu(x), T_x M) = T_x \mathbb{R}^{n+1} \) (under the canonical identifications of each term with subspaces or vectors in \( \mathbb{R}^{n+1} \)). Produce an orientation on \( M \).

Problems

These are meant to be more interesting, and are of varying but unstated levels of difficulty. You should do them.

Problem 11 (Milnor, Problem 3). If two maps \( f, g : X \to S^n \) satisfy \( |f(x) - g(x)| < 2 \) for all \( x \in X \), prove that \( f \) is homotopic to \( g \), the homotopy being smooth if \( f \) and \( g \) are smooth.

Problem 12. Show that \( S^n \) admits a nowhere vanishing vector field if and only if the antipodal map \( x \mapsto -x \) from \( S^n \) to itself is homotopic to the identity map.

Problem 13 (Milnor, Problem 5). If \( m < n \), show that every map \( M \to S^n \) from an \( m \)-manifold to an \( n \)-sphere is nullhomotopic (i.e. homotopic to a constant map).

Problem 14. Let \( O(n) \subset GL_n(\mathbb{R}) \) denote the subset of matrices \( A \) such that \( A^tA = \text{Id} \). Prove that it is a submanifold and compute its dimension. Hint: Use the fact that the preimage of a regular value of a smooth map is a submanifold with a known dimension. Exercise a modicum of caution about the target of the smooth map you’d like to write down.

Problem 15. Show that Gram-Schmidt orthogonalization defines a diffeomorphism

\[
O(n) \times \mathbb{R}^{n(n+1)/2} \cong GL_n(\mathbb{R}).
\]

In particular, the inclusion \( O(n) \hookrightarrow GL_n(\mathbb{R}) \) is a homotopy equivalence.

Problem 16. A topological group is a topological space with a group structure such that the multiplication and inverse maps are continuous. If \( G \) is a topological group and \( H \subset G \) is a subgroup, then the set of cosets \( G/H \) is a topological space with the quotient topology.

Let \( SO(n) \subset O(n) \) be the subgroup of orthogonal matrices with determinant 1.

Prove that there are homeomorphisms

\[
O(n)/O(n-1) \cong SO(n)/SO(n-1) \cong S^{n-1}.
\]

(Hint: This will be substantially easier if you know a few facts from topology, like the fact that a bijective continuous map between compact Hausdorff spaces is automatically a homeomorphism. Look up the ‘closed mapping theorem’. It’s useful.)

Use this fact to show that \( SO(n) \) is connected by induction on \( n \). Deduce using the previous problem that the subspace of \( GL_n(\mathbb{R}) \) consisting of matrices with positive determinant is connected.

Problem 17. Let \( Gr_k(\mathbb{R}^n) \) denote the set of \( k \)-dimensional vector subspaces of \( \mathbb{R}^n \).

(a) Show that there is a canonical bijection between \( Gr_k(\mathbb{R}^n) \) and \( n \times n \), symmetric matrices \( A \) of trace \( k \) satisfying \( A^2 = A \). So we may identify \( Gr_k(\mathbb{R}^n) \) as a subset of \( Mat_n(\mathbb{R}) \cong \mathbb{R}^{n^2} \).
(b) Show that $\text{Gr}_k(\mathbb{R}^n)$ is a smooth manifold by showing it is the preimage of a regular value of a suitable smooth map. What is its dimension?

**Problem 18.** Let $M$ be an $n$-manifold with tangent bundle $\pi : TM \rightarrow M$, and let $\text{Fr}(TM)$ denote the subspace of $TM \times \mathbb{R}^n$ consisting of tuples $((x_1, v_1), ..., (x_n, v_n))$ such that $x_1 = \cdots = x_n = x$ and $(v_1, ..., v_n)$ is an ordered basis of $T_x M$. Recall that the orientation space $\text{Or}(M)$ denotes the quotient space of $\text{Fr}(TM)$ formed by identifying bases related by a transition matrix of positive determinant.

(a) Show that there is a diffeomorphism $S^2 \cong \text{Or}(\mathbb{R}P^2)$ making the diagram

\[
\begin{array}{ccc}
S^2 & \longrightarrow & \text{Or}(\mathbb{R}P^2) \\
\downarrow & & \downarrow \\
\mathbb{R}P^2 & \longrightarrow & \mathbb{R}P^2
\end{array}
\]

commute. Here the left diagonal arrow sends a point $(x, y, z) \in S^2$ to the line it spans.

(b) Use the previous step to deduce that $\text{Or}(\mathbb{R}P^2) \rightarrow \mathbb{R}P^2$ does not admits a continuous section. Hint: Restrict $S^2 \rightarrow \mathbb{R}P^2$ to the hemisphere $S^1$ and then reduce to showing that $z \mapsto z^2$, $S^1 \rightarrow S^1$, does not have a continuous section.

**Problem 19.** Recall that if $M$ is an $n$-manifold, then $TM$ has the canonical structure of a $2n$-manifold. Show that $TM$ is always an orientable $2n$-manifold.

**Problem 20.** Let $\text{Sph}(TS^2)$ denote the subspace of $TS^2$ consisting of pairs $(x, v)$ where $v \in T_x S^2 \subset \mathbb{R}^3$ has length 1. Show that $\text{Sph}(TS^2)$ is diffeomorphic to $\text{SO}(3)$. Describe the resulting projection map $\text{SO}(3) \rightarrow S^2$. If you made good choices, this will look good.