# REU 2017 Algebraic Topology Exercises

June 27, 2017

## Exercises

These are meant to solidify definitions in your head. You should do them.

**Exercise 1.** Prove in detail that $S^2$ is a smooth 2-manifold.

**Exercise 2.** Let $U \subset \mathbb{R}^n$ be open and let $f : U \longrightarrow \mathbb{R}^m$ be a smooth function.

(a) Show that the graph of $f$

$$\Gamma_f := \{(x, y) \in U \times \mathbb{R}^m | y = f(x)\} \subset \mathbb{R}^{n+m}$$

is a smooth $n$-manifold.

(b) Identify the tangent space $T_{(a,f(a))}\Gamma_f$ as a subspace of $\mathbb{R}^{n+m}$.

(c) Show that the projection map $\pi : \Gamma_f \longrightarrow U$ is a diffeomorphism.

**Exercise 3.** Let $M \subset \mathbb{R}^k$ be an $n$-manifold. Show that $TM \subset \mathbb{R}^{2k}$ is a $2n$-manifold. Hint: If $\phi_\alpha : U_\alpha \to M$ is a chart for $M$, write down a chart for $TM$ with source $U_\alpha \times \mathbb{R}^n$.

**Exercise 4.** Let $\det : \mathsf{GL}_n(\mathbb{R}) \longrightarrow \mathbb{R}$ denote the determinant. Prove that, under the identification $T_{\mathrm{Id}}\mathsf{GL}_n(\mathbb{R}) = T_{\mathrm{Id}}\mathsf{Mat}_n(\mathbb{R}) = \mathsf{Mat}_n(\mathbb{R})$, we have

$$d(\det)_{\mathrm{Id}} = \mathrm{tr}.$$

Note: In case it's not clear, $\mathsf{Mat}_n$ denotes the vector space of $n \times n$ matrices, and $\mathrm{tr} : \mathsf{Mat}_n \longrightarrow \mathbb{R}$ denotes the trace.

**Exercise 5.** Let $m : \mathsf{GL}_n(\mathbb{R})^{\times 2} \to \mathsf{GL}_n(\mathbb{R})$ denote matrix multiplication. Compute

$$dm_{(\mathrm{Id},\mathrm{Id})} : \mathsf{Mat}_n(\mathbb{R}) \times \mathsf{Mat}_n(\mathbb{R}) \longrightarrow \mathsf{Mat}_n(\mathbb{R}).$$

**Exercise 6.** Let $\mathbb{R}P^n$ denote the set of lines through the origin in $\mathbb{R}^{n+1}$.

(a) Show that there is a canonical bijection between $\mathbb{R}P^n$ and $(n + 1) \times (n + 1)$, symmetric matrices $A$ of trace 1 satisfying $A^2 = A$. So we may identify $\mathbb{R}P^n$ as a subset of $\mathsf{Mat}_{n+1}(\mathbb{R}) \cong \mathbb{R}^{(n+1)^2}$.

(b) Show that $\mathbb{R}P^n$ is a smooth $n$-manifold by using the charts

$$\phi_i : \mathbb{R}^n \longrightarrow \mathbb{R}P^n$$

given by sending $(x_1, ..., x_n)$ to the line spanned by $(x_1, ..., x_{i-1}, 1, x_{i+1}, ..., x_n)$.

**Exercise 7.** Show that $\mathsf{SO}(2)$ (defined below) is diffeomorphic to $S^1$.

**Exercise 8.** Show that $S^2$ is orientable by writing down an explicit orientation.

**Exercise 9.** An **atlas** for a smooth $n$-manifold $M$ is a set $\{(U_\alpha, \phi_\alpha)\}$ of open subsets $U_\alpha \subset \mathbb{R}^n$ together with smooth maps $\phi_\alpha : U_\alpha \longrightarrow M$ which are diffeomorphisms onto open subsets of $M$ (i.e. charts). Show that $M$ is orientable if and only if it is possible to find an atlas $\{(U_\alpha, \phi_\alpha)\}$ such that the **transition functions**

$$\phi_\beta^{-1} \circ \phi_\alpha : U_\alpha \cap \phi_\alpha^{-1}\phi_\beta(U_\beta) \longrightarrow U_\beta \cap \phi_\beta^{-1}\phi_\alpha(U_\alpha)$$

are orientation-preserving for the standard orientation on open subsets of $\mathbb{R}^n$ (i.e. the determinant of the Jacobian is positive). Moreover, a choice of such an atlas determines an orientation.

**Exercise 10.** Suppose that $M \subset \mathbb{R}^{n+1}$ is an $n$-manifold which admits a smooth function $\nu : M \to \mathbb{R}^{n+1}$ with the property that, for all $x \in M$, $\operatorname{span}(\nu(x), T_x M) = T_x \mathbb{R}^{n+1}$ (under the canonical identifications of each term with subspaces or vectors in $\mathbb{R}^{n+1}$). Produce an orientation on $M$.

# Problems

These are meant to be more interesting, and are of varying but unstated levels of difficulty. You should do them.

**Problem 11** (Milnor, Problem 3)**.** If two maps $f, g : X \to S^n$ satisfy $|f(x) - g(x)| < 2$ for all $x \in X$, prove that $f$ is homotopic to $g$, the homotopy being smooth if $f$ and $g$ are smooth.

**Problem 12.** Show that $S^n$ admits a nowhere vanishing vector field if and only if the antipodal map $x \mapsto -x$ from $S^n$ to itself is homotopic to the identity map.

**Problem 13** (Milnor, Problem 5)**.** If $m < n$, show that every map $M \to S^n$ from an $m$-manifold to an $n$-sphere is nullhomotopic (i.e. homotopic to a constant map).

**Problem 14.** Let $\mathsf{O}(n) \subset \mathsf{GL}_n(\mathbb{R})$ denote the subset of matrices $A$ such that $A^{\mathrm{tr}} A = \mathrm{Id}$. Prove that it is a submanifold and compute its dimension. Hint: Use the fact that the preimage of a regular value of a smooth map is a submanifold with a known dimension. Exercise a modicum of caution about the target of the smooth map you'd like to write down.

**Problem 15.** Show that Gram-Schmidt orthogonalization defines a diffeomorphism

$$\mathsf{O}(n) \times \mathbb{R}^{n(n+1)/2} \cong \mathsf{GL}_n(\mathbb{R}).$$

In particular, the inclusion $O(n) \hookrightarrow \mathsf{GL}_n(\mathbb{R})$ is a homotopy equivalence.

**Problem 16.** A **topological group** is a topological space with a group structure such that the multiplication and inverse maps are continuous. If $G$ is a topological group and $H \subset G$ is a subgroup, then the set of cosets $G/H$ is a topological space with the quotient topology.

Let $\mathsf{SO}(n) \subset \mathsf{O}(n)$ be the subgroup of orthogonal matrices with determinant 1.

Prove that there are homeomorphisms

$$\mathsf{O}(n)/\mathsf{O}(n-1) \cong \mathsf{SO}(n)/\mathsf{SO}(n-1) \cong S^{n-1}.$$

(Hint: This will be substantially easier if you know a few facts from topology, like the fact that a bijective continuous map between compact Hausdorff spaces is automatically a homeomorphism. Look up the 'closed mapping theorem'. It's useful.)

Use this fact to show that $\mathsf{SO}(n)$ is connected by induction on $n$. Deduce using the previous problem that the subspace of $\mathsf{GL}_n(\mathbb{R})$ consisting of matrices with positive determinant is connected.

**Problem 17.** Let $\mathsf{Gr}_k(\mathbb{R}^n)$ denote the set of $k$-dimensional vector subspaces of $\mathbb{R}^n$.

(a) Show that there is a canonical bijection between $\mathsf{Gr}_k(\mathbb{R}^n)$ and $n \times n$, symmetric matrices $A$ of trace $k$ satisfying $A^2 = A$. So we may identify $\mathsf{Gr}_k(\mathbb{R}^n)$ as a subset of $\mathsf{Mat}_n(\mathbb{R}) \cong \mathbb{R}^{n^2}$.

(b) Show that $\mathsf{Gr}_k(\mathbb{R}^n)$ is a smooth manifold by showing it is the preimage of a regular value of a suitable smooth map. What is its dimension?

**Problem 18.** Let $M$ be an $n$-manifold with tangent bundle $\pi : TM \to M$, and let $\mathsf{Fr}(TM)$ denote the subspace of $TM^{\times n}$ consisting of tuples $((x_1, v_1), ..., (x_n, v_n))$ such that $x_1 = \cdots = x_n = x$ and $(v_1, ..., v_n)$ is an ordered basis of $T_xM$. Recall that the orientation space $\mathsf{Or}(M)$ denotes the quotient space of $\mathsf{Fr}(TM)$ formed by identifying bases related by a transition matrix of positive determinant.

(a) Show that there is a diffeomorphism $S^2 \cong \mathsf{Or}(\mathbb{R}P^2)$ making the diagram

$$\begin{tikzcd} S^2 \arrow[rr] \arrow[dr] & & \mathsf{Or}(\mathbb{R}P^2) \arrow[dl] \\ & \mathbb{R}P^2 & \end{tikzcd}$$

commute. Here the left diagonal arrow sends a point $(x, y, z) \in S^2$ to the line it spans.

(b) Use the previous step to deduce that $\mathsf{Or}(\mathbb{R}P^2) \to \mathbb{R}P^2$ does not admits a continuous section. Hint: Restrict $S^2 \to \mathbb{R}P^2$ to the hemisphere $S^1$ and then reduce to showing that $z \mapsto z^2$, $S^1 \to S^1$, does not have a continuous section.

**Problem 19.** Recall that if $M$ is an $n$-manifold, then $TM$ has the canonical structure of a $2n$-manifold. Show that $TM$ is always an orientable $2n$-manifold.

**Problem 20.** Let $\mathsf{Sph}(TS^2)$ denote the subspace of $TS^2$ consisting of pairs $(x, v)$ where $v \in T_xS^2 \subset \mathbb{R}^3$ has length 1. Show that $\mathsf{Sph}(TS^2)$ is diffeomorphic to $\mathsf{SO}(3)$. Describe the resulting projection map $\mathsf{SO}(3) \to S^2$. If you made good choices, this will look good.