REU 2017 Algebraic Topology Exercises: Weeks 3-4

July 11, 2017

Exercises

These are meant to solidify definitions in your head. You should do them.

Exercise 1. Viewing S^1 as a subspace of the complex numbers $\mathbb{C}$, show that the map $S^1 \to S^1$ given by $z \mapsto z^k$ has degree k.

Exercise 2. Suppose

$$X = \bigcup_{i \geq 0} X_i$$

is the union of a sequence of T_1-spaces $X_0 \subset X_1 \subset X_2 \subset \cdots \subset X$ such that each $X_i \subset X$ is closed. We also require that $U \subset X$ be open if and only if $U \cap X_i$ is open for all $i \geq 0$. Let A be a compact space and $f : A \to X$ a map. Show that the image of A is contained in some X_n. As a corollary, deduce that the closure of any cell in a CW complex intersects only finitely many other cells.

Exercise 3. Show that CW complexes are Hausdorff (even normal.)

Exercise 4. Produce a CW structure on the torus T^2 with one 0-cell, five 1-cells, and two 2-cells. Compute the cellular homology for this CW structure. Now produce a different CW structure with one 0-cell, two 1-cells, and one 2-cell. Compute the cellular homology again using this new CW structure. Observations?

Problems

These are meant to be more interesting, and are of varying but unstated levels of difficulty. You should do them.

Problem 5. (Milnor Problem 5) If $\dim(M) < n$, show that every map $M \to S^n$ is homotopic to a constant map. In particular, $[S^m, S^n] = 0$ when $m < n$.

Problem 6. Let M and N be compact, oriented manifolds with boundary such that $\partial M = \partial N = X$. Show that

$$\chi(M \cup_X N) = \chi(M) + \chi(N) - \chi(X)$$

by constructing an appropriate vector field and counting the zeros. [Don't get bogged down verifying various spaces or vector fields you'd like to consider are smooth, that's not the point of this problem.]

Problem 7. Recall that $\mathbb{R}P^2$ is homeomorphic to the quotient of the closed disk $\mathbb{D}^2$ by the equivalence relation where $x \sim -x$ when $x \in \partial\mathbb{D}^2$. Give $\mathbb{R}P^2$ the structure of a CW complex with one 0-cell, one 1-cell, and one 2-cell. Compute the boundary maps in the cellular chain complex, and compute the cellular homology.

Problem 8. Compute the cellular homology of $\mathbb{R}P^2 \times \mathbb{R}P^2$ corresponding to the cell structure on $\mathbb{R}P^2$ in the previous problem, and using the fact proved in class that $C_*^{\text{cell}}(\mathbb{R}P^2 \times \mathbb{R}P^2) \cong C_*^{\text{cell}}(\mathbb{R}P^2) \otimes C_*^{\text{cell}}(\mathbb{R}P^2)$. Compare your answer to $H_*^{\text{cell}}(\mathbb{R}P^2 \times \mathbb{R}P^2)$.

Problem 9. Let X be a CW-complex. Prove that $H_0^{\text{cell}}(X) \cong \mathbb{Z}^{\oplus |\pi_0 X|}$ is a free abelian group generated by the path components of X. [Hint: First show that every point lies in the path component of a vertex, by inducting on the minimal skeleton at which that point appears. Next, show that if two vertices are connected by a path, then they are connected by a path contained in the 1-skeleton (induct on the minimal skeleton containing the original path). From there, it shouldn't be so bad.]

Problem 10. Suppose X is a CW complex with a single 0-cell $x_0 \in X$ (this isn't really necessary, but it simplifies things). Recall that $\pi_1 X$ denotes the set of pointed homotopy classes of pointed maps $S^1 \to X$. Construct a group homomorphism

$$h : \pi_1(X, x_0) \longrightarrow H_1^{\text{cell}}(X).$$

Prove that this map is surjective, and that the kernel of this homomorphism is precisely the commutator subgroup of $\pi_1(X, x_0)$. Deduce that $H_1^{\text{cell}}(X)$ is the abelianization of the fundamental group.