REU 2017 AlgTop+ Exercises

June 30, 2017

Category theory

Exercise 1. Convince yourself that the following data are equivalent:

- A (unital) monoid, i.e. a set M equipped with a binary operation $M \times M \to M$ and a distinguished element $e \in M$ which is associative and where e behaves as a two-sided identity.
- A category with one object.

We will denote the category associated to a monoid M, abusively, by M.

(a) A monoid is a group (i.e. has inverses) if and only if the associated category has the property that every morphism is an isomorphism.

(b) What is a functor $M \to M'$ in more elementary terms?

(c) If G and G' are the categories associated to the similarly named groups, what is a natural transformation $\eta : f \to h$ between two functors $f, h : G \to G'$, in more elementary terms?

(d) With notation as in the previous part, what's another name for the set of natural isomorphisms from the identity functor on G to itself?

Exercise 2 (Baby Eilenberg-Steenrod)**.** Show that, up to natural isomorphism, there is a unique functor

$$h : \mathsf{Spaces}^{op} \longrightarrow \mathsf{Sets}$$

satisfying the following properties:

(i) If $f : X \to Y$ is a map inducing a bijection

$$f_* : \pi_0 X \to \pi_0 Y,$$

on the set of path components, then $f^* : h(Y) \to h(X)$ is a bijection.

(ii) For any indexing set Λ, and any collection of pointed spaces $\{X_\lambda\}_{\lambda \in \Lambda}$, the evident map

$$h\left(\coprod_{\Lambda} X_\lambda\right) \longrightarrow \prod_{\Lambda} h(X_\lambda)$$

is a bijection.

(iii) $h(*) = \mathbb{Z}$.

Finally, prove that such a functor actually exists, and it is of the form

$$X \mapsto \mathrm{Hom}(X, Y)$$

for a fixed space Y. What is the space? Can you work out the analogous story for pointed spaces?

Exercise 3. Look up the statement of the Yoneda lemma (both covariant and contravariant versions). Then prove it.

Exercise 4. Let Ab denote the category of abelian groups and fix an abelian group A. Show that the functor $A \otimes (-) : \mathsf{Ab} \to \mathsf{Ab}$ is left adjoint to the functor $\mathrm{Hom}(A, -) : \mathsf{Vect} \to \mathsf{Vect}$. Use this fact together with the Yoneda lemma to verify the following computations (i.e. show both sides (co)represent the same functor):

(a) $\mathbb{Z}/2 \otimes \mathbb{Z}/3 \cong 0$,

(b) $\mathbb{Z}/2 \otimes \mathbb{Z}/2 \cong \mathbb{Z}/2$,

(c) $\mathbb{Z}/n \otimes (\mathbb{Q}/\mathbb{Z}) \cong \mathbb{Z}/n$.

Exercise 5. Let S be a set, $\mathcal{C}$ a category, and $\{X_s\}_{s\in S}$ a collection of elements in $\mathcal{C}$ indexed by S. We say that a collection of maps $\{i_s : X_s \to Y\}_{s\in S}$ with common target **exhibits Y as a coproduct of the** $\{X_s\}$ if, for every $Z \in \mathcal{C}$, the function

$$\mathrm{Hom}_{\mathcal{C}}(Y, Z) \xrightarrow{(i_s^*)} \prod_{s\in S} \mathrm{Hom}_{\mathcal{C}}(X_s, Z)$$

is a bijection. Colloquially: to describe a map out of Y it is equivalent to describe a map out of each of the X_s.

(a) Given $\{i_s : X_s \to Y\}$ and $\{j_s : X_s \to Y'\}$ exhibiting both Y and Y' as coproducts of the X_s, show that there is a unique isomorphism between Y and Y' compatible with the maps i_s, j_s. Because of this, if some coproduct exists, we often denote it by $\coprod_{s\in S} X_s$, since there is an 'essentially unique' choice of coproduct. When $|S| = 2$ we often write $X_1 \amalg X_2$ and so on.

(b) Show that the coproduct of sets is given by the disjoint union. (Hence the notation.)

(c) Show that coproduct of spaces is given by the disjoint union, together with the evident topology.

(d) Show that the coproduct of *pointed* spaces is modeled by the wedge.

(e) Show that the coproduct of groups is modeled by the 'amalgamated product', i.e. $G * G'$ is the free group on $G \times G'$ modulo the relations that elements in G multiply as they should, and elements in G' multiply as they should, but with no relations on multiplying elements of G and G' together.

(f) Show that the coproduct in the category of commutative rings is given by the tensor product (over $\mathbb{Z}$).

(g) Show that the coproduct in the category of abelian groups is given by direct sum $\oplus$. (Notice that this means $A \coprod A'$ could mean different things depending on whether we regard A, A' as objects in Ab or objects in Grp. In other words, the inclusion functor $\mathsf{Ab} \hookrightarrow \mathsf{Grp}$ 'does not preserve coproducts'.)

Exercise 6. Suppose $F : \mathcal{C} \to \mathcal{D}$ is a functor which admits a right adjoint G (so that F is a left adjoint). Show that, if $\coprod_{s\in S} X_s$ is a coproduct in $\mathcal{D}$ then the maps $F(i_s) : F(X_s) \to F\left(\coprod_{s\in S} X_s\right)$ exhibit $F\left(\coprod_{s\in S} X_s\right)$ as a coproduct of the $F(X_s)$ in $\mathcal{D}$. That is: F preserves all coproducts that exist in $\mathcal{C}$.

Topology

Exercise 7. Let V and W be vector spaces, and denote by S^V and S^W their one-point compactifications (i.e. the topological space $V \amalg \{\infty\}$ where the open neighborhoods of ∞ are those subsets of the form $(V \setminus K) \amalg \{\infty\}$ for compact subsets $K \subset V$). Show that there is a natural homeomorphism:

$$S^V \wedge S^W \cong S^{V\oplus W}.$$

(We're using ∞ as the basepoint.) Deduce that $S^n \wedge S^m \cong S^{n+m}$.

Exercise 8. Let A be a set with two associative, unital binary operations: $\boxtimes, \star : A \times A \to A$. Suppose they distribute past each other in the following way:

$$(a \star a') \boxtimes (b \star b') = (a \boxtimes b) \star (a' \boxtimes b'),$$

and suppose their units coincide. Then show that $\boxtimes = \star$ and the operation is commutative. (This is called the *Eckmann-Hilton argument.*)

Exercise 9. Use the previous problem to show that $\pi_2(X)$ is an abelian group. (Hint: Think of $\pi_2(X)$ as $[(I^2, \partial I^2), (X, x_0)]$ and build two ways of concatenating maps from the square: one via stacking vertically and the other horizontally. Also, draw pictures.)

Exercise 10. Show that S^1 is a $K(\mathbb{Z}, 1)$ and that $\mathbb{R}P^\infty$ is a $K(\mathbb{Z}/2, 1)$. Hint: Use some facts you know about universal covers, assuming you know these facts. Otherwise, take this opportunity to go and learn about universal covers.

Exercise 11. Use the cofiber sequence $\mathbb{C}P^n \hookrightarrow \mathbb{C}P^{n+1} \longrightarrow S^{2n}$ to inductively compute the cohomology of $\mathbb{C}P^n$ for all n straight from the Eilenberg-Steenrod axioms.

Exercise 12. Prove that $\mathbb{C}P^\infty$ is a $K(\mathbb{Z}, 2)$.

Exercise 13. Construct $K(\mathbb{Z}/p, 1)$ by analogy with the construction of $K(\mathbb{Z}/2, 1)$ as $\mathbb{R}P^\infty$.

Sadly, S^1, $\mathbb{R}P^\infty$, and $\mathbb{C}P^\infty$ are the only really familiar examples of $K(\pi, n)$'s.