# DIFFERENTIAL EQUATIONS: EXISTENCE AND UNIQUENESS OF SOLUTIONS IN ECONOMIC MODELS

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Abstract. The following paper aims to create an integration between differential equations and economics by proving the existence and uniqueness of solutions in ordinary differential equations, then taking what we've proved and apply it to standard economic models. First, an overview of ordinary differential equations will be given through definitions, a basic example, and its applications in various fields of study. Next, using the theorems of Ascoli-Arzel, Peano, and Banach Fixed Point Theorem, we will construct the proof for the existence and uniqueness of solutions in differential equations. Afterwards, we will apply the proof to the Harrod-Domar Model of the effect of investment on economic productivity and aggregate demand.

Contents



## 1. Introduction and Background

First studied and documented in the 17th century by Issac Newton and Gottfried Leibniz, the subject of differential equations is a rich topic in mathematics, with applications to diverse fields such as physics, biology, and even social sciences like economics. With its versatility and compatabilitiy, differential equations grant one predictive power to use in modeling motions and systems whose evolution over time can be described. A differential equation is simply an equation which contains derivatives of a function. The two types of differential equations are *ordinary differential equations* and *partial differential equations*. The former type - and the focus of this paper - contains derivatives for one or more dependent variables with respect to one independent variable. Examples of ordinary differential equations:

$$\frac{dy}{dx} = u(y)$$

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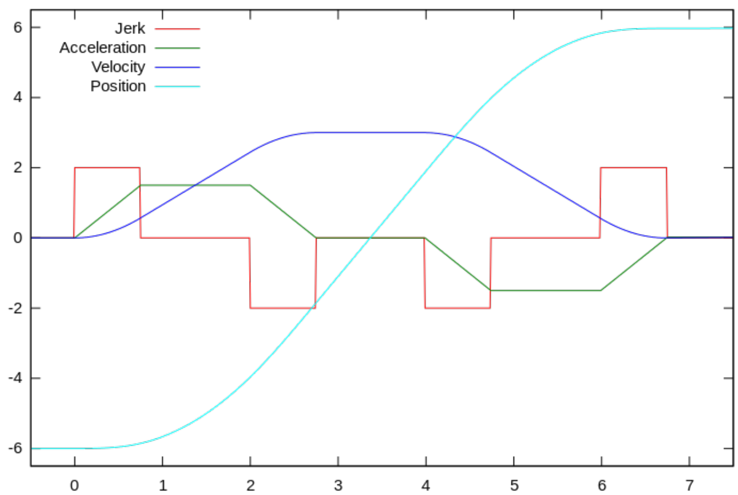
or

$$\frac{dy}{dx} = u(x, y)$$

The latter type involves a special type of derivative called the *partial derivative*, where the derivative of one dependent variable is taken with respect to each other variable. Example of a partial differential equation:

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = u$$

From these examples, one can begin to see how differential equations are utilized in other fields (think velocity, acceleration, and jerk used in tandem to locate an object's position in the study of physics).




Moving back to our focus in this paper, we study the theory of differential equations and their applications to economics. to make precise the definition of our object of study, we make the following definition:

**Definition 1.1.** An ordinary differential equation expresses, at each point of an interval that is the domain of some function $u$, a relationship between a function $u(x)$ and its derivatives $u^{(k)}(x)$. The order of the maximum derivative is called the *order* of the equation.

We also provide an example to complement this definition:

$$\frac{dx}{dt} = cx(t)$$

where $c$ is a constant; it is often treated as a parameter.

Now we can try to find a solution for the equation above. Using derivatives from calculus, we can determine that the solution is $x(t) = ke^{(ct)}$. We can also check the solution. Let $u(t)$ be any solution and then compute the derivative of $u(t)e^{(-ct)}$:

$$\frac{d}{dt}u(t)e^{(-ct)} = cu(t)e^{(-ct)} - cu(t)e^{(-ct)} = 0$$

From this equation, we've found that $u(t)e^{(-at)}$ is constant, therefore $u(t)e^{(-ct)} = k$, where $k$ is any real number. We have found the general solution for our basic differential equation, meaning that is it the only - or unique - solution.

The existence and uniqueness of a solution for a differential equation is useful in our purpose of finding applications. By finding a unique solution, questions raised in physics, biology, and economics can all be answered with significance. Furthermore, equilibria solutions can also be determined, which we will go into detail about later.

The existence and uniqueness of solutions to some ordinary differential equations is the consequence of the following theorem:

**Theorem 1.2** (Existence and Uniqueness Theorem)**.** *Suppose that $X$ is an open subset of $\mathbb{R}_{n+1}$, and suppose that $f$ is a continuous function from $X$ to $\mathbb{R}_n$ that satisfies a Lipschitz condition with respect to $y$. Then, for each point $(x_0, y_0)$ in $X$, the equation $u(x) = y_0 + \int_x^{x_0} f(t, u(t))dt$ has a solution on some open interval containing $x_0$, and any two solutions are equivalent on their common domain.*

Next we will go through the definitions and theorems necessary to construct our proof.

## 2. Definitions and Theorems

In order to construct our proofs, we must first introduce some definitions that are useful for us in understanding how functions behave. Introducing the space $C(I)$: Let $A$ be a compact set in a metric space $S$. Let $C(A;\mathbb{R})$ denote the set whose elements are the continuous functions from $A$ to $\mathbb{R}$. Similarly, $C(A;\mathbb{C})$ denotes the set whose elements are the continuous functions from $A$ to $\mathbb{C}$.

Now, let $I = [x \in \mathbb{R} : a \leq x \leq b]$. Then:

Let us introduce the concept of completeness:

**Definition 2.1.** A space $C(I)$is called complete if every Cauchy sequence of points in $C(I)$ has a limit that is also in $C(I)$.

Next we define a norm:

**Definition 2.2.** If $I$ is a bounded closed interval $[a, b]$, then $C(I)$ denotes the space of continuous real-valued functions defined on $I$ , with norm

$$|u|_{sup} = \sup_{x \in I} |u(x)|$$

Convergence in norm is the same as uniform convergence.

**Theorem 2.3.** *For any compact set $A$, the spaces $C(A;\mathbb{R})$ and $C(A;\mathbb{C})$ are complete.*

*Proof.* Let $f_n$ be a Cauchy sequence in $C(A;\mathbb{C})$. For each $p \in A$, the sequence $f_n(p)$ is a Cauchy sequence of complex numbers:

$$|f_n(p) - f_m(p)| \leq |f_n - f_m| = d(f_n, f_m)$$

For any $\epsilon > 0$, there exists an $N$ so large that $n$, $m \geq N$ implies $d(f_n, f_m) < \epsilon$. Therefore we may define $f(p) = \lim_{n\to\infty} f_n(p)$. For each $p$ and each $m$, we have:

$$|f(p) - f_m(p)| = \lim_{n\to\infty} |f_n(p) - f_m(p)| \leq \epsilon$$

if $m \geq N$, then $d(f, f_m) \leq \epsilon$ if $m \geq N$. To complete the proof, we must show that $f$ is continuous. For any $\epsilon > 0$, choose a large $N$ same as before. The function $f_N$ is uniformly continuous, so there is $\delta > 0$ such that $dS(p, q) < \delta$ implies $|f_N(p) - f_N(q)| < \epsilon$. Suppose that $dS(p, q) < \delta$. Then we can show:

$$|f(p) - f(q)| \leq |f(p) - f_N(p)| + |f_N(p) - f_N(q)| + |f_N(q) - f_(q)| \leq \epsilon + \epsilon + \epsilon = 3\epsilon$$

Therefore $f$ is continuous as required. The proof is the same for $C(A; \mathbb{R})$. □

Now we make some definitions which distinguish a special class of functions in $C(I)$:

**Definition 2.4.** A collection $F$ of functions in $C(I)$ is is said to be *equicontinuous* if for each $\epsilon > 0$ there is $\delta > 0$ such that, for every $u \in F$, $|u(x) - u(y)| < \epsilon$ if $x, y \in I, |x - y| < \delta$.

The $\delta$ does not depend on the function $u$. This is significant because sequences of functions with increasing slope values are ruled out, thus allowing us to bound any sequence of functions. Lastly, the concept of the *Lipschitz condition* is important and is addressed:

**Definition 2.5.** A function $f(x, y)$ defined for certain values of $x \in \mathbb{R}_n$ and $y \in \mathbb{R}_m$ and having values in $\mathbb{R}_l$ is said to satisfy a Lipschitz condition (the values $x$, $y$, and the function $f(x, y)$ can be in any space) with respect to $y$ if there exists a constant $K$ such that

$$|f(x, y') - f(x, y)| \leq K|y - y'|$$

whenever the left side is defined. The constant $K$ is called a *Lipschitz constant.*

Keep these definitions and theorems in mind; they will be important soon. Now that we have our definition and theorem groundwork in place, we can move on to hooking the big fish: proving the existence and uniqueness of solutions of differential equations.

## 3. Proofs for Theorems

The first theorem that is important in our path to proving the existence and uniqueness of solutions in differential equations is the *Ascoli-Arzel Theorem.* This theorem allows us to observe how a space such as $C(I)$ can be used as a way to confine an infinite set of functions, which is important in allowing a equilibrium point or solution to be found.

**Theorem 3.1** (Ascoli-Arzel)**.** *If $F$ is a bounded, equicontinuous set of functions in $C(I)$, then every sequence in $F$ contains a subsequence that converges in norm to an element of $C(I)$.*

*Proof.* Let $(u_k)$ with $1 \leq k < \infty$ be a sequence in $F$. Next partition the interval $I$ into $2^n$ ($n$ is a positive integer) equal subintervals, and do the same for interval $[-M, M]$, where $M$ is a bound for $F$. From this we get that the large rectangle $IX[-M, M]$ has $4^n$ subrectangles. If $u$ belongs to $F$, then its graph is a subset of the large rectangle $IX[-M, M]$. Let the *n-pattern* of $u$ be a union of the $4^n$ subrectangles that are interesected by the graph of $u$, and proceed by choosing patterns ie. the first pattern, $n = 1$ (displayed below) shows that there are nine possible patterns and at least one of them is a the pattern of $u_k$ for $k \to \infty$. Choose such a pattern $P_1$. Then choose a pattern $P_2$ that is a 2-pattern ($n = 2$) of $u_k$ for $k \to \infty$, which itself is among various other 2-patterns that are subsets of $P_1$. Hence, we get the sequence of patterns:

$$P_1 \supset P_2 \supset P_3 ...$$

where $P_n$ is the *n-pattern* of $u_{k_n}$.

Now, using the equicontinuity assumption in the statement of the proof, we know that for any given $\epsilon > 0$ there exists an $N$ large such that for any $u \in F$ varies at most by $\epsilon$ with intervals of length $2^{-N}|I|$. Choose $N$ such that $2^{-N}M \leq \epsilon$, and we get, for $n, m \geq N$:

$$|u_{k_n} - u_{k_m}|_{sup} \leq 5\epsilon$$

Therefore the subsequence $u_k$ converges in norm to a function $u$ in $C(I)$. □

With our first main theorem out of the way, we now move onto the *Theorem of Peano*, which helps us solidify the existence of a solution.

**Theorem 3.2** (Peano)**.** *Suppose that $\Omega$ is an open subset of $\mathbb{R}^2$ and suppose that $f$ is a continuous real-valued function on $\Omega$. For each point $(x_0, y_0)$ in $\Omega$, there is a continuously differentiable function $u$, defined in an open interval containing $x_0$, such that $u(x) = f(x, u(x))$ and $u(x_0) = y_0$.*

*Proof.* Since $\Omega$ is open and $f$ is continuous, we can choose positive constants $K$ and $\delta$ such that if $|x - x_0| \leq \delta$ and $|y - y_0| \leq K\delta$, then $(x, y) \in \Omega$ and $|f(x, y)| \leq K$.

Let $I = [x_0 - \delta, x_0 + \delta]$ and let $F$ be the subset of $C(I)$ that consists of the functions $u_k$. Partition $I$ into $2^k$ equal subintervals and take the unique continuous function whose graph goes through $(x_0, y_0)$, which has a constant slope on each subinterval. The slope on a subinterval is the value of $f$ at the right endpoint of this portion of the graph if the subinterval is left of $x_0$, and at the left endpoint if the subinterval is to the right. Because of $|x - x_0| \leq \delta$ and $|y - y_0| \leq K\delta$, these graphs stay in the open subet $\Omega$ and the family of functions is bounded and the limitation on slopes implies that it is equicontinuous, so some subsequence converges uniformly to a function $u \in C(I)$. Now each $u_k$ is piecewise, continuously differentiable and its derivative at a given point converges uniformly to the value of $f$ at that point of the graph as $k$ increases, because of continuity of $f$ and the choice of the $u_k$ . Therefore, for $x$ in $I$,

$$u(x) = \lim_{n\to\infty} u_{k_n}(x) = \lim_{n\to\infty}\left[y_0 + \int_x^{x_0} u'_{k_n}(t))dt\right] = y_0 + \int_x^{x_0} f(t, u(t))dt$$

which is equivalent to $u(x_0) = y_0$ as required.

□

The equation $u(x) = y_0 + \int_x^{x_0} f(t, u(t))dt$ is called a *Picard iterate* and is used to pinpoint a solution after making an initial attempt to find one by slowly iterating functions towards the most accurate solution, and it is important to our main proof. Lastly, we will approach our proof through fixed points through the Banach Fixed Point Theorem, which is useful in confirming the uniqueness of our solutions in the main proof.

**Theorem 3.3** (Banach Fixed Point Theorem)**.** *If $X$ is a nonempty, complete metric space with metric $d$ and if the function $S$ from $X$ to itself is a strict contraction, meaning that for some positive constant $\rho < 1$, and any $x$ and $y$ in $S$,*

$$dS(x), S(y)d(x, y),$$

*Then $S$ has a unique fixed point in $X$.*

*Proof.* Choose a point $x_1 \in X$ and define a sequence as $x_{k+1} = S(x_k)$. The equation from our theorem statement, $dS(x), S(y)d(x, y)$, implies that

$$d(x_{k+2}, x_{k+1}) \leq \rho d(x_{k+1}, x_k) \leq \rho^2 d(x_k, x_{k1}) \leq ... \leq \rho^k d(x_2, x_1)$$

Therefore

$$d(x_{k+m}, x_{k+1}) \leq d(x_{k+m}, x_{k+m1}) + d(x_{k+m1}, x_{k+m2}) + ...$$

$$+d(x_{k+2}, x_{k+1}) \leq [\rho^{k+m2} + \rho^{k+m3} + ... + \rho^k]d(x_2, x_1) \leq \frac{\rho^k}{1} d(x_2, x_1)$$

From this we get that $(x_k)_{k=1}^{\infty}$ is a Cauchy sequence in $X$, so it converges to a point $x \in X$. Let $S$ be continuous so that

$$d(x, S(x)) = \lim_{k\to\infty} d(x_k, S(x_k)) = \lim_{k\to\infty} d(x_k, x_{k+1}) = 0$$

If $x'$ is a fixed point, then

$$d(x, x') = d(S(x), S(x')) \leq \rho d(x, x')$$

Therefore $d(x, x') = 0$ and $x = x'$ as required. □

3.1. **Existence and Uniqueness Theorem.** As written in the introduction before, the proof, also know as the *proof of Picard-Lindelf* is as given:

**Theorem 3.4** (Existence and Uniqueness Theorem). *Suppose that $X$ is an open subset of $\mathbb{R}_{n+1}$, and suppose that $f$ is a continuous function from $X$ to $\mathbb{R}_n$ that satisfies a Lipschitz condition with respect to $y$. Then, for each point $(x_0, y_0)$ in $X$, the equation $u(x) = y_0 + \int_x^{x_0} f(t, u(t))dt$ has a solution on some open interval containing $x_0$, and any two solutions are equivalent on their common domain.*

*Proof.* Given a point $(x_0, y_0) \in \Omega$, choose $\epsilon > 0$ and $r > 0$ small enough such that

$$\{(x, y) : |x - x_0| \leq \epsilon, |y - y_0| \leq r\} \subset \Omega$$

Let

$$N = \sup_{|x-x_0|\leq\epsilon} |f(x, y_0)|$$

Let $K$ be a Lipschitz constant for $f$ and let $J = [x_0 - \delta, x0 + \delta]$, where

$$\delta = min\{\epsilon, r/2N, 1/2K\}$$

Let $u_0 y_0$ be the first Picard iterate, and let $S$ be the mapping defined by

$$[S(u)](x) = y_0 + \int^x x_0 f(t, u(t))dt$$

Then let $X \subset C(I)$ be the closed ball

$$X = \{u \in C(I) : |u - u_0|_{sup} \leq r\}$$

By our choice of $\delta$, we get

$$|S(u_0) - u_0|_{sup} = \sup_{|x-x_0|\leq\delta} |\int_{x_0}^x f(t, y0)dt \leq N\delta \leq \frac{r}{2}$$

If $u$ and $v$ belong to $X$, then $S(u)$ and $S(v)$ are defined, and using the Lipschitz condition inequality

$$|f(x, y') - f(x, y)| \leq K|y - y'|$$

shows that

$$|S(u) - S(v)|_{sup} \leq K\delta|u - v|_{sup} \leq \frac{1}{2}|u - v|_{sup}$$

Finally, we can take $v = u_0$ in from our inequality above and use the inequality

$$|S(u_0) - u_0|_{sup} \leq \frac{r}{2}$$

to conclude that $S$ maps $X$ to itself. Thus, by the Banach Fixed Point Theorem proves that the solutions found from the Picard iterates are unique as required. $\square$

## 4. Application to the Harrod-Domar Model

With the mathematics in place, we can now go on to applying our findings. The Harrod-Domar model is a model relating the change in the rate of investments per year, which we can denote as $I(t)$, with the growth of the economy, represented by the rate of income flow per year, which we can denote as $Y(t)$. The Harrod-Domar model, under the theory of Keynesian economics, was first hypothesized by Roy F. Harrod and Evsey Domar in the 1940s, and used to explain economic growth. The relationship between the the rate of investments per year and the rate of income flow per year is expressed as a ratio:

$$Y'(t) = \frac{I'(t)}{s}$$

The constant $s$ is the constant fraction of total output in the economy that is saved to be used for capital stock; the savings are considered the investment. This equation suggests that the growth of the economy is determined by the rate by which investments are made over a predetermined constant that states how much of the investment goes toward growth of the economy. Put simply, the equation states how much of investment leads to growing the economy. There is the necessary assumption that the entire economy is treated as a single good, which eliminates the need to account for relative prices, substitution and income effects, as well as the factors that contribute to the capital of the economy. Thus, the relationship between investments and economics growth can be captured by just two variables; otherwise, several other variables may be needed.

Now, in order to examine the rate of maximum capacity of production - or how much the economy is capable of producing at its maximum potential at a given point in time - we need to use another equation. The capacity of production can be displayed by the ratio:

$$\rho = \frac{k(t)}{K(t)}$$

The function $k(t)$ represents the maximum capacity output flow and $K(t)$ represents the capital stock present in the economy, both of which are used in a ratio to represent $\rho$, which is unoriginally called the *constant capacity-capital ratio.* This equation implies that the economy with capital stock $K(t)$ can produce an annual output flow of $k(t)$. Next, by taking the derivative of our second equation with respect to $t$ (usually representing years in macroeconomics):

$$k'(t) = \rho K'(t)$$

And, if one were to possess some knowledge in economics and mathematics (which is everyone after they've read this paper), we realize that $K(t)$ is equivalent to $I(t)$, as the rate of investments is equivalent to the rate of capital stock present in the economy, as the investment contributes to the capital present in the economy:

$$\rho K'(t) = \rho I$$

Lastly, in the model, we define equilibrium as the situation where at a specific rate of investments, the maximum capacity outflow is reached. So, if the investments, $I(t)$, allows the income flow, Y(t), to reach a maximum, the value of the maximum capacity outflow , k(t), is also obtained. Thus, by taking the derivative, we have the equation:

$$Y'(t) = k'(t)$$

With all these qualitative equations in place, we can construct a differential equation using the rate of investments per year $I(t)$. With equations of $Y'(t) = \frac{I'(t)}{s}$ and $\rho k'(t) = \rho I$ substituted into $Y'(t) = k'(t)$, we get:

$$I'(t) = s\rho I(t)$$

The equation above now looks more mathematically familiar (an ordinary differential equations perhaps?) and also significant, as now all our qualitative statements from the model grant us now with a mathematical and economic conclusion. So let us apply what we've proved before.

The purpose of the Harrod-Domar model is to determine the best rate of investments to ensure the highest rate of growth in the economy. The differential equation $I'(t) = s\rho I(t)$ mathematically is not difficult to understand as it is familiar in form to our example given in the introduction of the paper. Thus, the solution of the differential equation is:

$$I(t) = I(0)e^{(\rho st)}$$

Where $I(0)$ is the initial value, or rather initial investment in this context. Now we can more rigorously delve into and verify our solution.

If we consider the investments per year $I(t)$ to be be an open subset of all the possible investment amounts per year (this can serve as an analogous example of $\mathbb{R}_{n+1}$ from our theorem, we can consider the different rates of investment $I'(t)$ to be a continuous function within the possible investment rates with respect to $t$, which in our case is year, that provides us the optimal rate of investment for maximum economic growth. The different rates of investment can satisfy the Lipschitz condition

$$|f(t, I'(t) - f(t, I(t))| \leq K|(I(t) - I'(t)|$$

as the left side will always be defined in the non-abstract world, and the right side is always present as due to the nature of economics, where resoures are scarce, so a Lipschitz constant $K$ can be extrapolated from data present in the economy. Thus, using Picard iterates, the individual solutions of the model can be proven to exist and be unique.

From our solution(s), we can conclude mathematically that the function $I(t)$ increases at a rate of $e^{(s\rho t)}$ given the initial value $I(t)$, and we can conclude economically using our mathematics that, given an intial investment, we can model the greatest rate of growth of the economy. Even more so from our equations we can extrapolate and find what amount of capital stock $K(t)$ and rate of income flow $Y(t)$ are required to obtain our solution. By substituting $I(t) = I(0)e^{(s\rho t)}$ into $K'(t) = I(t)$, we get:

$$K'(t) = I(0)e^{(s\rho t)}$$

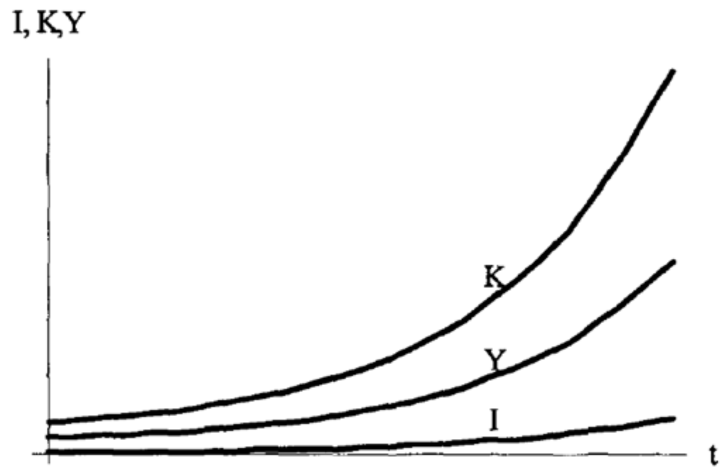
which we can then integrate, using the initial stock present and the initial investment rate, $K(0)$ and $I(0)$ respectively, to get:

$$K(t) = \frac{I(0)e^{\rho st}}{\rho s} + K(0) - I(0)$$

Furthermore, using $Y(t) = \rho K(t)$, we can get:

$$Y(t) = \frac{I(0)e^{\rho st}}{s} + \rho(K(0) - I(0))$$

These two equations, along with the first, $I(t) = I(0)e^{(s\rho t)}$, can then be graphed to give a pictorial representation of the solution in the model.




## 5. Conclusion and Afterword

At this point, the reader (as well as the author!) should come to see that there are some synergistic forces between mathematics and economics. One would be remiss to think that our previous application even comes close in demonstrating fully how differential equations can be used in economic modeling. I hope that the paper was informative as well as a tad bit interesting, enough so that the reader will be provoked enough to try and explore more of what differential equations have to offer in other fields.

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