BROWNIAN MOTION AND ITÔ'S CALCULUS

SHANGXI ZHANG

Abstract. In this paper we will prove several properties of Brownian Motion which serve as the motivation for Itô's calculus. Then, we will construct the Itô integral and then prove the Itô Formula. Furthermore, we will use Itô Formula to derive the Black-Scholes Equation.

Contents

1. Introduction 1
2. Brownian Motion 3
3. Construction of Itô Integral 9
4. Itô formula 11
5. Black-Scholes Equation 13
Acknowledgments 14
References 14

1. Introduction

Brownian motion, stochastic calculus and their applications in Black-Scholes model for option pricing have been well-studied topics in mathematical probabilities, and there are already several expository papers about those topics in the REU for previous years[1]. With this in mind, this paper will approach from a more theoretical and comprehensive perspective about the mathematical foundation of the topics involved.

Definition 1.1. A *σ-algebra* Σ over a set X is a non-empty collection of subsets of X such that

(1) $X \in \Sigma$;
(2) if $A \in \Sigma$, then $X \setminus A \in \Sigma$;
(3) if $\{A_n\}$ is a countable subset of Σ, then $\bigcup_{n=1}^{\infty} A_n \in \Sigma$.

Definition 1.2. The *Borel σ-algebra* $\mathcal{B}(\mathbb{R})$ is the σ-algebra generated by all open sets in $\mathbb{R}$. Sets in $\mathcal{B}(\mathbb{R})$ are called *Borel sets*.

Definition 1.3. A *probability space* is a triple $(\Omega, \mathcal{F}, \mathbb{P})$ where Ω is a set, $\mathcal{F}$ is a σ-algebra over Ω, and $\mathbb{P}$ is a function from $\mathcal{F}$ to $[0,1]$ such that $\mathbb{P}(\Omega) = 1$ and for any countable collection of pairwise disjoint $\{A_n\} \subset \mathcal{F}$,

$$\mathbb{P}(\bigcup_{n=1}^{\infty} A_n) = \sum_{n=1}^{\infty} \mathbb{P}(A_n).$$

[1]See [4][5][6].

Elements in Ω are called *outcomes* and sets in $\mathcal{F}$ are called *events*.

Lemma 1.4 (Borel-Cantelli)**.** *Suppose $A_1, A_2, \dots, A_n$ is a collection of events such that*

$$\sum_{n=1}^{\infty} \mathbb{P}\{A_n\} < \infty.$$

Then, $P\{A_n \text{ i.o.}\} = 0$, where i.o. stands for "infinitely often" and $\{A_n \text{ i.o.}\} = \limsup_{n\to\infty} A_n = \bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} A_k$.

Proof.

$$\begin{aligned}\mathbb{P}\{A_n \text{ i.o.}\} &= \mathbb{P}\{\limsup_{n\to\infty} A_n\} \\ &= \lim_{n\to\infty} \mathbb{P}\{\bigcup_{k=n}^{\infty} A_k\} \\ &\leq \lim_{n\to\infty} \sum_{k=n}^{\infty} \mathbb{P}\{A_k\}\end{aligned}$$

The tail sum goes to 0 since by assumption the series converges. □

Definition 1.5. Given a function $X : \Omega \to \mathbb{R}$ and a σ-algebra $\mathcal{F}$, X is called $\mathcal{F}$*-measurable* if $X^{-1}(B) \in \mathcal{F}$ for any Borel sets B.

Definition 1.6. A *random variable* X on $(\Omega, \mathcal{F}, \mathbb{P})$ is a function from Ω to $\mathbb{R}$ that is $\mathcal{F}$-measurable.

The *distribution* of random variable X is the measure μ_X defined by $\mu_X(B) = \mathbb{P}(X^{-1}(B))$. The function defined by $F_X(x) = \mu_x(-\infty, x]$ is called the distribution function of X.

The function $f : \mathbb{R} \to [0, \infty)$ such that

$$\mathbb{P}\{a \leq X \leq b\} = \int_a^b f(x)dx,$$

if exists for X, is called the *density* function of X.

Definition 1.7. Let X be a random variable. We say $\mathcal{F}_X$ is a σ-algebra *generated by X* if

$$\mathcal{F}_X = \{X^{-1}(B) : B \text{ is Borel}\}.$$

Definition 1.8. Two events A, B are *independent* if

$$\mathbb{P}(A \cap B) = \mathbb{P}(A)\mathbb{P}(B).$$

Definition 1.9. A finite collection of σ-algebras $\mathcal{F}_1, \dots, \mathcal{F}_n$ are independent if for any $A_1 \in \mathcal{F}_1, \dots, A_n \in \mathcal{F}_n$,

$$\mathbb{P}(A_1 \cap \dots \cap A_n) = \mathbb{P}(A_1) \dots \mathbb{P}(A_n).$$

Random variables $X_1, X_2, \dots$ are independent if the σ-algebras they generated $\mathcal{F}_{X_1}, \mathcal{F}_{X_2}, \dots$ are independent.

Definition 1.10. The *expectation* of a random variable X, denoted $\mathbb{E}[X]$, is

$$\mathbb{E}[X] = \int_\Omega X d\mathbb{P}.$$

We say X is *integrable* if $\mathbb{E}[X] < \infty$.

Definition 1.11. The *variance* of a random variable X is

$$\mathrm{Var}(X) = \mathbb{E}[X^2] - (\mathbb{E}[X])^2.$$

The *covariance* of two random variables X, Y is

$$\mathrm{Cov}(X, Y) = \mathbb{E}[(X - \mathbb{E}[X])(Y - \mathbb{E}[Y])].$$

Theorem 1.12 (Chebyshev's Inequality)**.** *Let X be a random variable. If $a > 0$,*

$$\mathbb{P}\{|X - \mathbb{E}[X]| \geq a\} \leq \frac{\mathrm{Var}(X)}{a^2}.$$

Definition 1.13. Let X be an integrable random variable. Let $\mathcal{G} \subset \mathcal{F}$ be a σ-algebra. The *conditional expectation* of X given $\mathcal{G}$ is a random variable, denoted $\mathbb{E}[X \mid \mathcal{G}]$, such that

(1) $\mathbb{E}[X \mid \mathcal{G}]$ is $\mathcal{G}$-measurable;
(2) For every $A \in \mathcal{G}$,

$$\int_A \mathbb{E}[X \mid \mathcal{G}] d\mathbb{P} = \int_A X d\mathbb{P}.$$

Definition 1.14. A random variable X is *normally distributed* with mean μ and variance σ^2 if its density function is

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-(x-\mu)^2/2}.$$

The function f is called the *normal distribution.* The standard normal distribution is a normal distribution with mean 0 and variance 1. We say $X \sim N(\mu, \sigma^2)$ if X is normally distributed with mean μ and variance σ^2.

Proposition 1.15. *If $Z \sim N(0, 1)$, then $\mathbb{E}[Z^2] = 1$ and $\mathbb{E}[Z^4] = 3$.*

Lemma 1.16. *Suppose X, Y are independent $N(0, 1)$ random variables and*

$$Z = \frac{X + Y}{2}, \ W = \frac{X - Y}{2}.$$

Then Z and W are independent, $N(0, 1/2)$ random variables.

2. Brownian Motion

Definition 2.1. A *stochastic process* $\{X_t\}_{t \in T}$ is a collection of random variables on $(\Omega, \mathcal{F}, \mathbb{P})$ indexed by set T.

Definition 2.2. Given a stochastic process $\{X_t\}_{t \in T}$ and a fixed outcome ω, a *path* of $\{X_t\}$ is a function from T to $\mathbb{R}$.

Definition 2.3. A *filtration* $\{\mathcal{F}_t\}_{t \in T}$ on $(\Omega, \mathcal{F}, \mathbb{P})$ is a sequence of σ-algebras such that

$$\mathcal{F}_1 \subset \mathcal{F}_2 \subset \dots.$$

Definition 2.4. A stochastic process $\{X_t\}$ is *adapted* to the filtration $\{\mathcal{F}_t\}$ if each X_t is $\mathcal{F}_t$-measurable.

Definition 2.5. A *Brownian motion* $\{B_t\}$ is a stochastic process satisfying the following properties:

(1) $B_0 = 0$;
(2) For $s < t$, the distribution of $B_t - B_s$ is normal with mean $\mu(t - s)$ and variance $\sigma^2(t - s)$;

(3) the probability distribution of $B_t - B_s$ depends only on $t-s$ for $0 \leq s \leq t$;
(4) with probability one, the function $t \to B_t$ is a continuous function of t.

If a Brownian motion has mean $\mu = 0$ and variance $\sigma^2 = 1$ we call it a *standard Brownian motion.* In this paper for Brownian motions we refer to standard Brownian motions.

Definition 2.6. $\mathcal{D}_n = \{\frac{k}{2^n} : k = 0, 1, \ldots, 2^n\}$ is a *dyadic rational* in [0,1]. Denote

$$\mathcal{D} = \bigcup_{n=1}^{\infty} \mathcal{D}_n.$$

Lemma 2.7. *Brownian motion $\{B_t\}_{t\in\mathcal{D}}$ exists.*

Proof. For each $q \in \mathcal{D}$ we pick a random variable Z_q with distribution $N(0,1)$, such that the collection

$$\{Z_q : q \in \mathcal{D}\}$$

is an independent collection of random variables.

We define B_t on $\mathcal{D}$ recursively. First, we define $B_0 = 0$, $B_1 = Z_1$ which are clearly $N(0,1)$. Then, let

$$B_{1/2} = \frac{B_1}{2} + \frac{Z_{1/2}}{2}.$$

Then,

$$B_1 - B_{1/2} = \frac{B_1}{2} - \frac{Z_{1/2}}{2}.$$

So, by Lemma 1.17, $B_{1/2}$ and $B_1 - B_{1/2}$ are independent random variables each with distribution $N(0, 1/2)$. We continue this splitting. If $q = \dfrac{2k+1}{2^{n+1}} \in \dfrac{\mathcal{D}_{n+1}}{\mathcal{D}_n}$, we define

$$B_q = B_{k/2^n} + \frac{B_{(k+1)/2^n} - B_{k/2^n}}{2} + \frac{Z_q}{2^{(n+2)/2}}$$

which is the same as

$$B_q - B_{2k/2^{n+1}} = \frac{B_{(k+1)/2^n} - B_{k/2^n}}{2} + \frac{Z_q}{2^{(n+2)/2}}.$$

By repeating this process, we see that for each n, the random variables

$$\{B_{k/2^n} - B_{(k-1)/2^n} : k = 1, \ldots, 2^n\}$$

are independent and each with $N(0, 2^{-n})$ distribution. From this, $\{B_q : q \in \mathcal{D}\}$ satisfies the properties of Brownian motion.

□

Lemma 2.8. *$\{B_t\}_{t\in\mathcal{D}}$ have uniformly continuous paths.*

Proof. Define

$$K_n = \sup\{|B_s - B_t| : s, t \in \mathcal{D}, |s-t| \leq 2^{-n}\}.$$

It suffices to show that $K_n \to 0$ as $n \to \infty$ with probability 1.

Consider a sequence of random variables

$$J_n = \max_{j=1,\ldots,2^n} Y(j,n)$$

where

$$Y(j,n) = \sup\{\left|B_q - B_{(j-1)2^{-n}}\right| : q \in \mathcal{D},\ (j-1)2^{-n} \leq q \leq j2^{-n}\}$$

We restrict K_n by triangle inequality. For any $s, t \in \mathcal{D}$, and $|s-t| \leq 2^{-n}$, we have

$$K_n \leq |B_s - B_t| \leq \left|B_s - B_{(j-1)2^{-n}}\right| + \left|B_t - B_{j2^{-n}}\right| + \left|B_{j2^{-n}} - B_{(j-1)2^{-n}}\right| \leq 3J_n.$$

Note that for $\epsilon > 0$,

$$\mathbb{P}\{J_n \geq \epsilon\} \leq \sum_{j=1}^{2^n} \mathbb{P}\{Y(j,n) \geq \epsilon\} = 2^n \mathbb{P}\{Y(1,n) \geq \epsilon\}.$$

Since the distribution of B_t is the same as $\sqrt{t}B_1$, the distribution of $Y(1,n)$ is the same as the distribution of $2^{-n/2}Y(1,0)$. Also,

$$Y(1,0) = \sup\{|B_q| : q \in \mathcal{D}\} = \lim_{m\to\infty} \max\{B_q : q \in \mathcal{D}_m\}.$$

For fixed $\delta > 0$, consider

$$\mathbb{P}\{\max\{B_q : q \in \mathcal{D}_m\} > \delta\}.$$

We define a sequence of events A_k to be

$$A_k = \{B_{j2^{-m}} \leq \delta, B_{k2^{-m}} > \delta, j < k\}$$

and then write

$$\bigcup_{k=1}^{2^m} A_k = \{\max\{B_q : q \in \mathcal{D}_m\} > \delta\}.$$

Note that if $B_{k2^{-m}} > \delta$, then with probability at least $\frac{1}{2}$, $B_1 > \delta$. Therefore,

$$\mathbb{P}(A_k \cap \{B_1 > \delta\}) \geq \frac{1}{2}\mathbb{P}(A_k).$$

Since A_k are disjoint events, we have

$$\begin{aligned}
\mathbb{P}\{B_1 > \delta\} &= \sum_{k=1}^{2^m} \mathbb{P}(A_k \cap \{B_1 > \delta\}) \\
&\geq \frac{1}{2}\sum_{k=1}^{2^m} \mathbb{P}(A_k) \\
&= \frac{1}{2}\mathbb{P}\{\max\{B_q : q \in \mathcal{D}_m\} > \delta\}.
\end{aligned}$$

Since $B_1 \sim N(0,1)$, for $\delta > \sqrt{\frac{\pi}{2}}$,

$$\begin{aligned}
\mathbb{P}\{\max\{B_q : q \in \mathcal{D}_m\} > \delta\} &\leq 2\mathbb{P}\{B_1 > \delta\} \\
&= \sqrt{\frac{2}{\pi}} \int_\delta^\infty e^{-x^2/2} dx \\
&\leq \sqrt{\frac{2}{\pi}} \int_\delta^\infty e^{-x\delta/2} dx \\
&\leq e^{-\delta^2/2}.
\end{aligned}$$

Since the inequality above holds for all m, whe have

$$\begin{aligned}\mathbb{P}\{K_n \geq 3\epsilon\} &\leq \mathbb{P}\{J_n \geq \epsilon\} \\ &= 2^n \mathbb{P}\{Y(1,n) \geq \epsilon\} \\ &= 2^n \mathbb{P}\{2^{-n/2} Y(1,0) \geq \epsilon\}\end{aligned}$$

Therefore,

$$\sum_{n=1}^{\infty} \mathbb{P}\{K_n \geq 3\epsilon\} \leq \infty,$$

and then by Lemma 1.4 (Borel-Cantelli),

$$\mathbb{P}\{K_n > 3\epsilon \ i.o\} = 0.$$

Since ϵ is arbitrary, we have thus shown $K_n \to 0$ in probability. □

Theorem 2.9. *Brownian motion exists.*

Proof. Pick ϵ and by uniform continuity, we have δ such that $|B_s - B_t| \leq \frac{\epsilon}{2}$ for all $s, t \in \mathcal{D}$. Pick $N \in \mathbb{N}$ such that $2^{-N} < \delta$.

Pick $x \in \mathcal{D}$. Pick $n, m > N$ and $k_N = 0, 1, \ldots, 2^N$ such that $0 < x - \dfrac{k_N}{2^n} < \frac{1}{2^n}$. We pick k_n and k_m in the same way and have

$$|k_n - k_N| < 2^{-N}$$

$$|k_m - k_N| < 2^{-N}$$

Then by triangle inequality and uniform continuity,

$$|B_{k_n} - B_{k_m}| \leq |B_{k_n} - B_{k_N}| + |B_{k_m} - B_{k_N}| < 2\epsilon.$$

Then, $\{B_{k_n}\}$ is a Cauchy sequence in $\mathbb{R}$ and we can extend our definition of Brownian motion from $\mathcal{D}$ to $\mathbb{R}$. □

Proposition 2.10. *For every Brownian motion $\{B_t\}$, there exists a filtration $\{\mathcal{F}_t\}$ such that*

(1) *for each t, B_t is $\mathcal{F}_t$-measurable;*
(2) *for all t, $\mathbb{E}[B_t] < \infty$;*
(3) *for all $0 \leq s \leq t$, $\mathbb{E}[B_t \mid \mathcal{F}_s] = B_s$.*

Proof. Take each $\mathcal{F}_t$ to be the σ-algebra generated by B_t. Then, (1) and (2) are immediate. (3) is shown in Theorem 2.13. □

Definition 2.11. A *martingale* with respect to a filtration $\{\mathcal{F}_t\}$ is a sequence of integrable random variables $\{X_t\}$ such that each X_t is $\mathcal{F}_t$-measurable and for all $s < t$,

$$\mathbb{E}[X_t \mid \mathcal{F}_s] = X_s.$$

Example 2.12. Consider a betting game with a series of bets with winning probability 1/2. One strategy is to bet \$$d$ for the first game. If won, stop betting. If lost, bet \$$2d$ for next round. Suppose the initial wealth is X_0. Then the wealth after n rounds X_n is a martingale.

Theorem 2.13. *Brownian motion is a martingale.*

Proof. Fix $0 \leq s \leq t$. Then

$$\begin{aligned}\mathbb{E}[B_t|\mathcal{F}_s] &= \mathbb{E}[B_t - B_s + B_s|\mathcal{F}_s] \\ &= \mathbb{E}[B_t - B_s|\mathcal{F}_s] + \mathbb{E}[B_s|\mathcal{F}_s] \\ &= \mathbb{E}[B_t - B_s] + B_s \\ &= \mathbb{E}[B_t] - \mathbb{E}[B_s] + B_s \\ &= B_s\end{aligned}$$

□

Theorem 2.14. *The path generated by Brownian motion is nowhere differentiable, with probability 1.*

Proof. It suffices to prove the claim on $[0, 1]$. Fix $\omega \in \Omega$ and denote B_t the path generated by t. Define

$$f(n,k) = \max\{\left|B_{(k+2)/n} - B_{(k+1)/n}\right|, \left|B_{(k+1)/n} - B_{k/n}\right|, \left|B_{k/n} - B_{(k-1)/n}\right|\}$$

$$f_n = \min\{f(1,n), \dots, f(n,n)\}.$$

If there were a point t in $[0, 1]$ such that B_t is differentiable, then, for some $M \in \mathbb{R}$,

$$\lim_{r \to t} \frac{B_r - B_t}{r - t} = M.$$

So for any $\epsilon > 0$, there exits $\delta > 0$ such that $|B_r - B_t| < (M + \epsilon)|r - t|$ given $|r - t| < \delta$. Given an ϵ, pick n_0 such that $\frac{2}{n_0} < \delta$. Then, for all $n \geq n_0$, $\frac{2}{n} \leq \frac{2}{n_0} < \delta$. Then, for t and $\frac{k}{n}$ such that $0 < t - \frac{k}{n} < \frac{1}{n}$, we have

$$\begin{aligned}\left|B_{k/n} - B_{(k-1)/n}\right| &\leq \left|B_t - B_{(k-1)/n}\right| + \left|B_t - B_{k/n}\right| \\ &\leq (M + \epsilon)\frac{2}{n} + (M + \epsilon)\frac{2}{n} \\ &\leq \frac{4(M + \epsilon)}{n}.\end{aligned}$$

Let $C = 4(M + \epsilon)$, we can use similar argument to bound the other two terms. Then, we conclude that if there exists $t \in [0, 1]$ where B_t is differentiable, then there exists $C, N \in \mathbb{R}$ such that for all $n \geq N$, $\frac{C}{n} \geq f_n$.

For each $C \in \mathbb{N}$, define

$$A_C = \{\omega \in \Omega : \exists N \text{ s.t. } \forall n \geq N, f_n < C/n\}.$$

The theorem is proven if we show

$$\mathbb{P}\{\bigcup_{C=1}^{\infty} A_C\} = 0.$$

From the definition of Brownian motion, we see that $f(k, n)$ takes the maximum of a collection of independent, normally distributed random variables. This implies that for any Brownian motion and any k, n,

$$
\begin{aligned}
\mathbb{P}\{f_n(k) < \frac{C}{n}\} &= (\mathbb{P}\{\left|B_{1/n}\right| < \frac{C}{n}\})^3 \\
&= (\int_{-\frac{C}{n}}^{\frac{C}{n}} \sqrt{\frac{n}{2\pi}} e^{-\frac{x^2}{2}n}\,dx)^3 \\
&\leq (\frac{2C}{N}\sqrt{\frac{n}{2\pi}})^3 \\
&\leq \frac{C'^3}{n^{3/2}}.
\end{aligned}
$$

Thus, there exists a constant C' such that $\mathbb{P}\{f_n(k) < \frac{C}{n}\} \leq (\frac{C'}{\sqrt{n}})^3$. Then,

$$\mathbb{P}\{f_n < \frac{C}{n}\} \leq \mathbb{P}\bigcup_{k=1}^{n}\{f(k,n) < \frac{M}{n}\} \leq \sum_{k=1}^{n}\mathbb{P}\{f(k,n) < \frac{C}{n}\} \leq \frac{C'^3}{n^{1/2}},$$

which goes to 0 as $n \to \infty$. Therefore, $\lim_{n\to\infty} \mathbb{P}\{f_n \geq \frac{C}{n}\} = 1$. That is, $\mathbb{P}\{A_C\} = 0$ for any fixed C. Therefore,

$$\mathbb{P}\{\bigcup_{C=1}^{\infty} A_C\} = 0.$$

□

One motivation for Itô's Calculus is to deal with the non-differentiability of Brownian motions.

Definition 2.15. Let $f : [0,\infty) \to \mathbb{R}$ be a function. Define

$$Q_{\Pi_n}(f,T) = \sum_{j=0}^{n-1}(f(t_{j+1}) - f(t_j))^2$$

where $\Pi_n = \{t_0, t_1, \ldots, t_n\}$ with $t_i = iT/n$. The *quadratic variation* of f up to T is

$$Q(f,T) = \lim_{n\to\infty} Q_{\Pi_n}(f,T)$$

provided the limit exists.

Theorem 2.16. *For each $T \geq 0$, $Q(B,T) = T$ with probability 1, where B_t is a sample path of Brownian motion.*

Proof. First, we compute

$$
\begin{aligned}
\mathbb{E}[Q_{\Pi_n}(B_t,T)] &= \sum_{j=0}^{n-1}\mathbb{E}[(B_{t_{j+1}} - B_{t_j})^2] \\
&= \sum_{j=0}^{n-1}\mathrm{Var}(B_{t_{j+1}} - B_{t_j})] \\
&= \sum_{j=0}^{n-1}(t_{j+1} - t_j) \\
&= T.
\end{aligned}
$$

For partition Π_n, define $\|\Pi_n\| = \max_{0\le j\le n}\{t_j - t_{j-1}\}$. We know that $(B_{t_{j+1}} - B_{t_j}) \sim \sqrt{(t_{j+1}-t_j)}Z$ where $Z \sim N(0,1)$. Since $\mathbb{E}[Z^4] = 3$, we have

$$\mathbb{E}[(B_{t_{j+1}} - B_{t_j})^4] = \mathbb{E}[(t_{j+1}-t_j)^2 Z^4] = 3(t_{j+1}-t_j)^2.$$

Thus,

$$\begin{aligned}\mathrm{Var}[Q_{\Pi_n}(B_t,T)] &= \sum_{j=0}^{n-1}\mathrm{Var}[(B_{t_{j+1}} - B_{t_j})^2]\\ &= \sum_{j=0}^{n-1} 2(t_{j+1}-t_j)^2\\ &\le \sum_{j=0}^{n-1} 2\|\Pi_n\|(t_{j+1}-t_j)\\ &= 2\|\Pi_n\|T.\end{aligned}$$

By Chebyshev's inequality, for each integer k,

$$\mathbb{P}\{|Q_{\Pi_n}(B_t,T) - T| > \frac{1}{k}\} \le \frac{\mathrm{Var}[Q_{\Pi_n}(B_t,T)]}{(1/k)^2} \le 2k^2\|\Pi_n\|T$$

the right hand side goes to 0 as $n\to\infty$. Then, this gives convergence in probability. □

3. Construction of Itô Integral

Suppose $f(t)$ is a continuous function and we wish to define

$$\int_0^1 f(t)dt.$$

We partition $[0,1]$ into small intervals $0 = t_0 < t_1 < \cdots < t_n = 1$, and approximate it by a step function

$$f_n(t) = f(s_j),\ t_{j-1} < t < t_j$$

where s_j is some point chosen in $[t_{j-1}, t_j]$. We define

$$\int_0^1 f_n(t)dt = \sum_{j=1}^{n} f(s_j)(t_j - t_{j-1}).$$

By calculus, take a sequence of partitions such that the maximum length of $[t_j - 1, t_j] \to 0$, the limit

$$\int_0^1 f(t)dt = \lim_{n\to\infty}\int_0^1 f_n(t)dt$$

exists and the value is independent of the choice of n. We could define Riemann's integral by this construction.

The construction of Itô's integral follows a similar process.

Definition 3.1. Let $V = V(S,T), S,T \in \mathbb{R}_+$ be the set of stochastic processes $X : [0,\infty)\times\Omega \to \mathbb{R}$ satisfying the following properties:

(1) $X(t,\omega)$ is $\mathcal{B}\times\mathcal{F}$ measurable;
(2) $X(t,\omega)$ is adapted by $\mathcal{F}_t$;
(3) $\mathbb{E}[\int_S^T (X(t,\omega))^2 dt] < \infty$.

Definition 3.2. $H_n \in V$ is a *simple process* if for each ω, the path can be written as the form

$$H_n(t,\omega) = \sum_{j=0}^{k-1} e_j(\omega)\mathbb{1}_{[t_j,t_{j+1})}(t)$$

where e_j are indexed random variables, n is a fixed positive integer, $\mathbb{1}_X$ is the indicator function of set X defined by

$$\mathbb{1}_X(x) = \begin{cases} 1, & x \in X \\ 0, & x \notin X \end{cases}$$

and

$$t_j = \begin{cases} 2^{-n}j, & S \leq 2^{-n}j \leq T \\ S & 2^{-n}j < S \\ T & 2^{-n}j > T. \end{cases}$$

Simple process is the analogue of a step function for the stochastic integral.

Definition 3.3. We define the Itô integral for a simple process H_n with respect to a Brownian motion $\{B_t\}$ as

$$\int_S^T H_n(t,\omega)dB_t = \sum_{j\geq 0} e_j(\omega)(B_{t_{j+1}} - B_{t_j})(\omega)$$

Lemma 3.4. *If $H_n(t,\omega) \in V$ is a simple process, then*

$$\mathbb{E}[(\int_S^T H_n(t,\omega)dB_t)^2] = \mathbb{E}[\int_S^T (H_n(t,\omega))^2 dt]$$

Proof. For $i < j$,

$$\mathbb{E}[e_i(B_{t_{i+1}} - B_{t_i})e_j(B_{t_{j+1}} - B_{t_j})] = \mathbb{E}[e_i(B_{t_{i+1}} - B_{t_i})e_j]\mathbb{E}[(B_{t_{j+1}} - B_{t_j})] \quad = 0$$

since $B_{t_{j+1}} - B_{t_j}$ is independent to $\mathcal{F}_{t_j}$.

Thus,

$$\begin{aligned}
\mathbb{E}[(\int_S^T H_n(t,\omega)dB_t)^2] &= \sum_{i,j} \mathbb{E}[e_i(B_{t_{i+1}} - B_{t_i})e_j(B_{t_{j+1}} - B_{t_j})] \\
&= \sum_i \mathbb{E}[e_i^2(B_{t_{i+1}} - B_{t_i})^2] \\
&= \sum_i \mathbb{E}[e_i^2]\mathbb{E}[(B_{t_{i+1}} - B_{t_i})^2] \\
&= \sum_i \mathbb{E}[e_i^2](t_{i+1} - t_i) \\
&= \mathbb{E}[\int_S^T (H_n(t,\omega))^2 dt].
\end{aligned}$$

□

Proposition 3.5. *If $H \in V$ has a continuous path, then there exists a sequence of simple processes $\{H_n\} \subset V$ such that*

$$\mathbb{E}[\int_S^T (H - H_n)^2 dt] \to 0.$$

Proof. It suffices to prove the lemma for each fixed value of S and T and for ease we choose $S = 0, T = 1$. For $H \in V$, define

$$H_m(t) = \int_0^t me^{m(s-t)}H(s)ds.$$

Then for each m, $H_m \in V$, H_m has a continuous path, and

$$\int_0^t (H - H_m)^2 dt \to 0.$$

Then, since each H_m has a continuous paths, for each H_m, we can set

$$H_n(t) = H_m(k/n), \frac{k}{n} \leq t \leq \frac{k+1}{n}, k = 0, 1, \ldots, n$$

and see that by construction, H_n are simple processes that converge to H_m.

□

From Lemma 3.4 and Proposition 3.5, we can show that $\{\int_S^T H_n dB_t\}$ is a Cauchy sequence in $L^2(\Omega)$ space. Because $L^2(\Omega)$ is complete, $\lim_{n\to\infty} \int_S^T H_n dB_t$ is thus a well-defined limit.

Definition 3.6. Let $H \in V(S,T)$. We define its Itô integral by

$$\int_S^T H(t,\omega)dB_t = \lim_{n\to\infty} \int_S^T H_n(t,\omega)dB_t$$

where $\{H_n\}$ is a sequence of simple processes such that

$$\mathbb{E}[\int_S^T (H(t,\omega) - H_n(t,\omega))^2 dt] \to 0.$$

4. Itô formula

Itô's formula is the stochastic version of the fundamental theorem of calculus.

Theorem 4.1 (Itô formula)**.** *Suppose f is a C^2 function and $\{B_t\}_{t\in T}$ is a Brownian Motion. Then for each t,*

$$f(B_t) - f(B_0) = \int_0^t f'(B_s)dB_s + \frac{1}{2}\int_0^t f''(B_s)ds$$

In differential form,

$$df(B_t) = f'(B_t)dB_t + \frac{1}{2}f''(B_t)dt.$$

Proof. We define a partition of the interval $[0,T]$ by $t_i = iT/n$ for $0 \leq i \leq n$. Then,

$$f(B_T) - f(0) = \sum_{i=1}^n (f(B_{t_i}) - f(B_{t_{i-1}})).$$

We estimate each term $f(B_{t_i}) - f(B_{t_{i-1}})$ by a two-term Taylor expansion. Then, we have

$$
\begin{aligned}
f(B_T) - f(0) &= \sum_{i=1}^{n} (f(B_{t_i}) - f(B_{t_{i-1}})) \\
&= \sum_{i=1}^{n} f'(B_{t_{i-1}})(B_{t_i} - B_{t_{i-1}}) \\
&+ \frac{1}{2}\sum_{i=1}^{n} f''(B_{t_{i-1}})(B_{t_i} - B_{t_{i-1}})^2 \\
&+ \sum_{i=1}^{n} o((B_{t_i} - B_{t_{i-1}})^2)
\end{aligned}
$$

Since f' is continuous, the first term is the simple process approximation to an Itô integral so

$$
\sum_{i=1}^{n} f'(B_{t_{i-1}})(B_{t_i} - B_{t_{i-1}}) \to \int_0^T f'(B_t)dB_t, \text{ in probability.}
$$

If f'' were constant, say $f'' = b$, then the limit of the second term would be

$$
\lim_{n\to\infty} \frac{b}{2}\sum_{i=1}^{n}(B_{t_i} - B_{t_{i-1}})^2 = \frac{b}{2}Q(B_t, T) = \frac{bT}{2}.
$$

Let $h(t) = f''(B_t)$ which is a continuous function. For every $\epsilon > 0$, there exists a step function $h_\epsilon(t)$ such that $|h(t) - h_\epsilon(t)| < \epsilon$ for every t. For fixed ϵ, consider each interval on which h_ϵ is constant. Then, we have

$$
\lim_{n\to\infty} \sum_{i=1}^{n} h_\epsilon(t)(B_{t_i} - B_{t_{i-1}})^2 = \int_0^T h_\epsilon(t)dt.
$$

Also,

$$
\left|\sum_{i=1}^{n}(h(t) - h_\epsilon(t))(B_{t_i} - B_{t_{i-1}})^2\right| \le \epsilon \sum_{i=1}^{n}(B_{t_i} - B_{t_{i-1}})^2 \to \epsilon T.
$$

Then, the limit of the second term is the same as

$$
\lim_{\epsilon\to 0} \int_0^T h_\epsilon(t)dt = \int_0^T f''(B_t)dt.
$$

For the third term,

$$
\lim_{n\to\infty} \sum_{i=1}^{n} o((B_{t_i} - B_{t_{i-1}})^2) = \lim_{n\to\infty} \sum_{i=1}^{n} o(\frac{1}{n}) = 0.
$$

□

Definition 4.2. An *Itô process* is a stochastic process $\{X_t\}$ that satisfies

$$
dX_t = a_t dt + c_t dB_t,
$$

where a_t and c_t are adapted processes with continuous paths.

Definition 4.3. A stochastic process S_t is said to follow a *geometric Browian motion* if it satisfies the equation

$$
dS_t = \mu S_t dt + \sigma S_t dB_t.
$$

Geometric Browinan motion is an example of Itô process.

Theorem 4.4. *Suppose $f(t,x)$ is a function that is C^1 for t and C^2 for x, $\{X_t\}$ is an Itô process. Then*

$$f(t,X_t) = f(0,X_0) + \int_0^t \frac{\partial f}{\partial x}(s,X_s)a_s ds \int_0^t \frac{\partial f}{\partial s}(s,X_s)b_s dX_s + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}b_s^2(s,B_s)ds$$

in differential form,

$$df(t,X_t) = \frac{\partial f}{\partial t}dt + \frac{\partial f}{\partial x}dX_t + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}(dX_t)^2$$

This theorem allows us to integrate Itô processes and the proof for the theorem is similar for that of Itô formula, so it is omitted here.

5. Black-Scholes Equation

An *option* for an asset S (such as stock) that changes its value with respect to time t is the contract that gives the person a right to either buy or sell the asset at a predetermined price called the strike price K at or before a future date T, called the expiry time. A *call option* is an option to buy an asset. A *put option* is an option to sell an asset.

If the option can be only exercised at the expiry date, it is called a *European option*, if it can be exercised at any time prior to the expiry date, it is called an *American option.*

An *arbitrage* is the opportunity to profit from buying and selling an asset at the the same time by exploiting the price difference in the market. An *arbitrage free* market does not allow arbitrage.

Example 5.1. Consider a stock that has present price S, at a future time T, will either go up with price $\alpha S, \alpha > 1$ for probability p or go down with price $\beta S, \beta < 1$ for probability $1-p$. Suppose the risk-free rate is r. For a call option $C(S,t)$, we assume that the option follows risk neutral measure. That is, the price of the option is equal to the expectation of the price discounted by the risk-free rate under the measure. Mathematically, it is defined by

$$C(S,0) = e^{-rt}\mathbb{E}[C(S,t)].$$

Then, the arbitrage free price for the call option $C(S,T)$ with strike price K is

$$C(S,T) = \frac{(\alpha S - k)(1-\beta e^{-rT})}{\alpha - \beta}.$$

Theorem 5.2 (Black-Scholes Equation)**.** *Given a European call option $C(S,t)$ with expiry time T and strike price K, on a stock with price S following geometric Brownian motion, and the risk-free rate r, then*

$$\frac{\partial C}{\partial t} + rS\frac{\partial C}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 C}{\partial S^2} = rC.$$

Proof. To develop the arbitrage-free price of the option, we will build a portfolio V_t that consists of the stock with price S and the risk-free bond with rate r. We want the price of the option C to equal the value of the portfolio that is maintained to have value $C(S_T)$ at time T.

Let S_t model the price of the underlying stock, which is assumed to follow geometric Brownian motion

$$dS_t = \mu S_t dt + \sigma S_t dB_t.$$

Let R_t be the price of the risk-free bond, which is assumed to increase in value exponentially, satisfying

$$dR_t = rR_t dt.$$

Let a_t and b_t be adapted processes that represent the combination of stock and bonds we hold at time t, respectively. At any given time, the value of the portfolio is

$$V_t = aS_t + bR_t.$$

Next, we assume that our replicating portfolio is self-financing. That is, any change in the portfolio must come form either a change in value of the stock or a change in the value of the bond

$$dV_t = a_t dS_t + b_t dR_t.$$

Now, we get

$$dV_t = a_t(\mu S_t dt + \sigma S_t dB_t) + b_t(rR_t dt) = (a_t \mu S_t + (V_t - a_t S_t)rR_t)dt + a_t \sigma S_t dB_t$$

Let $C(t, S_t)$ be the desired price of the option at time t, assuming the stock price at t is S_t. We set the price of the option equal to the value of the portfolio, so $C(t, S_t) = V_t$. Thus, we apply the Itô formula to get that

$$\begin{aligned} dV_t &= \frac{\partial C}{\partial t}(t, S_t)dt + \frac{1}{2}\frac{\partial^2 C}{\partial S^2}(t, S_t)dS_t \cdot dS_t + \frac{\partial C}{\partial S}(t, S_t)dS_t \\ &= (\frac{\partial C}{\partial t} + \frac{\partial C}{\partial S}\mu S_t + \frac{1}{2}\frac{\partial^2 C}{\partial S^2}\sigma^2 S_t^2)dt + \frac{\partial C}{\partial S}\sigma S_t dB_t. \end{aligned}$$

By equating the coefficient terms, we have

$$a_t = \frac{\partial C}{\partial S}$$

$$\frac{\partial C}{\partial t} + \frac{1}{2}S_t^2\sigma^2\frac{\partial^2 C}{\partial S^2} = (V_t - a_t S_t)r$$

Solving the two equations gives us the Black-Scholes equation.

□

Acknowledgments. It is a pleasure to thank my mentor, Boming Jia, for the generous support on the paper. I would also thank Professor Peter May for hosting such a fantastic program.

References

[1] Gregory F. Lawler. Notes On Probability. http://www.math.uchicago.edu/ lawler/probnotes.pdf.
[2] Gregory F. Lawler. Stochastic Calculus: An Introduction with Applications. http://www.math.uchicago.edu/ lawler/finbook.pdf.
[3] Lawrence C. Evans. An Introduction to Stochastic Differential Equations. https://pdfs.semanticscholar.org/916a/5b03d0e6964f5efdfe6ef147815cfa2069c1.pdf.
[4] James Leiner. Brownian motion and the strong Markov property. http://math.uchicago.edu/ may/REU2014/REUPapers/Wang.pdf.
[5] Zachry Wang. An introduction to stochastic calculus and Black-Scholes option pricing. http://math.uchicago.edu/ may/REU2014/REUPapers/Wang.pdf.
[6] Effy Fang. An application of probability theory in finance: the Black-Scholes formula. http://math.uchicago.edu/ may/REU2015/REUPapers/Fang.pdf.