THE L^p CONVERGENCE OF FOURIER SERIES

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Abstract. In this expository paper, we show the L^p convergence of Fourier series of functions on the one-dimensional torus. We will first turn the question of convergence into a question regarding the uniform boundedness of the partial-sum operators of Fourier series, and then bound these partial-sum operators with the help of Hilbert transform.

Contents

1. Preliminaries

Let's begin with some definitions and conventions, followed by the motivating example of L^2-convergence of Fourier series, and some propositions that will be handy later in this paper.

1.1. **Basics about Fourier series.**

Definitions 1.1.

(1) Let $\mathbb{T} = \mathbb{R}/\mathbb{Z}$ denote the one-dimensional torus, that is, the circle. We will occasionally use the interval $\left(-\frac{1}{2}, \frac{1}{2}\right]$ to represent $\mathbb{T}$ for the sake of simplicity and of symmetry, especially when it comes to measuring subsets, integrating over a subset, or talking about the monotonicity of functions defined on $\mathbb{T}$.
(2) We mean by $e(\cdot)$ the function from $\mathbb{T}$ to $\mathbb{C}$ given by $e(\theta) := e^{2\pi i\theta}$, and we define, for each $n \in \mathbb{Z}$, the function $e_n : \mathbb{T} \to \mathbb{C}$ by $e_n(\theta) := e(n\theta)$.

(3) The notation $|\cdot|$, when applied to a measurable subset of $\mathbb{T}$, will denote its Lebesgue measure. For instance, $|\mathbb{T}| = 1$.
(4) Let $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\} = \{re(\theta) : |r| < 1, \theta \in \mathbb{T}\}$ denote the open unit disk, with its boundary $\partial\mathbb{D}$ identified with $\mathbb{T}$.

Observe that the set $\{e_n\}_{n\in\mathbb{Z}} \subseteq L^2(\mathbb{T})$ is orthonormal with respect to the usual inner product

$$\langle f, g\rangle := \int_{\mathbb{T}} f\bar{g}dm,$$

since $\int_{\mathbb{T}} e_n dm = \chi_{\{0\}}(n)$.

Definitions 1.2.

(1) We denote by $\mathcal{M}(\mathbb{T})$ the space of complex Borel measures of bounded variation on $\mathbb{T}$, and we mean by $\|\mu\|$ the total variation of a complex measure $\mu \in \mathcal{M}(\mathbb{T})$.
(2) The convolution of a function $g \in L^1(\mathbb{T})$ with a complex measure $\mu \in \mathcal{M}(\mathbb{T})$, written $g * \mu$, is defined by

$$(g * \mu)(\theta) := \int_{\mathbb{T}} g(\theta - \tau)\, d\mu(\tau).$$

In the case that $d\mu = f dm$ for some $f \in L^1(\mathbb{T})$, we may write $g * f$ in place of $g * \mu$.

By a change of variables, we can see that convolution, as a binary operation on $L^1(\mathbb{T})$, is commutative. By the Fubini-Tonelli theorem, convolution is associative. And the bilinearity of convolution follows from the linearity of integration.

It is worth noting that every $\mu \in \mathcal{M}(\mathbb{T})$ is regular, since $\mathbb{T}$ is a second-countable compact Hausdorff space. Additionally, by the Riesz-Markov theorem, there is a canonical isometric isomorphism between $\mathcal{M}(\mathbb{T})$ and $C(\mathbb{T})^*$, namely $\mu \mapsto \big(f \mapsto \int_{\mathbb{T}} f d\mu\big)$, which allows us to identify these two spaces and to equip the former with the weak* topology. Then, as an immediate result of the Banach-Alaoglu theorem, the unit ball of $\mathcal{M}(\mathbb{T})$ is compact in the weak* topology. Moreover, since $C(\mathbb{T})$ is a separable Banach space, the closed unit ball of its dual $C(\mathbb{T})^* = \mathcal{M}(\mathbb{T})$ is metrizable and therefore sequentially compact with respect to the weak* topology. (For detailed proof, see Corollary 7.6 and Corollary 7.18 in [2], and Theorem 3.29 in [1].)

Proposition 1.3. ***(Young's inequality)*** *If $f \in L^p(\mathbb{T})$, with $p \in [1, \infty]$, and if $\mu \in \mathcal{M}(\mathbb{T})$, then*

$$\|f * \mu\|_p \leqslant \|f\|_p \|\mu\|.$$

In particular, if $f \in L^p(\mathbb{T})$ and $g \in L^1(\mathbb{T})$, then

$$\|f * g\|_p \leqslant \|f\|_p \|g\|_1.$$

The above inequality essentially follows from Minkowski's inequality for integrals. Proof is provided in the appendix.

Proposition 1.4. *If $f \in L^\infty(\mathbb{T})$ and if $g \in L^1(\mathbb{T})$, then $f * g \in C(\mathbb{T})$.*

The idea of the proof is to first show that $f * g \in C(\mathbb{T})$ in the case that $g \in C(\mathbb{T})$, and then generalize to the case where $g \in L^1(\mathbb{T})$, using a density argument. Details of the proof are provided in the appendix.

Remark 1.5. While no integrable function can serve as an identity for the convolution, there is the Dirac measure δ_0 at 0 such that $g * \delta_0 = g$ for every $g \in L^1(\mathbb{T})$.

Definitions 1.6.

(1) For each $f \in L^1(\mathbb{T})$, we define its Fourier coefficients

$$\hat{f}(n) := \int_{\mathbb{T}} f\overline{e_n}dm = \int_{\mathbb{T}} fe_{-n}dm$$

for every $n \in \mathbb{Z}$.

(2) A trigonometric polynomial is a function of the form $f = \sum_{n=-\infty}^{\infty} a_n e_n$, where all but finitely many coefficients a_n are zero.

By the orthonormality of the collection $\{e_n\}_{n\in\mathbb{Z}}$, if $f : \mathbb{T} \to \mathbb{C}$ is a trigonometric polynomial, then $\hat{f}(n) = a_n$ for every $n \in \mathbb{Z}$, and

$$f = \sum_{n=-\infty}^{\infty} \hat{f}(n)e_n.$$

So, it is natural to ask if the right-hand side of the above equality is well-defined for other integrable functions on $\mathbb{T}$, and if the equality will still hold in that case. From now on, we will refer to the formal sum $\sum_{n=-\infty}^{\infty} \hat{f}(n)e_n$ as the Fourier series associated to a general $f \in L^1(\mathbb{T})$, despite that the actual series may be ill-defined.

Definition 1.7. For each $f \in L^1(\mathbb{T})$, we define its symmetric partial sums

$$S_N f := \sum_{n=-N}^{N} \hat{f}(n)e_n$$

for every $N \in \mathbb{N}$.

Expanding the formula in the above definition, we obtain that for every $N \in \mathbb{Z}^+$ and every $f \in L^1(\mathbb{T})$

$$\begin{aligned} S_N f(\theta) &= \sum_{n=-N}^{N} e(n\theta)\left(\int_{\mathbb{T}} f(\tau)e(-n\tau)d\tau\right) \\ &= \int_{\mathbb{T}} f(\tau)\left(\sum_{n=-N}^{N} e_n(\theta-\tau)\right)d\tau = \left(f * \sum_{n=-N}^{N} e_n\right)(\theta), \end{aligned}$$

which leads us to define the following.

Definition 1.8. For each $N \in \mathbb{N}$, we define a Dirichlet kernel $D_N := \sum_{n=-N}^{N} e_n$.

While the formulae for the Dirichlet kernels seem rather simple, these kernels are not so easy to analyze as one would like. For this reason, we will turn to the better-behaved Cesàro sums of the Dirichlet kernels, known as the Fejér kernels, in order to prove that the trigonometric polynomials are dense in $L^2(\mathbb{T})$, and consequently the L^2 convergence of the symmetric partial sums of Fourier series.

1.2. **The L^2 convergence of Fourier series.**

Definitions 1.9.

(1) For each $N \in \mathbb{Z}^+$, we define the Fejér kernel $K_N := \frac{1}{N}\sum_{n=0}^{N-1} D_n$.

(2) For each $f \in L^1(\mathbb{T})$, we define its Cesàro sums

$$\sigma_N f := \frac{1}{N}\sum_{n=0}^{N-1} S_n f = f * K_N.$$

(3) A family $\{\Phi_N\}_{N=1}^{\infty} \subseteq L^\infty(\mathbb{T})$ is said to form an approximate identity if it satisfies the following:
   (a) $\int_{\mathbb{T}} \Phi_N dm = 1$ for every index N;
   (b) $\sup_N \int_{\mathbb{T}} |\Phi_N| \, dm < \infty$;
   (c) given any $\epsilon > 0$, one has that $\int_{|x|>\epsilon} |\Phi_N| \, dm \to 0$ as $N \to \infty$.

The following proposition justify the appellation "approximate identity".

Proposition 1.10. *Let $\{\Phi_N\}_{N=1}^{\infty}$ be an approximate identity.*
(1) If $f \in C(\mathbb{T})$, then $\|\Phi_N * f - f\|_\infty \to 0$ as $N \to \infty$.
(2) If $f \in L^p(\mathbb{T})$, with $p \in [1, \infty)$, then $\|\Phi_N * f - f\|_p \to 0$ as $N \to \infty$.
(3) For every $\mu \in \mathcal{M}(\mathbb{T})$, $\Phi_N * \mu \to \mu$ in the weak sense as $N \to \infty$.*

Proof is provided in the appendix.

Examples 1.11.
(1) The box kernels $\left\{\frac{N}{2}\chi_{\left[-\frac{1}{N}, \frac{1}{N}\right]}\right\}_{N=1}^{\infty}$ form an approximate identity.
(2) The Fejér kernels $\{K_N\}_{N=1}^{\infty}$ form an approximate identity.
(3) The Dirichlet kernels $\{D_N\}_{N=1}^{\infty}$ do not form an approximate identity.

Let's justify briefly the second and the third examples. Expanding the formulae given in the definitions, we can show that for every positive integer N and every $x \in \mathbb{T} \setminus \{0\}$,

$$D_N(x) = \frac{\sin((2N+1)\pi x)}{\sin(\pi x)} \qquad \text{and} \qquad K_N(x) = \frac{1}{N}\left(\frac{\sin(N\pi x)}{\sin(\pi x)}\right)^2.$$

It then follows from the nonnegativity of each K_N that

$$\|K_N\|_1 = \int_{\mathbb{T}} K_N(x) dx = 1.$$

In addition, for every $\epsilon > 0$,

$$\limsup_{N\to\infty} \int_{|x|>\epsilon} |K_N(x)| dx \leqslant \limsup_{N\to\infty} \frac{1}{N} \int_{|x|>\epsilon} \frac{1}{\sin^2(\pi x)} dx = 0.$$

Hence, the Fejér kernels $\{K_N\}_{N=1}^{\infty}$ form an approximate identity.

On the other hand, for every positive integer N,

$$\begin{aligned}
\|D_N\|_1 = 2\int_0^{1/2} \frac{|\sin((2N+1)\pi x)|}{\sin(\pi x)} dx &\geqslant 2\int_0^{1/2} \frac{|\sin((2N+1)\pi x)|}{\pi x} dx \\
&\geqslant \frac{2}{\pi} \sum_{k=0}^{N-1} \int_{\frac{k}{2N+1}}^{\frac{k+1}{2N+1}} \frac{|\sin((2N+1)\pi x)|}{x} dx \\
&\geqslant \frac{2}{\pi} \sum_{k=0}^{N-1} \left(2 \cdot \frac{1}{2N+1} \cdot \frac{2N+1}{k+\frac{1}{2}}\right) \geqslant \frac{4}{\pi} \sum_{k=1}^{N} \frac{1}{k},
\end{aligned}$$

and whence that $\sup_N \|D_N\|_1 = \infty$. Therefore, the Dirichlet kernels $\{D_N\}_{N=1}^{\infty}$ do not form an approximate identity.

Corollary 1.12.
(1) If $f \in L^1(\mathbb{T})$ and if $\hat{f}(n) = 0$ for every integer n, then $f = 0$ a.e. on $\mathbb{T}$.
(2) The trigonometric polynomials are dense in $L^p(\mathbb{T})$ for every $p \in [1, \infty)$, and are dense in $C(\mathbb{T})$ with respect to the uniform norm.

Proof.

(1) Suppose that $f \in L^1(\mathbb{T})$ and that $\hat{f}(n) = 0$ for every integer n. Since the Fejér kernels $\{K_N\}_{N=1}^{\infty}$ form an approximate identity, it follows that
$$0 = \lim_{N\to\infty} \|\sigma_N f - f\|_1 = \lim_{N\to\infty} \|0 - f\|_1 = \|f\|_1,$$
which implies that $f = 0$ a.e.

(2) The second corollary follows from the fact that the Cesàro sums of any L^p function are trigonometric polynomials, and that the Fejér kernels form an approximate identity. □

The L^2 convergence of Fourier series now follows from the first collorary.

Theorem 1.13. *The collection $\{e_n\}_{n\in\mathbb{Z}}$ forms an orthonormal basis for $L^2(\mathbb{T})$, and the following holds.*

(1) For every $f \in L^2(\mathbb{T})$, $S_N f \to f$ in L^2 as $N \to \infty$.

(2) For any two $f, g \in L^2(\mathbb{T})$,
$$\langle f, g\rangle = \sum_{n\in\mathbb{Z}} \hat{f}(n)\overline{\hat{g}(n)}.$$
In particular, one has Parseval's identity
$$\|f\|_2^2 = \sum_{n\in\mathbb{Z}} \left|\hat{f}(n)\right|^2.$$

Proof. Suppose that f is an $L^2(\mathbb{T})$ function with $\langle f, e_n\rangle = 0$ for every $n \in \mathbb{N}$. Since $L^2(\mathbb{T}) \subseteq L^1(\mathbb{T})$, and since $\hat{f}(n) = \langle f, e_n\rangle = 0$ for every n, it follows that $f = 0$ a.e. Hence, the orthonormal system $\{e_n\}_{n=-\infty}^{\infty}$ is complete, i.e. $\{e_n\}_{n=-\infty}^{\infty}$ is an orthonormal basis for $L^2(\mathbb{T})$. The remainder of this theorem follows immediately from basic properties of an orthonormal basis for a Hilbert space, but can also be proven using the ensuing proposition, which states an equivalent but more accessible condition for the L^p convergence of Fourier series. □

Proposition 1.14. *For each $p \in [1, \infty]$, the following statements are equivalent:*

(1) $\sup_{N\in\mathbb{N}} \|S_N\|_{p\to p} < \infty$;

(2) For every $f \in L^p(\mathbb{T})$ (or for every $f \in C(\mathbb{T})$ if $p = \infty$),
$$\|S_N f - f\|_p \to 0 \quad \text{as } N \to \infty.$$

Proof. Fix an arbitrary $p \in [1, \infty]$. Firstly, let's suppose $\sup_N \|S_N\|_{p\to p} < \infty$. Then, for every $f \in L^p(\mathbb{T})$ (or for every $f \in C(\mathbb{T})$ if $p = \infty$) and for every trigonometric polynomial g,
$$\begin{aligned}\limsup_{N\to\infty} \|S_N f - f\|_p &\leqslant \limsup_{N\to\infty} \left(\|S_N f - S_N g\|_p + \|g - f\|_p + \|S_N g - g\|_p\right)\\ &\leqslant \limsup_{N\to\infty} \left(\|S_N\|_{p\to p}\|f - g\|_p + \|f - g\|_p\right) + 0\\ &\leqslant \left(\sup_{N\in\mathbb{N}} \|S_N\|_{p\to p} + 1\right) \|f - g\|_p\end{aligned}$$
Since the trigonometric polynomials are dense in $L^p(\mathbb{T})$ (and in $C(\mathbb{T})$ with respect to the uniform norm), taking the infimum over all trigonometric polynomials g, we obtain that $\limsup_{N\to\infty} \|S_N f - f\|_p \leqslant 0$ for every $f \in L^p(\mathbb{T})$, which implies that
$$\|S_N f - f\|_p \to 0 \quad \text{as } N \to \infty.$$

On the other hand, suppose that $\sup_N \|S_N\|_{p\to p} = \infty$. Since $L^p(\mathbb{T})$ is complete (and $C(\mathbb{T})$ is complete with respect to the uniform norm), by the uniform boundedness principle, there exists $f \in L^p(\mathbb{T})$ (or $f \in C(\mathbb{T})$ if $p = \infty$) such that $\sup_N \|S_N f\|_p = \infty$, and consequently $S_N f$ diverges in L^p as $N \to \infty$. $\square$

An alternative proof of Theorem 1.13.
For every $f \in L^2(\mathbb{T})$ and every $N \in \mathbb{N}$, we have

$$\begin{aligned} 0 &\leqslant \|f - S_N f\|_2^2 \\ &= \|f\|_2^2 - 2\,\mathrm{Re}\left(\int_{\mathbb{T}} f(x)\overline{S_N f(x)}\,dx\right) + \|S_N f(x)\|_2^2 \\ &= \|f\|_2^2 - 2\,\mathrm{Re}\left(\sum_{n=-N}^{N} \left|\hat{f}(n)\right|^2\right) + \sum_{n=-N}^{N} \left|\hat{f}(n)\right|^2 \\ &= \|f\|_2^2 - \sum_{n=-N}^{N} \left|\hat{f}(n)\right|^2, \end{aligned}$$

which implies that $\|S_N f\|_2^2 = \sum_{n=-N}^{N} \left|\hat{f}(n)\right|^2 \leqslant \|f\|_2^2$. Thus,

$$\sup_{N\in\mathbb{N}} \|S_N\|_{2\to 2} \leqslant 1 < \infty,$$

and it follows that $\lim_{N\to\infty} \|S_N f - f\|_2 = 0$ for every $f \in L^2(\mathbb{T})$. Furthermore, we deduce from the continuity of the inner product that

$$\begin{aligned} \langle f, g\rangle &= \lim_{N\to\infty}\lim_{M\to\infty} \langle S_N f, S_M g\rangle \\ &= \lim_{N\to\infty}\lim_{M\to\infty} \sum_{n=-\min\{N,M\}}^{\min\{N,M\}} \hat{f}(n)\overline{\hat{g}(n)} = \sum_{n=-\infty}^{\infty} \hat{f}(n)\overline{\hat{g}(n)}. \end{aligned}$$

$\square$

In the end, we will show that $\sup_{N\in\mathbb{N}} \|S_N\|_{p\to p} < \infty$ for every $p \in (1, \infty)$, and then invoke Proposition 1.14 to complete the proof of the L^p convergence of Fourier series. But before doing so, let's show that the Fourier series does not always converge to the original function in $L^1(\mathbb{T})$, nor in $C(\mathbb{T})$.

Examples 1.15.

(1) There exist $L^1(\mathbb{T})$ functions whose Fourier series do not converge to the original function.
(2) There exist $C(\mathbb{T})$ functions whose Fourier series do not converge to the original function with respect to the uniform norm.

Proof. By the previous proposition, it suffices to show that $\sup_{N\in\mathbb{N}} \|S_N\|_{1\to 1}$ and $\sup_{N\in\mathbb{N}} \|S_N\|_{\infty\to\infty}$ are both infinite.

(1) Since every Fejér kernel has L^1 norm $\|K_M\|_1 = 1$, and since the family of Fejér kernels form an approximate identity, we have that

$$\begin{aligned} \sup_{N\in\mathbb{N}} \|S_N\|_{1\to 1} &= \sup_{N\in\mathbb{N}} \sup_{f:\|f\|_1=1} \|D_N * f\|_1 \\ &\geqslant \sup_{N\in\mathbb{N}} \sup_{M\in\mathbb{N}} \|D_N * K_M\|_1 \geqslant \sup_{N\in\mathbb{N}} \|D_N\|_1 = \infty. \end{aligned}$$

(2) Observe that $\operatorname{sgn} \circ D_N \circ (-Id_{\mathbb{T}})(-x) = \operatorname{sgn}(D_N(x))$ for every $x \in \mathbb{T}$ and every $N \in \mathbb{N}$. Hence,

$$\begin{aligned} \sup_{N\in\mathbb{N}} \|S_N\|_{\infty\to\infty} &\geqslant \sup_{N\in\mathbb{N}} \sup_{f:\|f\|_\infty=1} |D_N * f(0)| \\ &\geqslant \sup_{N\in\mathbb{N}} |D_N * (\operatorname{sgn} \circ D_N \circ (-Id_{\mathbb{T}}))(0)| \\ &= \sup_{N\in\mathbb{N}} \|D_N\|_1 = \infty. \end{aligned}$$ □

1.3. **Some other useful facts.** To generalize the L^2 convergence of Fourier series to L^p convergence for every $p \in (1, \infty)$, we will utilize the Marcinkiewicz interpolation theorem, which provides us with boundedness properties on "intermediate spaces".

Definition 1.16. A complex valued measurable function g on a measure space (X, Σ, μ) is said to belong to weak-L^p, with $p \in [1, \infty)$, if there exists a finite constant $C > 0$ such that for every $\lambda > 0$,

$$\mu(\{x \in X : |g(x)| > \lambda\}) \leqslant C\lambda^{-p},$$

i.e. if the following quantity is finite:

$$[g]_p := \sup_{\lambda>0} \mu(\{x \in X : |g(x)| > \lambda\}) \cdot \lambda^p.$$

Note that by Chebyshev's inequality, every L^p function, with $p \in [1, \infty)$, belongs to weak-L^p.

Definitions 1.17. Let T be a map from some vector space $\mathcal{D}$ of complex valued measurable functions on a measure space (X, Σ, μ) to the space of all complex valued measurable functions on the space (X, Σ, μ).

(1) T is said to be sublinear if for all $f, g \in \mathcal{D}$ and $c > 0$,

$$|T(f+g)| \leqslant |Tf| + |Tg| \quad \text{and} \quad |T(cf)| = c|Tf|.$$

(2) A sublinear operator T is said to be of strong-type (p,p), with $p \in [1, \infty]$, if $L^p(X) \subseteq \mathcal{D}$, and if T is bounded on $L^p(X)$, meaning that there exists a constant $C > 0$ such that for every $f \in L^p(X)$,

$$\|Tf\|_p \leqslant C\|f\|_p.$$

(3) A sublinear operator T is said to be of weak-type (p,p), with $p \in [1, \infty)$, if $L^p(X) \subseteq \mathcal{D}$ and T is bounded from $L^p(X)$ to weak-$L^p(X)$, meaning that there exists a constant $C > 0$ such that for every $f \in L^p(X)$ and every $\lambda > 0$,

$$\mu(\{x \in X : |Tf(x)| > \lambda\}) \leqslant C\|f\|_p\lambda^{-p}.$$

We also define weak-type (∞, ∞) to mean the same as strong-type (∞, ∞).

Theorem 1.18. ***(A simplified version of the Marcinkiewicz interpolation theorem)*** *If a sublinear operator T is simultaneously of weak-type (p,p) and of weak-type (q,q), with $1 \leqslant p < q \leqslant \infty$, then T is of strong-type (r,r) for every $p < r < q$.*

Proof of this theorem can be found in [2], [3], and [4].

2. Poisson Kernel and Hardy Spaces

It is known that Cesàro summability does not imply the usual summability which we are seeking to prove. For this reason, we now introduce a different derivate of the Dirichlet kernels, called the Poisson kernel, which will allow us to invoke some nice properties of functions of complex variables, such as the mean-value property and the maximal principle of harmonic functions.

2.1. **Poisson kernel.** Observe that for each $f \in L^1(\mathbb{T},)$ the formal sum

$$\sum_{n=-\infty}^{\infty} \hat{f}(n)e_n(\theta)$$

can also be written as

$$\sum_{n=1}^{\infty} \hat{f}(-n)r^n e_{-n}(\theta) + \sum_{n=0}^{\infty} \hat{f}(n)r^n e_n(\theta),$$

with $r = 1$. Moreover, when $r \in [0,1)$, the two series, $\sum_{n=1}^{\infty} \hat{f}(-n)r^n e_{-n}(\theta)$ and $\sum_{n=0}^{\infty} \hat{f}(n)r^n e_n(\theta)$, are both absolutely convergent since $\left|\hat{f}(n)\right| \leqslant \|f\|_1$ for every $n \in \mathbb{Z}$, whence

$$\begin{aligned}
&\sum_{n=1}^{\infty} \hat{f}(-n)r^n e_{-n}(\theta) + \sum_{n=0}^{\infty} \hat{f}(n)r^n e_n(\theta) \\
&= \sum_{n=-\infty}^{\infty} \hat{f}(n)r^{|n|}e_n(\theta) = \sum_{n=-\infty}^{\infty} \int_{\mathbb{T}} f(t)r^{|n|}e_n(\theta - t)dt \\
&= \int_{\mathbb{T}} f(t)\left(\sum_{n=-\infty}^{\infty} r^{|n|}e_n(\theta - t)\right)dt = \left(f * \sum_{n=-\infty}^{\infty} r^{|n|}e_n\right)(\theta)
\end{aligned}$$

where the third equality holds by the Fubini-Tonelli theorem, considering the Lebesgue measure on $\mathbb{T}$ and the counting measure on $\mathbb{Z}$.

This observation leads us to define the following.

Definition 2.1. For each $r \in [0,1)$, we define the Poisson kernel $P_r : \mathbb{T} \to \mathbb{C}$ by $P_r(\theta) := \sum_{n=-\infty}^{\infty} r^{|n|}e_n(\theta)$.

From now on, whenever F is a function on $\mathbb{D}$ and r is in $[0,1)$, the notation F_r will refer to the function on $\mathbb{T}$ defined by $F_r(\theta) := F(re(\theta))$. Conversely, if a family $\{F_r\}_{r\in[0,1)}$ of functions on $\mathbb{T}$ is defined, with F_0 constant, then we will mean by F the function on $\mathbb{D}$ defined by $F(re(\theta)) := F_r(\theta)$.

Remark 2.2. Rearranging the formula for the Poisson kernel, we obtain that

$$\begin{aligned}
P(z) = P_r(\theta) &= \sum_{n=1}^{\infty} r^n e(-n\theta) + \sum_{n=0}^{\infty} r^n e(n\theta) \\
&= \frac{re(-\theta)}{1 - re(-\theta)} + \frac{1}{1 - re(\theta)} = \frac{1 - r^2}{1 - 2r\cos(2\pi\theta) + r^2} = \operatorname{Re}\left(\frac{1+z}{1-z}\right)
\end{aligned}$$

for every $z = re(\theta) \in \mathbb{D}$. Since $\frac{1+z}{1-z}$ is analytic on $\mathbb{D}$, the function P is harmonic on $\mathbb{D}$. Note also that for every $r \in [0,1)$,

$$P_r \geqslant \frac{1-r^2}{1+2r+r^2} = \frac{1-r}{1+r} > 0$$

and

$$\|P_r\|_1 = \sum_{n=-\infty}^{\infty} r^{|n|}\|e_n\|_1 = r^0\|e_0\|_1 = 1.$$

Proposition 2.3. *For every $\mu \in \mathcal{M}(\mathbb{T})$, the function $F : \mathbb{D} \to \mathbb{C}$ defined by $F_r := P_r * \mu$ for every $r \in [0,1)$ is harmonic on $\mathbb{D}$.*

Proof. Fix an arbitrary $\mu \in \mathcal{M}(\mathbb{T})$. Then, for every $z = re(\theta) \in \mathbb{D}$, we have

$$F(z) = F_r(\theta) = \sum_{n=1}^{\infty} \hat{\mu}(-n)r^n e_{-n}(\theta) + \sum_{n=0}^{\infty} \hat{\mu}(n)r^n e_n(\theta) = \overline{\sum_{n=1}^{\infty} \overline{\hat{\mu}(-n)} z^n} + \sum_{n=0}^{\infty} \hat{\mu}(n)z^n,$$

with

$$\sum_{n=1}^{\infty} \left|\overline{\hat{\mu}(-n)}\right| |z^n| \leqslant \|\mu\| \frac{|z|}{1-|z|} < \infty \quad \text{and} \quad \sum_{n=0}^{\infty} |\hat{\mu}(n)|\,|z^n| \leqslant \|\mu\| \frac{1}{1-|z|} < \infty.$$

From the absolute convergence of the two series $\sum_{n=1}^{\infty} \overline{\hat{\mu}(-n)} z^n$ and $\sum_{n=0}^{\infty} \hat{\mu}(n)z^n$, where $z \in \mathbb{D}$, it follows that the functions $\sum_{n=1}^{\infty} \overline{\hat{\mu}(-n)} z^n$ and $\sum_{n=0}^{\infty} \hat{\mu}(n)z^n$ are both analytic in $\mathbb{D}$. Hence, F is a sum of two harmonic functions, and is therefore itself harmonic. □

Proposition 2.4. *The Poisson kernels $\{P_r\}_{r\in(0,1)}$ form an approximate identity, with the index set $[0,1)$ in place of $\mathbb{Z}^+$. Moreover, all the convergence properties mentioned in Proposition 1.10 still hold, with the limit $r \to 1^-$ in place of $N \to \infty$.*

The proof of the above proposition is omitted, as the first half of this proposition can be shown via a simple calculation, and the proof of the second half is parallel to that of Proposition 1.10, which can be found in the appendix.

2.2. **Correspondence between $h^1(\mathbb{D})$ and $\mathcal{M}(\mathbb{T})$.**

Definition 2.5. For each harmonic function $u : \mathbb{D} \to \mathbb{C}$ and each $p \in [1,\infty]$, we define

$$|||u|||_p := \sup_{0<r<1} \|u(re(\cdot))\|_{L^p(\mathbb{T})}$$

which may attain the value ∞. Further, for each $p \in [1,\infty]$, we define the "little" Hardy space

$$h^p(\mathbb{D}) := \left\{u : \mathbb{D} \to \mathbb{C} \text{ harmonic} : |||u|||_p < \infty\right\},$$

and equip it with the norm $|||\cdot|||_p$.

Examples 2.6.

(1) Every non-negative harmonic function on $\mathbb{D}$ is in $h^1(\mathbb{D})$. In particular, the Poisson kernel P is in $h^1(\mathbb{D})$.
(2) If $f \in L^p(\mathbb{T})$, with $p \in [1, \infty]$, then the function $F : \mathbb{D} \to \mathbb{C}$ defined by $F_r = P_r * f$ is in $h^p(\mathbb{D})$, with $|||F|||_p \leqslant \|f\|_p$.

Proof.

(1) Let $F : \mathbb{D} \to [0, \infty)$ be any non-negative harmonic function. Then, by the mean-value theorem,
$$\|F_r\|_1 = \int_{\mathbb{T}} F(re(\theta))\, d\theta = F(0)$$
for every $r \in (0, 1)$, and thus that $|||F|||_1 = F(0) < \infty$.
(2) If $f \in L^p(\mathbb{T})$, then by Proposition 2.3 is the function $F : \mathbb{D} \to \mathbb{C}$ harmonic, and by Young's inequality,
$$|||F|||_p = \sup_{0<r<1} \|P_r * f\|_p \leqslant \sup_{0<r<1} \|P_r\|_1 \|f\|_p = \|f\|_p. \qquad \square$$

Lemma 2.7.

(1) If $F \in C\left(\overline{\mathbb{D}}\right)$, and if $\triangle F = 0$ in $\mathbb{D}$, then $F_r = P_r * F_1$ for every $r \in [0, 1)$.
(2) If $\triangle F = 0$ in $\mathbb{D}$, then $F_{rs} = P_r * F_s$ for any $r, s \in [0, 1)$.
(3) If $\triangle F = 0$ in $\mathbb{D}$, then for any $p \in [1, \infty]$, the norm $\|F_r\|_p$ is non-decreasing as a function of $r \in [0, 1)$.

Proof.

(1) Define a function $u : \mathbb{D} \to \mathbb{C}$ by $u(re(\theta)) := (P_r * F_1)(\theta)$. Since $\{P_r\}_{0 \leqslant r<1}$ forms an approximate identity as $r \to 1-$, the uniform norm $\|u(re(\cdot)) - F_1\|_\infty \to 0$ as $r \to 1^-$. So the harmonic function u can be extended to a continuous function on $\overline{\mathbb{D}}$ with the same boundary value as that of F. Since the continuous function F is also harmonic in $\mathbb{D}$, by the maximum principle, $F\restriction_{\mathbb{D}} = u$, which is to say that $F_r = P_r * F_1$ for every $r \in [0, 1)$.
(2) The idea is to rescale the unit disc so that part (1) applies. Fix an arbitrary $s \in [0, 1)$, and define a function $G : \overline{\mathbb{D}} \to \mathbb{C}$ by $G(re(\theta)) = F(sre(\theta))$. It follows that $G \in C\left(\overline{\mathbb{D}}\right)$, $\triangle G = 0$ in $\mathbb{D}$, and that $G_r = F_{rs}$ for every $r \in [0, 1)$. Hence, $F_{rs} = G_r = P_r * G_1 = P_r * F_s$ for every $r \in [0, 1)$.
(3) Fix any $p \in [1, \infty]$, and let $0 \leqslant r_1 < r_2 < 1$ be given arbitrarily. Since $r_1/r_2 \in [0, 1)$, we have that
$$\|F_{r_1}\|_p = \left\|P_{r_1/r_2} * F_{r_2}\right\|_p \leqslant \left\|P_{r_1/r_2}\right\|_1 \|F_{r_2}\|_p = \|F_{r_2}\|_p .$$
Thus, the norm $\|F_r\|_p$, as a function of $r \in [0, 1)$, is non-decreasing. $\square$

Lemma 2.8. *For each $F \in h^1(\mathbb{D})$, there exists a unique complex measure $\mu \in \mathcal{M}(\mathbb{T})$ such that $F_r = P_r * \mu$ for every $r \in [0, 1)$.*

Proof. Let $F \in h^1(\mathbb{D})$ be given arbitrarily. If $|||F|||_1 = 0$, then there is obviously the measure $\mu_{\text{null}} : \mathcal{B}_{\mathbb{T}} \to \{0\}$ such that $F_r = 0 = P_r * \mu_{\text{null}}$ for every $r \in [0, 1)$. So, we may assume that $|||F|||_1 > 0$. Since the closed unit ball of $\mathcal{M}(\mathbb{T})$ is sequentially compact in the weak* topology, the sequence $\left\{\frac{1}{|||F|||_1} F_{1-1/n}\right\}_{n=1}^{\infty}$ has a subsequence, say $\left\{\frac{1}{|||F|||_1} F_{1-1/n_k}\right\}_{k=1}^{\infty}$, that converges in the weak* sense to some $\mu_0 \in \mathcal{M}(\mathbb{T})$. Set

$\mu := \|F\|_1 \mu_0 \in \mathcal{M}(\mathbb{T})$. Then for every $r \in [0,1)$ and every $\theta \in \mathbb{T}$, since $P_r \in C(\mathbb{T})$, we obtain that

$$(P_r * \mu)(\theta) = \lim_{k\to\infty} \left(P_r * F_{1-1/n_k}\right)(\theta) = \lim_{k\to\infty} F_{r(1-1/n_k)}(\theta) = F_r(\theta).$$

Moreover, if there are two complex measures $\mu, \nu \in \mathcal{M}(\mathbb{T})$ such that $P_r * \mu = F_r = P_r * \nu$ for every $r \in [0,1)$, then

$$\int_{\mathbb{T}} f d\mu = \lim_{r\to 1^-} \int_{\mathbb{T}} f(\theta)(P_r * \mu)(\theta) d\theta = \lim_{r\to 1^-} \int_{\mathbb{T}} f(\theta)(P_r * \nu)(\theta) d\theta = \int_{\mathbb{T}} f d\nu$$

for every $f \in C(\mathbb{T})$, whence $\mu = \nu$.

From the above, we conclude that there is a unique complex measure $\mu \in \mathcal{M}(\mathbb{T})$ such that $F_r = P_r * \mu$ for every $r \in [0,1)$. □

Theorem 2.9. *There is a one-to-one correspondence between $\mathcal{M}(\mathbb{T})$ and $h^1(\mathbb{D})$, given by $\mu \mapsto (F_r(\theta) := (P_r * \mu)(\theta))$. Moreover, for every $\mu \in \mathcal{M}(\mathbb{T})$,*

$$\|\mu\| = \sup_{0<r<1} \|F_r\|_1 = \lim_{r\to 1^-} \|F_r\|_1 .$$

In addition, the following holds.

(1) A measure $\mu \in \mathcal{M}(\mathbb{T})$ is absolutely continuous with respect to Lebesgue measure iff $\{F_r\}_{0<r<1}$ converges in $L^1(\mathbb{T})$ as $r \to 1^-$. If this is the case, then $d\mu = f d\theta$, where f is the L^1 limit of F_r.

(2) For every $p \in (1,\infty]$ and every $\mu \in \mathcal{M}(\mathbb{T})$, the following are equivalent:
- *(a) $d\mu = f d\theta$ for some $f \in L^p$;*
- *(b) $\{F_r\}_{0<r<1}$ converges in L^p if $p \in (1,\infty)$ and converges in the weak* sense if $p = \infty$ as $r \to 1^-$;*
- *(c) $\{F_r\}_{0<r<1}$ is L^p-bounded.*

(3) For every $\mu \in \mathcal{M}(\mathbb{T})$, the following statements are equivalent:
- *(a) $d\mu = f d\theta$ for some $f \in C(\mathbb{T})$;*
- *(b) $\{F_r\}_{0<r<1}$ converges uniformly as $r \to 1^-$;*
- *(c) F extends to a continuous function on $\overline{\mathbb{D}}$.*

Proof. Recall from Proposition 2.3 that for each $\mu \in \mathcal{M}(\mathbb{T})$, the function $F : \mathbb{D} \to \mathbb{C}$ defined by $F_r := P_r * \mu$ is harmonic. Since

$$\sup_{0<r<1} \|P_r * \mu\|_1 \leqslant \sup_{0<r<1} \|P_r\|_1 \|\mu\| < \infty$$

for each $\mu \in \mathcal{M}(\mathbb{T})$, the function given by $\mu \mapsto (F_r(\theta) := (P_r * \mu)(\theta))$ is indeed a function from $\mathcal{M}(\mathbb{T})$ to $h^1(\mathbb{D})$. Additionally, given any $F \in h^1(\mathbb{D})$, by Lemma 2.8, there is a unique $\mu \in \mathcal{M}(\mathbb{T})$ such that $F_r(\theta) := (P_r * \mu)(\theta)\ \forall r \in [0,1)$. Hence, the mapping $\mu \mapsto (F_r(\theta) := (P_r * \mu)(\theta))$ gives a one-to-one correspondence between $\mathcal{M}(\mathbb{T})$ and $h^1(\mathbb{D})$.

Fix any $\mu \in \mathcal{M}(\mathbb{T})$. Since

$$\left|\int_{\mathbb{T}} f d\mu\right| = \lim_{r\to 1^-} \left|\int_{\mathbb{T}} f F_r d\theta\right| \leqslant \|f\|_\infty \sup_{0<r<1} \|F_r\|_1$$

for every $f \in C(\mathbb{T})$, we have $\|\mu\|_{TV} = \|\mu\|_{C(\mathbb{T})^*} \leqslant \sup_{0<r<1} \|F_r\|_1$. On the other hand, we also have

$$\sup_{0<r<1} \|F_r\|_1 = \sup_{0<r<1} \|P_r * \mu\|_1 \leqslant \sup_{0<r<1} \|P_r\|_1 \|\mu\| = \|\mu\|.$$

Thus, $\|\mu\| = \sup_{0<r<1} \|F_r\|_1 = \lim_{r\to 1^-} \|F_r\|_1$, with the last equality following from Proposition 2.7(3). We now proceed to prove the remainder of the theorem.

(1) Suppose that $\mu \ll m$. Then, by Radon-Nikodym theorem, there exists a function $f \in L^1(\mathbb{T})$ such that $d\mu = f d\theta$. By Proposition 2.4, $\lim_{r\to 1^-} \|F_r - f\|_1 = \lim_{r\to 1^-} \|P_r * f - f\|_1 = 0$, which is to say that $\{F_r\}_{0<r<1}$ converges in $L^1(\mathbb{T})$ to f as $r \to 1^-$.

Conversely, suppose that $\{F_r\}_{0<r<1}$ converges in $L^1(\mathbb{T})$ to some function f as $r \to 1^-$. Then, for every $g \in C(\mathbb{T})$,

$$\begin{aligned}\lim_{r\to 1^-} \left| \int_{\mathbb{T}} g(\theta) F_r(\theta) d\theta - \int_{\mathbb{T}} g(\theta) f(\theta) d\theta \right| &\leqslant \lim_{r\to 1^-} \int_{\mathbb{T}} \|g\|_\infty |F_r(\theta) - f(\theta)| \, d\theta \\ &= \|g\|_\infty \lim_{r\to 1^-} \|F_r - f\|_1 \\ &= 0,\end{aligned}$$

that is, $\lim_{r\to 1^-} \int_{\mathbb{T}} g(\theta) F_r(\theta) d\theta = \int_{\mathbb{T}} g(\theta) f(\theta) d\theta$. In other words, $\{F_r d\theta\}_{0<r<1}$ converges to $f d\theta$ in the weak* sense as $r \to 1^-$. By uniqueness, $d\mu = f d\theta$, and consequently $\mu \ll m$.

(2) Let q denote the conjugate exponent of p, that is, the number such that $\frac{1}{p} + \frac{1}{q} = 1$. Since $L^p(\mathbb{T})$ is isometrically isomorphic to $(L^q)^*$ via the map $\psi \mapsto \left(\varphi \mapsto \int_{\mathbb{T}} \varphi\psi \, d\theta\right)$, it makes sense to speak of weak* convergence in L^p.

Suppose that $d\mu = f d\theta$ for some $f \in L^p(\mathbb{T})$. Then it follows from Proposition 2.4 that, as $r \to 1^-$, $F_r = P_r * f$ converges in L^p to f if $p \in (1, \infty)$ and converges in the weak* sense to f if $p = \infty$.

Next, suppose that $p \in (1, \infty)$ and that $\{F_r\}_{0<r<1}$ converges in L^p to some f as $r \to 1^-$. Then, by Lemma 2.7(3),

$$\sup_{0<r<1} \|F_r\|_p = \lim_{r\to 1^-} \|F_r\|_p = \|f\|_p < \infty.$$

If $p = \infty$ and F_r converges in the weak* sense to some $f \in L^\infty(\mathbb{T})$ as $r \to 1^-$, then it follows from the inclusion $C(\mathbb{T}) \subseteq L^1(\mathbb{T})$ and the uniqueness mentioned in Lemma 2.8 that $d\mu = f\,dx$. Hence,

$$\sup_{0<r<1} \|F_r\|_\infty \leqslant \sup_{0<r<1} \|P_r\|_1 \|f\|_\infty = \|f\|_\infty < \infty.$$

Finally, suppose that the collection $\{F_r\}_{0<r<1}$ is L^p-bounded. Then, the sequence $\left\{F_{1-1/n}\right\}_{n=1}^\infty \subseteq (L^q)^*$ has a subsequence, say $\left\{F_{1-1/n_k}\right\}_{k=1}^\infty$, that converges in the weak* sense to some $f \in L^p(\mathbb{T})$, because the closed unit ball of $L^p(\mathbb{T})$ is compact and metrizable with respect to the weak* topology. As a result, we have that for every $g \in C(\mathbb{T})$,

$$\int_{\mathbb{T}} g(\theta) d\mu(\theta) = \lim_{k\to\infty} \int_{\mathbb{T}} g(\theta) \left(P_{1-1/n_k} * \mu\right)(\theta) \, d\theta = \int_{\mathbb{T}} g(\theta) f(\theta) \, d\theta,$$

and whence $d\mu = f d\theta$, with $f \in L^p(\mathbb{T})$.

(3) Suppose that $d\mu = f d\theta$ for some $f \in C(\mathbb{T})$. Then, by Proposition 2.4, the functions $F_r = P_r * f \to f$ uniformly as $r \to 1^-$.

Next, suppose that the functions F_r converge uniformly to some function $f : \mathbb{T} \to \mathbb{C}$ as $r \to 1^-$. Define a function $G : \overline{\mathbb{D}} \to \mathbb{C}$ by $G{\restriction}_{\mathbb{D}} := F$ and $G{\restriction}_{\mathbb{T}} := f$. Since the harmonic function F is continuous on $\mathbb{D}$, the continuity of f follows from uniform convergence. In order to prove that the function G is continuous on $\overline{\mathbb{D}}$, it suffices to show that G is continuous on $\overline{\mathbb{D}} \setminus \mathbb{D}$, that is, on $\mathbb{T}$. Fix an arbitrary $e(\theta) \in \mathbb{T}$, and an arbitrary $\epsilon > 0$. Then, there exist $\delta_1 \in (0,1)$ such that $\|F_r - f\|_\infty < \epsilon/2$ whenever $r \in (1-\delta_1, 1)$, and $\delta_2 \in (0, \delta_1)$ such that $|f(\tau) - f(\theta)| < \epsilon/2$ whenever $|\tau - \theta| < \delta_2$. Consequently, for every $re(\tau) \in \overline{\mathbb{D}}$ with $|re(\tau) - e(\theta)| < \delta_2$, we have $r > 1 - \delta_2 > 1 - \delta_1$ and $|\tau - \theta| < \frac{1}{2}\delta_2 < \delta_2$, whence

$$\begin{aligned} |G(re(\tau)) - G(e(\theta))| &\leqslant |F_r(\tau) - f(\tau)| + |f(\tau) - f(\theta)| < \epsilon \qquad &&\text{if } r < 1, \\ |G(re(\tau)) - G(e(\theta))| &= |f(\tau) - f(\theta)| < \epsilon/2 < \epsilon &&\text{if } r = 1. \end{aligned}$$

Therefore, the function $G : \overline{\mathbb{D}} \to \mathbb{C}$ is a continuous extension of F.

Lastly, suppose that F extends to a continuous function, say G, on $\overline{\mathbb{D}}$. Define a continuous function $f := G\restriction_{\mathbb{T}}$. Then, the continuity of G implies that $F_r \to f$ pointwise as $r \to 1^-$. In addition, the maximum principle tells us that

$$\sup_{0<r<1} \|F_r\|_\infty = \sup_{\mathbb{D}} |F| = \sup_{\mathbb{T}} |f| = \|f\|_\infty < \infty.$$

Thus, by the compactness of $\mathbb{T}$ and by the dominated convergence theorem, for every $g \in C(\mathbb{T})$,

$$\lim_{r \to 1^-} \int_{\mathbb{T}} g F_r \, d\theta = \int_{\mathbb{T}} g f \, d\theta,$$

which is to say that $f d\theta$ is the weak* limit of F_r as $r \to 1^-$. From the uniqueness of such limit, it follows that $d\mu = f d\theta$, with $f \in C(\mathbb{T})$. □

3. Two Maximal Operators

Definition 3.1. For each $\mu \in \mathcal{M}(\mathbb{T})$, we define its Hardy-Littlewood maximal function $M\mu : \mathbb{T} \to [0, \infty]$ by

$$(M\mu)(\theta) := \sup_{I \ni \theta} \left(\frac{|\mu|(I)}{|I|} \right),$$

where $I \subseteq \mathbb{T}$ is an open interval. When, $d\mu = f dm$ for some $f \in L^1(\mathbb{T})$, we may write Mf in place of $M\mu$. We will refer to the map $\mu \mapsto M\mu$ as the Hardy-Littlewood maximal operator.

Proposition 3.2. *For every $\mu \in \mathcal{M}(\mathbb{T})$ and every $\lambda > 0$, one has that*

$$|\{\theta \in \mathbb{T} : M\mu(\theta) > \lambda\}| \leqslant 3\lambda^{-1} \|\mu\|.$$

Proof. Let $\mu \in \mathcal{M}(\mathbb{T})$ and $\lambda > 0$ be given arbitrarily. Then, for each $\eta \in \{\theta \in \mathbb{T} : M\mu(\theta) > \lambda\}$, there exists some open interval $I_\eta \subseteq \mathbb{T}$ containing η such that $|\mu|(I_\eta) > \lambda \cdot |I_\eta|$. Fix any compact $K \subseteq \{\theta \in \mathbb{T} : M\mu(\theta) > \lambda\}$. Then, there is a finite subset $\{\eta_1, \ldots, \eta_N\} \subseteq K$ such that $K \subseteq \bigcup_{n=1}^N I_{\eta_n}$. By the finite Vitali covering lemma, there exist $I_{\eta_{n_1}}, \ldots, I_{\eta_{n_J}} \in \{I_{\eta_n} \mid n = 1, \ldots, N\}$ pairwise disjoint such that $\bigcup_{n=1}^N I_{\eta_n} \subseteq \bigcup_{j=1}^J 3I_{\eta_{n_j}}$, where $3I$ denotes the open interval with the same centre as I but triple the length. As a result,

$$|K| \leqslant \sum_{n=1}^N |I_{\eta_n}| \leqslant 3 \sum_{j=1}^J \left| I_{\eta_{n_j}} \right| \leqslant 3\lambda^{-1} \sum_{j=1}^J |\mu| \left(I_{\eta_{n_j}} \right) \leqslant 3\lambda^{-1} |\mu| (\mathbb{T}) = 3\lambda^{-1} \|\mu\|.$$

Thus, by the inner regularity of Lebesgue measure,

$$|\{\theta \in \mathbb{T} : M\mu(\theta) > \lambda\}| \leqslant 3\lambda^{-1} \|\mu\|.$$

□

Definition 3.3. For each function $F : \mathbb{D} \to \mathbb{C}$, we define its radial maximal function $F^* : \mathbb{T} \to [0, \infty]$ by

$$F^*(\theta) := \sup_{0<r<1} |F(re(\theta))| .$$

In order to relate the radial maximal operator to the Hardy-Littlewood maximal operator, we will need the following lemma.

Lemma 3.4. *Suppose that a function $K : \left[-\frac{1}{2}, \frac{1}{2}\right] \to \mathbb{R}$ is nonnegative, continuous, even and decreasing, that is, $0 \leqslant K(\theta) \leqslant K(\eta)$ whenever $|\theta| > |\eta|$. Then for every $\mu \in \mathcal{M}(\mathbb{T})$ and every $\theta \in \left[-\frac{1}{2}, \frac{1}{2}\right]$, one has that*

$$|(K * \mu)(\theta)| \leqslant \|K\|_1 \, M\mu(\theta).$$

The idea of the proof is to write K as an average of box kernels, for which the above inequality is easier to establish. Details can be found in the appendix.

Proposition 3.5. *For every $u \in h^1(\mathbb{D})$, one has $u^* \leqslant M\mu$, where μ is the complex Borel measure corresponding to u.*

Proof. Fix an arbitrary $u \in h^1(\mathbb{D})$, and let $\mu \in \mathcal{M}(\mathbb{T})$ be the complex measure corresponding to u, as mentioned in Lemma 2.8. Observe from the formula

$$P_r(\theta) = \frac{1 - r^2}{1 - 2r\cos(2\pi\theta) + r^2}$$

that every P_r is nonnegative, continuous, even and decreasing. So it follows from the previous lemma that for every $\theta \in \mathbb{T}$,

$$u^*(\theta) = \sup_{0<r<1} |(P_r * \mu)(\theta)| \leqslant \sup_{0<r<1} \|P_r\|_1 \, M\mu(\theta) = M\mu(\theta). \qquad \square$$

Corollary 3.6. *If $u \in h^1(\mathbb{D})$, then, for every $\lambda > 0$,*

$$|\{\theta \in \mathbb{T} : u^*(\theta) > \lambda\}| \leqslant 3\lambda^{-1} \,|||u|||_1 .$$

Proof. Fix any $u \in h^1(\mathbb{D})$, and let μ be the corresponding complex measure. Then, for every $\lambda > 0$,

$$|\{\theta \in \mathbb{T} : u^*(\theta) > \lambda\}| \leqslant |\{\theta \in \mathbb{T} : M\mu(\theta) > \lambda\}| \leqslant 3\lambda^{-1}\|\mu\| = 3\lambda^{-1} \,|||u|||_1 . \qquad \square$$

Corollary 3.7. *If $f \in L^1(\mathbb{T})$, then $P_r * f \to f$ a.e. as $r \to 1^-$.*

Proof. Let $\epsilon > 0$ be given arbitrarily. Then, there exists some $g \in C(\mathbb{T})$ such that $\|f - g\|_1 < \epsilon^2$. Set $h := f - g$, and define a function $u : \mathbb{D} \to \mathbb{C}$ by $u_r := P_r * h$. Since $\lim_{r\to 1^-} |(P_r * g)(\theta) - g(\theta)| = 0$ for every $\theta \in \mathbb{T}$, we have

$$\begin{aligned}
&\left|\left\{\theta \in \mathbb{T} : \limsup_{r\to 1^-} |(P_r * f)(\theta) - f(\theta)| > \epsilon\right\}\right| \\
&\quad\leqslant \left|\left\{\theta \in \mathbb{T} : \limsup_{r\to 1^-} |(P_r * h)(\theta)| > \frac{\epsilon}{2}\right\}\right| + \left|\left\{\theta \in \mathbb{T} : |h(\theta)| > \frac{\epsilon}{2}\right\}\right| \\
&\quad\leqslant \left|\left\{\theta \in \mathbb{T} : u^*(\theta) > \frac{\epsilon}{2}\right\}\right| + \left|\left\{\theta \in \mathbb{T} : |h(\theta)| > \frac{\epsilon}{2}\right\}\right| \\
&\quad\leqslant 3 \cdot \frac{2}{\epsilon} \cdot |||u|||_1 + \frac{2}{\epsilon} \cdot \|h\|_1 \leqslant \frac{6}{\epsilon}\|h\|_1 + \frac{2}{\epsilon}\|h\|_1 \leqslant 8\epsilon.
\end{aligned}$$

Since $\epsilon > 0$ is arbitrary, the above inequality implies that for every $\delta \in (0,1)$,

$$\begin{aligned}&\left|\left\{\theta \in \mathbb{T} : \limsup_{r\to 1^-} |(P_r * f)(\theta) - f(\theta)| > 0\right\}\right| \\ &\quad\leqslant \sum_{n=1}^{\infty} \left|\left\{\theta \in \mathbb{T} : \limsup_{r\to 1^-} |(P_r * f)(\theta) - f(\theta)| > \delta^n\right\}\right| \leqslant \frac{8\delta}{1-\delta}.\end{aligned}$$

Taking $\delta \to 0^+$, we obtain that

$$\left|\left\{\theta \in \mathbb{T} : \limsup_{r\to 1^-} |(P_r * f)(\theta) - f(\theta)| > 0\right\}\right| = 0.$$

Therefore, $P_r * f(\theta) \to f(\theta)$ as $r \to 1^-$ for a.e. $\theta \in \mathbb{T}$. □

The above corollaries will help us show that the Hilbert transform, to be defined in the next section, meets the premise of the Marcinkiewicz interpolation theorem.

4. Conjugate Function and Hilbert Transform

4.1. **Conjugate function.**

Definition 4.1. For each real-valued and harmonic function u on $\mathbb{D}$, we define its conjugate function $\tilde{u}$ to be the unique real-valued and harmonic function on $\mathbb{D}$ such that $u + i\tilde{u}$ is analytic and $\tilde{u}(0) = 0$. And, for each complex-valued and harmonic function u on $\mathbb{D}$, we define its conjugate function $\tilde{u} := \widetilde{\mathrm{Re}(u)} + i\widetilde{\mathrm{Im}(u)}$.

Proposition 4.2. *Let $u : \mathbb{D} \to \mathbb{C}$ be a harmonic function. Then the following holds.*

(1) $\tilde{\bar{u}} = \bar{\tilde{u}}$
(2) If u is constant, then $\tilde{u} = 0$.
(3) If u is analytic in $\mathbb{D}$, and if $u(0) = 0$, then $\tilde{u} = -iu$. If u is co-analytic, meaning that $\bar{u}$ is analytic, and if $u(0) = 0$, then $\tilde{u} = iu$.
(4) The function u can be written uniquely as $u = c + f + \bar{g}$ with c a constant, f, g analytic, and $f(0) = g(0) = 0$.

Proof.

(1) By the definition of conjugate function,
$$\tilde{\bar{u}} = \widetilde{\mathrm{Re}(\bar{u})} + i\widetilde{\mathrm{Im}(\bar{u})} = \widetilde{\mathrm{Re}(u)} + i(\widetilde{-\mathrm{Im}(u)}) = \widetilde{\mathrm{Re}(u)} + i\left(-\widetilde{\mathrm{Im}(u)}\right) = \bar{\tilde{u}}.$$
(2) Suppose that u is constant. Then, it is clear from the Cauchy-Riemann equations that $\tilde{u}$ is constant. Since $\tilde{u}(0) = 0$, we deduce that $\tilde{u} = 0$.
(3) Suppose that u is analytic, and that $u(0) = 0$. Then,
$$\tilde{u} = \widetilde{\mathrm{Re}(u)} + i\widetilde{\mathrm{Im}(u)} = \mathrm{Im}(u) + i\,(-\,\mathrm{Re}(u)) = -iu.$$
Next, suppose that u is co-analytic, and that $u(0) = 0$. Then, it follows from above that $\tilde{u} = \bar{\tilde{\bar{u}}} = \bar{\tilde{\bar{u}}} = \overline{-i\bar{u}} = iu$.
(4) Set $f := \frac{1}{2}(u + i\tilde{u} - u(0))$, $g := \frac{1}{2}\overline{(u - i\tilde{u} - u(0))} = \frac{1}{2}\left(\bar{u} + i\tilde{\bar{u}} - \overline{u(0)}\right)$, and $c := u(0)$. Since $u + i\tilde{u}$ and $\bar{u} + i\tilde{\bar{u}}$ are analytic, the functions f and g are both analytic. Thus, we can write u as a sum $u = c + f + \bar{g}$, with c a constant, f, g analytic, and $f(0) = g(0) = 0$.

Now suppose that there exist a constant γ and two analytic functions φ and ψ such that $\varphi(0) = \psi(0) = 0$ and $u = \gamma + \varphi + \bar{\psi}$. Then,

$$\gamma = u(0) - \varphi(0) - \overline{\psi(0)} = u(0) = c,$$

thus $\varphi + \bar{\psi} = u - u(0) = f + \bar{g}$, and consequently $\varphi - f = \overline{g - \psi}$. Since $\varphi - f$ is analytic, while $\overline{g - \psi}$ is co-analytic, $\varphi - f = \overline{g - \psi} = 0$, or in other words, $\varphi = f$ and $\psi = g$. This proves the uniqueness of the decomposition. □

Notation 4.3. Hereinafter, we will sometimes write $\mathcal{F}f(n)$ in place of $\hat{f}(n)$ for the sake of clarity.

Proposition 4.4. *Let $u : \mathbb{D} \to \mathbb{C}$ be a harmonic function. Then, for every $r \in (0, 1)$ and every $n \in \mathbb{Z}^\times$,*

$$\mathcal{F}(\tilde{u}_r)(n) = -i \operatorname{sgn}(n)\widehat{u_r}(n)$$

Proof. Since the Fourier transform is a linear operator, by the last part of the previous proposition, it suffices to prove the above equality in the cases that u is constant, analytic, or co-analytic.

To begin with, let's suppose that u is constant. Then, $\tilde{u} = 0$. So, for every $r \in (0, 1)$, u_r and $\tilde{u}_r$ are both constant. Hence, for every $n \in \mathbb{Z}^\times$,

$$\mathcal{F}(\tilde{u}_r)(n) = 0 = -i \operatorname{sgn}(n) \cdot 0 = -i \operatorname{sgn}(n)\widehat{u_r}(n).$$

Next, suppose that u is analytic. Then, $\tilde{u} = -iu$ is also analytic. Thus, for every $r \in (0, 1)$, we have that

$$\mathcal{F}(\tilde{u}_r)(n) = 0 = -i \operatorname{sgn}(n) \cdot 0 = -i \operatorname{sgn}(n)\widehat{u_r}(n) \qquad \forall n \in \mathbb{Z}^-,$$

since u and $\tilde{u}$ are both analytic on the simply connected region $\mathbb{D}$, and that

$$\mathcal{F}(\tilde{u}_r)(n) = \mathcal{F}(-iu_r)(n) = -i\mathcal{F}(u_r)(n) = -i \operatorname{sgn}(n)\widehat{u_r}(n) \qquad \forall n \in \mathbb{Z}^+.$$

Finally, suppose u is co-analytic. Then, for every $r \in (0, 1)$ and every $n \in \mathbb{Z}^\times$,

$$\mathcal{F}(\tilde{u}_r)(n) = \overline{\mathcal{F}(\overline{\tilde{u}_r})(-n)} = \overline{-i \operatorname{sgn}(-n)\widehat{\bar{u}_r}(-n)} = -i \operatorname{sgn}(n)\widehat{u_r}(n). \qquad \square$$

Proposition 4.5. *Let $u \in h^2(\mathbb{D})$. Then, $\|\tilde{u}_r\|_2^2 = \|u_r\|_2^2 - |u(0)|^2$ for every $r \in (0, 1)$, and thus $\tilde{u} \in h^2(\mathbb{D})$.*

Proof. By Theorem 1.13, for every $r \in (0, 1)$,

$$\begin{aligned} \|\tilde{u}_r\|_2^2 &= \sum_{n\in\mathbb{Z}} |\mathcal{F}(\tilde{u}_r)(n)|^2 = \sum_{n\in\mathbb{Z}^\times} |-i \operatorname{sgn}(n)\widehat{u_r}(n)|^2 \\ &= \sum_{n\in\mathbb{Z}} |\widehat{u_r}(n)|^2 - |\widehat{u_r}(0)|^2 = \|u_r\|_2^2 - |u_r(0)|^2. \end{aligned}$$

Hence,

$$\|\|\tilde{u}\|\|_2 = \sup_{0<r<1} \|\tilde{u}_r\|_2 = \sup_{0<r<1} \left(\|u_r\|_2^2 - |u_r(0)|^2\right)^{1/2} \leqslant \sup_{0<r<1} \|u_r\|_2 = \|\|u\|\|_2 < \infty,$$

implying that $\tilde{u} \in h^2(\mathbb{D})$. □

Corollary 4.6. *If $u \in h^2(\mathbb{D})$, then the functions $\tilde{u}_r$ converges in $L^2(\mathbb{T})$ as $r \to 1^-$, and thus $\lim_{r\to1^-} \tilde{u}_r(\theta)$ exists for a.e. $\theta \in \mathbb{T}$ as an L^2 function.*

Proof. Since the collection $\{\tilde{u}_r\}_{0<r<1}$ is L^2-bounded, by Theorem 2.9, there exists $g \in L^2(\mathbb{T})$ such that $\tilde{u}_r \to g$ in L^2 as $r \to 1^-$, and $\tilde{u}_r = P_r * g$ for every r. Therefore, by Corollary 3.7, $\lim_{r\to1^-} \tilde{u}_r(\theta)$ exists and equals $g(\theta)$ for a.e. $\theta \in \mathbb{T}$. □

4.2. **Harmonic measures and conjugate radial maximal operator.**

Definition 4.7. For each $\lambda > 0$, we define a function $\omega_\lambda : \mathbb{R} \times \mathbb{R}^+ \to \mathbb{R}$ by

$$\omega_\lambda(x, y) := \frac{1}{\pi} \int_{(-\infty,-\lambda]\cup[\lambda,\infty)} \frac{y}{(x-t)^2 + y^2} dt,$$

which some may recognize as the harmonic measure of $(-\infty, -\lambda] \cup [\lambda, \infty)$ centred at (x, y) with respect to the open upper half-plane.

Proposition 4.8. *For every $\lambda > 0$, the function ω_λ is harmonic in the open upper half-plane $\mathbb{H} := \{z \in \mathbb{C} : \operatorname{Im}(z) > 0\} \cong \mathbb{R} \times \mathbb{R}^+$.*

Proof. Fix any $\lambda > 0$. Then, for every $z = x + iy \in \mathbb{H}$,

$$\begin{aligned}\omega_\lambda(z) &= \frac{1}{\pi} \int_{(-\infty,-\lambda]\cup[\lambda,\infty)} \frac{1}{\left(\frac{t-x}{y}\right)^2 + 1} \cdot \frac{1}{y}\, dt \\ &= 1 - \frac{1}{\pi} \left(\arctan\left(\frac{x+\lambda}{y}\right) - \arctan\left(\frac{x-\lambda}{y}\right)\right).\end{aligned}$$

And a routine calculation shows that $\triangle \omega_\lambda = 0$ in $\mathbb{H}$. □

Proposition 4.9. *Let $u \in h^1(\mathbb{D})$. Then, for every $\lambda > 0$, one has that*

$$\left|\left\{\theta \in \mathbb{T} : (\tilde{u})^*(\theta) > \lambda\right\}\right| \leqslant C\lambda^{-1} \left\|\!\left|u\right|\!\right\|_1 \lambda^{-1},$$

with C some absolute constant.

Proof. Fix some $\lambda > 0$. To start with, suppose that u is real-valued and non-negative. Then, by the maximum principle, u is either constantly zero or strictly positive on $\mathbb{D}$. The case that u is constantly zero is trivial, so let's assume in the rest of the paragraph that u is strictly positive. Set $F := i(u + i\tilde{u}) = -\tilde{u} + iu$. Then, F is an analytic function from $\mathbb{C}$ to $\mathbb{H}$, and thus the composition $\omega_\lambda \circ F$ is harmonic in $\mathbb{D}$. In addition, for every $\theta \in \mathbb{T}$ with $(\tilde{u})^*(\theta) > \lambda$, we have that

$$\begin{aligned}&\sup_{0<r<1} (\omega_\lambda \circ F)(re(\theta)) \\ &\quad= 1 + \frac{1}{\pi} \sup_{0<r<1} \left(\arctan\left(\frac{\tilde{u}(re(\theta)) - \lambda}{u(re(\theta))}\right) - \arctan\left(\frac{\tilde{u}(re(\theta)) + \lambda}{u(re(\theta))}\right)\right) \\ &\quad\geqslant 1 + \frac{1}{\pi} \sup_{0<r<1} \left(\arctan\left(\frac{\tilde{u}(re(\theta)) - \lambda}{u(re(\theta))}\right) - \frac{\pi}{2}\right) \\ &\quad= \frac{1}{2} + \frac{1}{\pi} \arctan\left(\sup_{0<r<1} \frac{\tilde{u}(re(\theta)) - \lambda}{u(re(\theta))}\right) > \frac{1}{2}.\end{aligned}$$

Hence,

$$\begin{aligned}\left|\left\{\theta \in \mathbb{T} : (\tilde{u})^*(\theta) > \lambda\right\}\right| &\leqslant \left|\left\{\theta \in \mathbb{T} : (\omega_\lambda \circ F)^*(\theta) > \frac{1}{2}\right\}\right| \\ &\leqslant 6 \left\|\!\left|\omega_\lambda \circ F\right|\!\right\|_1 = 6\,(\omega_\lambda \circ F)(0) = 6\, \omega_\lambda(0, u(0)),\end{aligned}$$

where the second inequality follows from Corollary 3.6, and the first equality follows from the mean-value property of harmonic functions. Furthermore, since

$$\omega_\lambda(0, u(0)) = \frac{2}{\pi} \int_{\lambda/u(0)}^{\infty} \frac{1}{t^2+1} dt \leqslant \frac{2}{\pi} \int_{\lambda/u(0)}^{\infty} \frac{1}{t^2} dt = \frac{2}{\pi} \cdot \frac{u(0)}{\lambda} = \frac{2}{\pi} \lambda^{-1} \left\|\!\left|u\right|\!\right\|_1,$$

we deduce that

$$\left|\left\{\theta \in \mathbb{T} : (\tilde{u})^* (\theta) > \lambda\right\}\right| \leqslant \frac{12}{\pi}\lambda^{-1} \left\|\!|u|\!\right\|_1 .$$

Now, allow u to be any function in $h^1(\mathbb{D})$. By Lemma 2.8, there is a unique $\mu \in \mathcal{M}(\mathbb{T})$ such that $u_r = P_r * \mu$ for every $r \in (0,1)$. Decompose the complex measure μ into $\mu = \mu_{\mathrm{Re}}^+ - \mu_{\mathrm{Re}}^- + i\mu_{\mathrm{Im}}^+ - i\mu_{\mathrm{Im}}^-$, with $\mu_{\mathrm{Re}}^+, \mu_{\mathrm{Re}}^-, \mu_{\mathrm{Im}}^+, \mu_{\mathrm{Im}}^-$ being the unique positive measures for which the above equality holds. Let v^+, v^-, w^+, and w^- be the harmonic functions on $\mathbb{D}$ corresponding to $\mu_{\mathrm{Re}}^+, \mu_{\mathrm{Re}}^-, \mu_{\mathrm{Im}}^+$, and μ_{Im}^-, respectively, as specified in Theorem 2.9. Then, $u = v^+ - v^- + iw^+ - iw^-$, and thus $\tilde{u} = \widetilde{v^+} - \widetilde{v^-} + i\widetilde{w^+} - i\widetilde{w^-}$. Consequently, for every $\theta \in \mathbb{T}$,

$$\begin{aligned}
(\tilde{u})^* (\theta) &= \sup_{0<r<1} \left|\widetilde{v^+}(re(\theta)) - \widetilde{v^-}(re(\theta)) + i\widetilde{w^+}(re(\theta)) - i\widetilde{w^-}(re(\theta))\right| \\
&\leqslant \sup_{0<r<1} \left(\left|\widetilde{v^+}(re(\theta))\right| + \left|\widetilde{v^-}(re(\theta))\right| + \left|\widetilde{w^+}(re(\theta))\right| + \left|\widetilde{w^-}(re(\theta))\right|\right) \\
&\leqslant \left(\widetilde{v^+}\right)^* (\theta) + \left(\widetilde{v^-}\right)^* (\theta) + \left(\widetilde{w^+}\right)^* (\theta) + \left(\widetilde{w^-}\right)^* (\theta).
\end{aligned}$$

Furthermore, since $\mu_{\mathrm{Re}}^+, \mu_{\mathrm{Re}}^-, \mu_{\mathrm{Im}}^+, \mu_{\mathrm{Im}}^-$ are positive measures, and since $P_r > 0$ for every $r \in (0,1)$, the functions v^+, v^-, w^+, w^- are obviously real-valued and non-negative. Therefore,

$$\begin{aligned}
&\left|\left\{\theta \in \mathbb{T} : \left|(\tilde{u})^* (\theta)\right| > \lambda\right\}\right| \\
&\quad \leqslant \left|\left\{\theta \in \mathbb{T} : \left|\left(\widetilde{v^+}\right)^* (\theta)\right| > \frac{\lambda}{4}\right\}\right| + \left|\left\{\theta \in \mathbb{T} : \left|\left(\widetilde{v^-}\right)^* (\theta)\right| > \frac{\lambda}{4}\right\}\right| \\
&\qquad + \left|\left\{\theta \in \mathbb{T} : \left|\left(\widetilde{w^+}\right)^* (\theta)\right| > \frac{\lambda}{4}\right\}\right| + \left|\left\{\theta \in \mathbb{T} : \left|\left(\widetilde{w^-}\right)^* (\theta)\right| > \frac{\lambda}{4}\right\}\right| \\
&\quad \leqslant \frac{12}{\pi} \cdot \frac{4}{\lambda} \left(\left\|\!|v^+|\!\right\|_1 + \left\|\!|v^-|\!\right\|_1 + \left\|\!|w^+|\!\right\|_1 + \left\|\!|w^-|\!\right\|_1\right) \\
&\quad = \frac{48}{\pi}\lambda^{-1} \left(\|\mu_{\mathrm{Re}}^+\| + \|\mu_{\mathrm{Re}}^-\| + \|\mu_{\mathrm{Im}}^+\| + \|\mu_{\mathrm{Im}}^-\|\right) = \frac{48}{\pi}\lambda^{-1}\|\mu\| = \frac{48}{\pi}\lambda^{-1} \left\|\!|u|\!\right\|_1 . \square
\end{aligned}$$

4.3. **Hilbert transform.**

Definition 4.10. Given any $f \in L^1(\mathbb{T})$, we define the Hilbert transform of f by

$$Hf(\theta) := \lim_{r \to 1^-} \left(\widetilde{u_f}\right)_r (\theta),$$

where u_f is the $h^1(\mathbb{D})$ function defined by $(u_f)_r := P_r * f$.

It is not obvious that the Hilbert transform of each $f \in L^1(\mathbb{T})$ is well-defined, so we will justify the above definition in the following proposition.

Proposition 4.11. *Given any $f \in L^1(\mathbb{T})$, the limit $\lim_{r\to 1^-} \left(\widetilde{u_f}\right)_r (\theta)$ exists for a.e. $\theta \in \mathbb{T}$.*

Proof. Fix an $f \in L^1(\mathbb{T})$, and let $\epsilon > 0$ be given arbitrarily. Since $C(\mathbb{T})$ is a dense subset of $L^1(\mathbb{T})$, there exists some $g \in C(\mathbb{T})$ such that $\|f - g\|_1 < \epsilon$. Put

$h := f - g \in L^1(\mathbb{T})$, and for each $\delta > 0$, define

$$E_\delta := \left\{ \theta \in \mathbb{T} : \limsup_{r,s\to 1^-} \left| \left(\widetilde{u_f}\right)_r(\theta) - \left(\widetilde{u_f}\right)_s(\theta) \right| > \delta \right\},$$

$$F_{\epsilon,\delta} := \left\{ \theta \in \mathbb{T} : \limsup_{r,s\to 1^-} \left| \left(\widetilde{u_h}\right)_r(\theta) - \left(\widetilde{u_h}\right)_s(\theta) \right| > \delta \right\}.$$

Since $\widetilde{u_f} = \widetilde{u_h + u_g} = \widetilde{u_h} + \widetilde{u_g}$, and since $g \in L^2(\mathbb{T})$, by Corollary 4.6, $E_\delta = F_{\epsilon,\delta}$ for every $\delta > 0$, and consequently

$$|E_\delta| = |F_{\epsilon,\delta}| \leqslant \left\{ \theta \in \mathbb{T} : \left| \left(\widetilde{u_h}\right)^*(\theta) \right| > \frac{\delta}{2} \right\} \leqslant C\delta^{-1} \left\| u_h \right\|_1 = C\delta^{-1}\|h\|_1 < C\delta^{-1}\epsilon.$$

As $\epsilon > 0$ may be arbitrarily small, we deduce from the above inequality that $|E_\delta| = 0$ for every $\delta > 0$. Thus, for a.e. $\theta \in \mathbb{T}$, the net $\left\{ \left(\widetilde{u_f}\right)_r(\theta) \right\}_{r\in(0,1)}$ is Cauchy. It then follows from the completeness of $\mathbb{C}$ that $\lim_{r\to 1^-} \left(\widetilde{u_f}\right)_r(\theta)$ exists a.e. on $\mathbb{T}$. □

Corollary 4.12. *The Hilbert transform is bounded from L^1 to weak-L^1.*

Proof. For any $f \in L^1(\mathbb{T})$ and any $\lambda > 0$,

$$|\{\theta \in \mathbb{T} : |Hf(\theta)| > \lambda\}| \leqslant \left| \left\{ \theta \in \mathbb{T} : \left| \left(\tilde{u_f}\right)^*(\theta) \right| > \lambda \right\} \right| \leqslant C \left\| u_f \right\|_1 \lambda^{-1} = C\|f\|_1\lambda^{-1},$$

where C is the same constant as in Proposition 4.9. □

Proposition 4.13. *For any $p \in (1,\infty)$, the Hilbert transform is bounded on L^p.*

Proof. It is clear from Proposition 4.5 that for any $f \in L^2(\mathbb{T})$,

$$\|Hf\|_2 \leqslant \left\| \widetilde{u_f} \right\|_2 \leqslant \left\| u_f \right\|_2 = \|f\|_2,$$

which implies that the Hilbert transform H is bounded on L^2. Then, by Marcinkiewicz interpolation theorem, H is bounded on L^p for every $p \in (1,2]$.

Further, by the dominated convergence theorem and by Proposition 4.4, for every $f \in L^2(\mathbb{T})$ and every $n \in \mathbb{Z}$,

$$\begin{aligned} \widehat{Hf}(n) &= \int_{\mathbb{T}} e(-nx) \lim_{r\to 1^-} \left(\widetilde{u_f}\right)_r(\theta) d\theta = \lim_{r\to 1^-} \int_{\mathbb{T}} e(-nx) \left(\widetilde{u_f}\right)_r(\theta) d\theta \\ &= -i\,\mathrm{sgn}(n) \lim_{r\to 1^-} \widehat{u_f}(n) = -i\,\mathrm{sgn}(n)\hat{f}(n). \end{aligned}$$

Hence, by Theorem 1.13, for any two $f, g \in L^2(\mathbb{T})$,

$$\begin{aligned} \langle Hf, g\rangle &= \sum_{n\in\mathbb{Z}} \widehat{Hf}(n)\overline{\hat{g}(n)} = -\sum_{n\in\mathbb{Z}} \hat{f}(n) \left(\overline{-i\,\mathrm{sgn}(n)\hat{g}(n)} \right) \\ &= -\sum_{n\in\mathbb{Z}} \hat{f}(n)\overline{\widehat{Hg}(n)} = -\langle f, Hg\rangle, \end{aligned}$$

which implies that the adjoint $H^* : L^2(\mathbb{T}) \to L^2(\mathbb{T})$ satisfies $H^* = -H$. Now, fix any $p \in (2,\infty)$, and let q denote its conjugate exponent. Since $L^p(\mathbb{T}) \subseteq L^2(\mathbb{T})$, we may consider the Hilbert transform on L^p as a linear map from L^p to L^2. Because $L^2(\mathbb{T})$ is a dense subset of $L^q(\mathbb{T})$, the adjoint $H^* : L^q(\mathbb{T}) \to L^2(\mathbb{T}) \supseteq L^q(\mathbb{T})$ satisfies $H^* = -H : L^q(\mathbb{T}) \to L^q(\mathbb{T})$, and is thus bounded on $L^q(\mathbb{T})$. Consequently, the Hilbert transform $H = H^{**} : L^p(\mathbb{T}) \to L^p(\mathbb{T})$ is bounded. □

5. The L^p Convergence of Fourier Series

Theorem 5.1. *For every $p \in (1, \infty)$, the partial sum operators S_N are uniformly bounded on $L^p(\mathbb{T})$, and thus $S_N f \to f$ in $L^p(\mathbb{T})$ for every L^p function f.*

Proof. Fix any $p \in (1, \infty)$, and define a linear operator $T : L^p(\mathbb{T}) \to L^p(\mathbb{T})$ by $Tf := \frac{1}{2}f + \frac{i}{2}Hf - \frac{1}{2}\hat{f}(0)$. Then, $\|T\|_{p\to p} \leqslant \frac{1}{2} + \frac{1}{2}\|H\|_{p\to p} + \frac{1}{2} < \infty$. Moreover, using the same argument as in the proof of Proposition 4.13, we can show that for every $f \in L^p(\mathbb{T})$ and every $n \in \mathbb{Z}$, $\widehat{Hf}(n) = -i\,\mathrm{sgn}(n)\hat{f}(n)$, and consequently $\widehat{Tf}(n) = \chi_{[1,\infty)}\hat{f}(n)$. Thus, for every $f \in L^p(\mathbb{T})$ and every $N \in \mathbb{Z}^+$,

$$S_N f = \sum_{n=-\infty}^{\infty} \chi_{[-N,N]}\hat{f}(n)e_n = e_{-(N+1)}T(e_{N+1}f) - e_N T(e_{-N}f),$$

and

$$\|S_N f\|_p \leqslant \|T\|_{p\to p}\left(\|e_{N+1}f\|_p + \|e_{-N}f\|_p\right) = 2\|T\|_{p\to p}\|f\|_p\,.$$

It follows that $\sup_{N\in\mathbb{Z}^+}\|S_N\|_{p\to p} \leqslant 2\|T\|_{p\to p} < \infty$. Therefore, by Proposition 1.14, $S_N f \to f$ in $L^P(\mathbb{T})$ for every L^p function f. □

6. Generalization: Multiplier Operators

Fix an arbitrary $p \in (1, \infty)$. Since the Hilbert transform is bounded on $L^p(\mathbb{T})$, repeating the argument in the proof of Proposition 4.13, we can show that

$$\widehat{Hf}(n) = -i\,\mathrm{sgn}(n)\hat{f}(n)$$

for every $n \in \mathbb{Z}$ and every $f \in L^p(\mathbb{T})$. Let $m_H : \mathbb{Z} \to \mathbb{C}$ denote the sequence $\{-i\,\mathrm{sgn}(n)\}_{n=-\infty}^{\infty}$. Then,

$$Hf = \lim_{N\to\infty} S_N Hf = \lim_{N\to\infty}\sum_{n=-N}^{N} m_H(n)\hat{f}(n)e_n$$

in L^p for every $f \in L^p(\mathbb{T})$, and we call m_H the multiplier associated to the Hilbert transform H. Now, consider an arbitrary sequence $m : \mathbb{Z} \to \mathbb{C}$, and associate to it, formally, an "operator" T_m on $L^p(\mathbb{T})$ given by

$$\widehat{T_m f}(n) = m(n)\hat{f}(n) \qquad \forall n \in \mathbb{Z} \quad \forall f \in L^p. \tag{6.1}$$

Under what condition is there an actual (bounded) operator T_m with the above property?

Notation 6.2. We will mean by ℓ^∞ the space of all bounded functions from $\mathbb{Z}$ to $\mathbb{C}$, equipped with the supremum norm.

Let's first look for some necessary conditions. Assume that there is an operator T_m such that (6.1) holds. Then, since $\|e_k\|_p = 1$ and $T_m e_k = m(k)e_k$ for every $k \in \mathbb{Z}$, we have that

$$\|T_m\|_{p\to p} \geqslant \sup_{k\in\mathbb{Z}}\|T_m e_k\|_p = \sup_{k\in\mathbb{Z}}|m(k)| = \|m\|_\infty.$$

Hence, in order for there to exist a bounded operator T_m on L^p such that (6.1) holds, it is necessary that $m \in \ell^\infty$. In the special case that $p = 2$, if $m \in \ell^\infty$, then for every $f \in L^2$,

$$\sum_{n=-\infty}^{\infty} |m(n)|^2 \left|\hat{f}(n)\right|^2 \leqslant \|m\|_\infty^2 \sum_{n=-\infty}^{\infty} \left|\hat{f}(n)\right|^2 = \|m\|_\infty^2 \|f\|_2^2 < \infty,$$

which implies that the sequence $\left\{\sum_{n=-N}^{N} m(n)\hat{f}(n)e_n\right\}_{N=0}^{\infty}$ is Cauchy and whence convergent. Therefore, for each multiplier $m \in \ell^\infty$, there is the bounded operator $T_m : L^2 \to L^2$, defined by

$$T_m f := \lim_{N\to\infty} \sum_{n=-N}^{N} m(n)\hat{f}(n)e_n$$

for each $f \in L^2(\mathbb{T})$, such that (6.1) holds. So, we have found an equivalent condition, namely that $m \in \ell^\infty$, for the existence of an operator T_m that is bounded on L^2.

Next, observe that for every $g \in L^1(\mathbb{T})$, $\mu \in \mathcal{M}(\mathbb{T})$ and $n \in \mathbb{Z}$, the equality

$$\widehat{(g * \mu)}(n) = \hat{g}(n) \cdot \hat{\mu}(n)$$

holds as an easy consequence of the Fubini-Tonelli theorem. Thus, if $m = \mathcal{F}\mu$ for some $\mu \in \mathcal{M}(\mathbb{T})$, then there is the bounded operator $T_m : L^p \to L^p$, defined by

$$T_m f := f * \mu$$

for each $f \in L^2(\mathbb{T})$, such that (6.1) holds, and $\|T_m\|_{p\to p} \leqslant \|\mu\| < \infty$. The converse is also true in the special cases that $p = 1$ or $p = \infty$, though I will only give a proof of the former in the appendix.

Theorem 6.3. *If $m : \mathbb{Z} \to \mathbb{C}$ is a multiplier for which there exists a bounded operator $T_m : L^1 \to L^1$ such that* (6.1) *holds, then $m = \hat{\mu}$ for some $\mu \in \mathcal{M}(\mathbb{T})$.*

For other value of p, the condition $m = \mathcal{F}\mu$ is clearly not necessary for the existence of the multiplier operator T_m, because there exists the bounded multiplier operator H whose associated multiplier m_H is not of the form $\mathcal{F}\mu$. The equivalent condition for the existence of a bounded operator T_m satisfying (6.1) remains largely an open question in these cases.

But why do we even care about these multipliers and multiplier operators, apart from the fact that they seem fun? Since Fourier coefficients are preserved under translation, every bounded multiplier operator is translation-invariant; and it turns out that these multiplier operators are exactly the translation-invariant operators. Moreover, the theory of singular integrals, when initially developed by Calderón and Zygmund, was built upon Mikhlin's work on multipliers. For further discussion on multipliers, see Section 7.3 of [4], and Chapter 8 of [3].

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Appendix

Proof of Proposition 1.3. Suppose that $f \in L^p(\mathbb{T})$, with $p \in [1, \infty]$, and that $\mu \in \mathcal{M}(\mathbb{T})$. We define for each $\theta \in \mathbb{T}$ a function $f_\theta : \mathbb{T} \to \mathbb{C}$ by $f_\theta(\tau) := f(\theta - \tau)$. Then, $\|f_\theta\|_p = \|f\|_p$ for every $\theta \in \mathbb{T}$.

If $p = \infty$, then

$$\|f*\mu\|_\infty \leqslant \sup_{\theta\in\mathbb{T}} |f*\mu(\theta)| \leqslant \sup_{\theta\in\mathbb{T}} \int_{\mathbb{T}} |f(\theta-\tau)|\,|\mu|(\tau) \leqslant \sup_{\theta\in\mathbb{T}} \int_{\mathbb{T}} \|f\|_\infty |\mu|(\tau) = \|f\|_\infty \|\mu\|.$$

If $p \in [1, \infty)$, define for each $\theta \in \mathbb{T}$ a function $f_\theta : \mathbb{T} \to \mathbb{C}$ by $f_\theta(\tau) := f(\theta - \tau)$. Then, $\|f_\theta\|_p = \|f\|_p$ for every $\theta \in \mathbb{T}$, whence

$$\begin{aligned}
\|f*\mu\|_p &\leqslant \left(\int_{\mathbb{T}} \left(\int_{\mathbb{T}} |f_\theta(\tau)|\, d|\mu|(\tau) \right)^p d\theta \right)^{1/p} \\
&\leqslant \int_{\mathbb{T}} \left(\int_{\mathbb{T}} |f_\theta(\tau)|^p \, d\theta \right)^{1/p} d|\mu|(\tau) \\
&= \int_{\mathbb{T}} \|f_\theta\|_p d|\mu|(\tau) = \|f\|_p \|\mu\|,
\end{aligned}$$

where the second inequality holds by Minkowski's integral inequality.

Therefore, the inequality $\|f*\mu\|_p \leqslant \|f\|_p \|\mu\|$ holds in all possible cases. □

Proof of Proposition 1.4.
To begin with, suppose that $f \in L^\infty(\mathbb{T})$ and $g \in C(\mathbb{T})$. Since g is uniformly continuous, given an arbitrary $\epsilon > 0$, there exists $\delta > 0$ such that $|g(x) - g(x')| < \epsilon$ whenever $|x - x'| < \delta$. Then, for any two $x, x' \in \mathbb{T}$ with $|x - x'| < \delta$,

$$|(f*g)(x) - (f*g)(x')| = \int_{\mathbb{T}} |f(y)| \cdot |g(x-y) - g(x'-y)|\, dy \leqslant \epsilon \|f\|_\infty.$$

Since $\|f\|_\infty < \infty$, we conclude that $f*g$ is continuous.

Now, suppose that $f \in L^\infty(\mathbb{T})$ and $g \in L^1(\mathbb{T})$. Since $C(\mathbb{T})$ is a dense subset of $L^1(\mathbb{T})$, for any given $\epsilon > 0$, there exists $h \in C(\mathbb{T})$ such that $\|g - h\|_1 < \epsilon$, and there exsits $\delta > 0$ such that $|(f*h)(x) - (f*h)(x')| < \epsilon$ whenever $|x - x'| < \delta$. Thus, for any two $x, x' \in \mathbb{T}$ with $|x - x'| < \delta$,

$$\begin{aligned}
&|(f*g)(x) - (f*g)(x')| \\
&\quad \leqslant |f*(g-h)(x)| + |f*(h-g)(x')| + |(f*h)(x) - (f*h)(x')| \\
&\quad \leqslant 2\|f\|_\infty \|g-h\|_1 + |(f*h)(x) - (f*h)(x')| < (2\|f\|_\infty + 1)\,\epsilon
\end{aligned}$$

It then follows that $f*g$ is continuous. □

Proof of Proposition 1.10.

(1) Firstly, suppose that $f \in C(\mathbb{T})$, and let $\epsilon > 0$ be given arbitrarily. Since f is uniformly continuous, there exists $\delta > 0$ such that $|f(x-y) - f(x)| < \epsilon$ for any two $x, y \in \mathbb{T}$ with $|y| \leqslant \delta$. Then, for every index N and every $x \in \mathbb{T}$,

$$\begin{aligned}
&|(\Phi_N * f)(x) - f(x)| \\
&\quad = \left| \int_{\mathbb{T}} (f(x-y) - f(x))\, \Phi_N(y) dy \right| \\
&\quad \leqslant \int_{|y|>\delta} |f(x-y) - f(x)| \cdot |\Phi_N(y)|\, dy + \int_{|y|\leqslant\delta} \epsilon\, |\Phi_N(y)|\, dy
\end{aligned}$$

$$\leqslant 2\|f\|_\infty \int_{|y|>\delta} |\Phi_N(y)|\,dy + \epsilon \int_{\mathbb{T}} |\Phi_N(y)|\,dy$$
$$\leqslant 2\|f\|_\infty \int_{|y|>\delta} |\Phi_N(y)|\,dy + \epsilon \sup_{K\in\mathbb{Z}^+} \int_{\mathbb{T}} |\Phi_K(y)|\,dy,$$

which implies that

$$\|\Phi_N * f - f\|_\infty \leqslant 2\|f\|_\infty \int_{|y|>\delta} |\Phi_N(y)|\,dy + \epsilon \sup_{K\in\mathbb{Z}^+} \|\Phi_K\|_1.$$

Taking $N \to \infty$, we obtain that

$$\limsup_{N\to\infty} \|\Phi_N * f - f\|_\infty \leqslant \epsilon \sup_{K\in\mathbb{Z}^+} \|\Phi_K\|_1.$$

Since $\epsilon > 0$ may be arbitrarily small, we deduce from the inequality above that $\lim_{N\to\infty} \|\Phi_N * f - f\|_\infty = 0$.

(2) Next, suppose that $f \in L^p(\mathbb{T})$, with $p \in [1,\infty)$. Then, for any $g \in C(\mathbb{T})$,

$$\begin{aligned}
&\|\Phi_N * f - f\|_p \\
&\quad\leqslant \|\Phi_N * f - \Phi_N * g\|_p + \|g - f\|_p + \|\Phi_N * g - g\|_p \\
&\quad= \|\Phi_N (f-g)\|_p + \|f-g\|_p + \left(\int_{\mathbb{T}} |(\Phi_N * g)(x) - g(x)|^p\,dx\right)^{1/p} \\
&\quad\leqslant \|\Phi_N\|_1 \|f-g\|_p + \|f-g\|_p + \left(\int_{\mathbb{T}} \|\Phi_N * g - g\|_\infty^p\,dx\right)^{1/p} \\
&\quad\leqslant \left(\sup_{K\in\mathbb{Z}^+} \|\Phi_K\|_1 + 1\right) \|f-g\|_p + \|\Phi_N * g - g\|_\infty,
\end{aligned}$$

for every index N, and whence that

$$\limsup_{N\to\infty} \|\Phi_N * f - f\|_p \leqslant \left(\sup_{K\in\mathbb{Z}^+} \|\Phi_K\|_1 + 1\right) \|f-g\|_p.$$

Since $C(\mathbb{T})$ is a dense subset of $L^p(\mathbb{T})$, taking the infimum over all $g \in C(\mathbb{T})$, we obtain that $\lim_{N\to\infty} \|\Phi_N * f - f\|_p = 0$.

(3) Finally, suppose that $\mu \in \mathcal{M}(\mathbb{T})$. Fix any $f \in C(\mathbb{T})$, and define a function $g : \mathbb{T} \to \mathbb{C}$ by $g(x) := f(-x)$. Then, g is continuous, and

$$\begin{aligned}
&\limsup_{N\to\infty} \left| \int_{\mathbb{T}} f(x)(\Phi_N * \mu)(x)dx - \int_{\mathbb{T}} f(x)\,d\mu(x) \right| \\
&\quad= \limsup_{N\to\infty} \left| \int_{\mathbb{T}} f(x) \int_{\mathbb{T}} \Phi_N(x-y)\,d\mu(y)\,dx - \int_{\mathbb{T}} f(x)\,d\mu(x) \right| \\
&\quad= \limsup_{N\to\infty} \left| \int_{\mathbb{T}} \int_{\mathbb{T}} f(x)\Phi_N(x-y)\,dx\,d\mu(y) - \int_{\mathbb{T}} f(x)\,d\mu(x) \right| \\
&\quad= \limsup_{N\to\infty} \left| \int_{\mathbb{T}} \int_{\mathbb{T}} f(x)\Phi_N(x-y)\,dx\,d\mu(y) - \int_{\mathbb{T}} f(x)\,d\mu(x) \right| \\
&\quad= \limsup_{N\to\infty} \left| \int_{\mathbb{T}} \int_{\mathbb{T}} g(-x)\Phi_N\left(-y-(-x)\right)\,dx\,d\mu(y) - \int_{\mathbb{T}} g(-x)\,d\mu(x) \right| \\
&\quad= \limsup_{N\to\infty} \left| \int_{\mathbb{T}} \left(\Phi_N * g\right)(-y)\,d\mu(y) - \int_{\mathbb{T}} g(-y)\,d\mu(y) \right|
\end{aligned}$$

$$\leqslant \limsup_{N\to\infty} \int_{\mathbb{T}} \left|\left(\Phi_N * g\right)(-y) - g(-y)\right| \, d|\mu|(y)$$
$$\leqslant \limsup_{N\to\infty} \left\|\Phi_N * g\right\|_\infty \|\mu\| = 0,$$

which implies that $\lim_{N\to\infty} \int_{\mathbb{T}} f(x)(\Phi_N * \mu)(x)dx = \int_{\mathbb{T}} f(x)\, d\mu(x)$. Therefore, $\Phi_N * \mu \to \mu$ in the weak* sense as $N \to \infty$. $\square$

Proof of Lemma 3.4.
It is clear from the piecewise monotonicity of K that K is of bounded variation. Since K is continuous, there exists a unique $\kappa \in \mathcal{M}(\mathbb{T})$ such that $\kappa([a,b]) = K(b) - K(a)$ for all $-\frac{1}{2} \leqslant a \leqslant b \leqslant \frac{1}{2}$. Set $\nu := -\kappa + K\left(\frac{1}{2}\right)\delta_{1/2}$, where $\delta_{1/2}$ is the Dirac mass at $\frac{1}{2}$. Then, for every $x \in \left[-\frac{1}{2}, \frac{1}{2}\right]$,

$$\begin{aligned}
K(x) = K(|x|) &= -\left(K\left(\frac{1}{2}\right) - K\left(|x|\right)\right) + K\left(\frac{1}{2}\right) \\
&= -\kappa\left(\left[|x|, \frac{1}{2}\right]\right) + K\left(\frac{1}{2}\right)\cdot 1 \\
&= -\int_0^{1/2} \chi_{[-\varphi,\varphi]}\left(|x|\right) d\kappa(\varphi) + K\left(\frac{1}{2}\right)\int_0^{1/2} \chi_{[-\varphi,\varphi]}\left(|x|\right)\, d\delta_{1/2}(\varphi) \\
&= -\int_0^{1/2} \chi_{[-\varphi,\varphi]}(x)\, d\kappa(\varphi) + K\left(\frac{1}{2}\right)\int_0^{1/2} \chi_{[-\varphi,\varphi]}(x)\, d\delta_{1/2}(\varphi) \\
&= \int_0^{1/2} \chi_{[-\varphi,\varphi]}(x)\, d\left(-\kappa + K\left(\frac{1}{2}\right)\delta_{1/2}\right)(\varphi) \\
&= \int_0^{1/2} \chi_{[-\varphi,\varphi]}(x)\, d\nu(\varphi),
\end{aligned}$$

whence

$$\begin{aligned}
\|K\|_1 &= 2\int_0^{1/2} K(x)\, dx = 2\int_0^{1/2}\int_0^{1/2} \chi_{[-\varphi,\varphi]}(x)\, d\nu(\varphi)\, dx \\
&= 2\int_0^{1/2}\int_0^{1/2} \chi_{[-\varphi,\varphi]}(x)\, dx\, d\nu(\varphi) = 2\int_0^{1/2} \varphi\, d\nu(\varphi),
\end{aligned}$$

Furthermore, since K is decreasing on $\left[0, \frac{1}{2}\right]$, the restriction of the signed measure μ to $\mathcal{B}_{[0,1/2]}$ is a negative measure. Consequently, the restriction of $\nu = -\mu + K\left(\frac{1}{2}\right)\delta_{1/2}$ to $\mathcal{B}_{[0,1/2]}$ is a sum of two positive measures, and thus itself a positive measure. Therefore, for every $\mu \in \mathcal{M}(\mathbb{T})$ and every $\theta \in \left[-\frac{1}{2}, \frac{1}{2}\right]$,

$$\begin{aligned}
\left|(K*\mu)(\theta)\right| &= \left|\int_{-1/2}^{1/2}\int_0^{1/2} \chi_{[-\varphi,\varphi]}(\theta-\tau)\, d\nu(\varphi)\, d\mu(\tau)\right| \\
&= \left|\int_0^{1/2}\int_{-1/2}^{1/2} \chi_{[-\varphi,\varphi]}(\theta-\tau)\, d\mu(\tau)\, d\nu(\varphi)\right| \\
&= \left|\int_0^{1/2} 2\varphi \cdot \frac{1}{\left|[\theta-\varphi, \theta+\varphi]\right|}\cdot \mu([\theta-\varphi,\theta+\varphi])\, d\nu(\varphi)\right|
\end{aligned}$$

$$\leqslant \int_0^{1/2} 2\varphi \cdot \frac{1}{|[\theta-\varphi,\theta+\varphi]|} \cdot |\mu|([\theta-\varphi,\theta+\varphi])\, d\nu(\varphi)$$
$$\leqslant \int_0^{1/2} 2\varphi \cdot M\mu(\theta)\, d\nu(\varphi) = \|K\|_1\, M\mu(\theta). \qquad \square$$

Proof of Theorem 6.3. Let $m : \mathbb{Z} \to \mathbb{C}$ be any multiplier for which there is a bounded operator $T_m : L^1 \to L^1$ such that (6.1) holds. We define a family $\{f_t\}_{t>0}$ of functions from $\mathbb{T}$ to $\mathbb{C}$ by setting

$$f_t := \sum_{n=-\infty}^{\infty} e^{-2\pi|n|t} e_n$$

for each $t > 0$. Then, by the Fubini-Tonelli theorem, $\|f_t\|_1 = 1$ for every $t > 0$, and whence the family $\{T_m f_t\}_{t>0} \subseteq L^1(\mathbb{T}) \subseteq \mathcal{M}(\mathbb{T})$ is bounded by $\|T_m\|_{1\to 1}$. Using a compactness argument similar to that used in the proof of Lemma 2.8, there exists a sequence $\{t_k\}_{k=1}^{\infty} \subset (0,\infty)$ decreasing to 0, and a complex measure $\mu \in \mathcal{M}(\mathbb{T})$ with $\|\mu\| \leqslant \|T_m\|_{1\to 1}$ such that $T_m f_{t_k} \to \mu$ in the weak* sense as $k \to \infty$. Since

$$\int_{\mathbb{T}} e_{-n}(x) T_m f_t(x)\, dx = m(n) e^{-2\pi|n|t}$$

for every $t > 0$ and every $n \in \mathbb{Z}$, it follows that for every $n \in \mathbb{Z}$,

$$m(n) = m(n) \lim_{t\to 0} e^{-2\pi|n|t} = \lim_{k\to\infty} m(n) e^{-2\pi|n|t_k}$$
$$= \lim_{k\to\infty} \int_{\mathbb{T}} e_{-n}(x)\, T_m f_{t_k}(x)\, dx = \int_{\mathbb{T}} e_{-n}(x)\, d\mu(x) = \hat{\mu}(n). \qquad \square$$

References

[1] H. Brezis, *Functional Analysis, Sobolev Spaces and Partial Differential Equations*, Springer, 2011.

[2] G. B. Folland, *Real Analysis: Modern Techniques and their Applications*, John Wiley & Sons Inc., Second Edition, 1999.

[3] C. Muscalu, W. Schlag, *Classical and Multilinear Harmonic Analysis*, Cambridge University Press, Volume I, 2013.

[4] E. M. Stein, G. Weiss, *Introduction to Fourier Analysis on Euclidean Spaces*, Princeton University Press, 1971.