WEYL CURVATURE AS A CROSS-RATIO OF POINTS ON THE CELESTIAL SPHERE

MOLLY WOLFSON

Abstract. The Weyl curvature appears in Einstein's field equations, and is one of the two main types of space-time curvature considered in general relativity. It is possible to view the Weyl curvature as a ridge system on a 2-sphere with index four singularities representing the principle null directions. I am interested in the cross ratio of these four singularities in terms of the Weyl curvature. This paper explores the mathematical foundation to look into this question.

Contents

1. Background

I am interested in studying the space of world-vectors, which are the position vectors from special relativity originating from an arbitrary event. This space is called a *Minkowski* vector space and makes up tangent spaces in the curved space-time of general relativity.

1.1. Basic Definitions.

To help understand this topic there are a few basic definitions that will be necessary.

Definition 1.1. The *celestial sphere* is an imaginary 2-sphere with an arbitrarily large radius onto which all celestial objects can be projected.

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Definition 1.2. A k-dimensional *manifold* is a subset of $\mathbb{R}^n$ that is locally diffeomorphic to $\mathbb{R}^k$. This means that the manifold can be embedded into a larger Euclidean space but at each point there is a neighborhood that "looks like" $\mathbb{R}^k$.

For this paper, the most important manifold is a 2-sphere. This is the set $\{(x, y, z) \mid x^2 + y^2 + z^2 = 1\}$, which shows the 2-sphere is nicely embedded in $\mathbb{R}^3$. However, the 2-sphere is in fact a compact 2-manifold.

Definition 1.3. The *tangent space* on a manifold X at $x \in X$, $T_x(X)$, is the closest linear approximation of X at x.

Definition 1.4. A *vector field* on a manifold $X \subset \mathbb{R}^N$ is a smooth map $\vec{v} : X \to \mathbb{R}^N$ such that $\vec{v}(x) \in T_x(X)$ for every x.

Definition 1.5. A *ridge system* is a smooth choice of an un-oriented direction at each point. It can be thought of as a vector space, but instead of a vector with one direction, we have a sign ambiguity where we consider $\pm$ the tangent vector, resulting in a line. However, it is important to note that there are ridge systems that cannot give rise to smooth vector spaces. Ridge systems are found in finger and palm prints.[1]

Definition 1.6. A *Minkowski* vector space is a four-dimensional vector space $\mathbb{V}$ over the field of real numbers. $\mathbb{V}$ is endowed with an orientation, a (bilinear) inner product of signature (+ - - -), and a time orientation.

Let $(\boldsymbol{t}, \boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z})$ be a basis of $\mathbb{V}$. We will denote a vector $\mathbf{U} \in \mathbb{V}$ as

$$\mathbf{U} = T\boldsymbol{t} + X\boldsymbol{x} + Y\boldsymbol{y} + Z\boldsymbol{z}.$$

Definition 1.7. The *null vectors* are the vectors whose coordinates satisfy

$$T^2 - X^2 - Y^2 - Z^2 = 0.$$

1.2. Derivation of Spinors from the Null Cone.

Definition 1.8. The *null cone* is the set of all null vectors.

It is often important to only consider the null *directions* of Minkowski vector space because the magnitude itself is not important only the relative geometry. There is both the future null directions and the past null directions. For now I will focus on the future null directions which can be represented in any coordinatization as the intersection of the null cone with the hyperplane $T = 1$. This leads to a set with equation of a 2-sphere in Euclidean space:

$$X^2 + Y^2 + Z^2 = 1$$

I will denote the future null directions as S^+. It is now possible to project S^+ onto a 2-plane using stereographic projection. This will take each null direction from S^+ to a complex number, except the north pole will go to ∞. Therefore, each null direction can be represented by one $\zeta \in \mathbb{C} \cup \infty$. To determine the ζ from the initial coordinates $(1, X, Y, Z)$ use:

$$\zeta = \frac{X + iY}{1 - Z}.$$

[1] Refer to [5] for a more detailed discussion of Ridge Systems.

Using the fact that:

$$\zeta\bar{\zeta} = \frac{X^2+Y^2}{(1-Z)^2} = \frac{1-Z^2}{(1-Z)^2} = \frac{1+Z}{1-Z}.$$

I can derive the inverse relations for $X, Y, Z \in \mathbb{R}$ as:

$$X = \frac{\zeta+\bar{\zeta}}{\zeta\bar{\zeta}+1},\ Y = \frac{\zeta-\bar{\zeta}}{i(\zeta\bar{\zeta}+1)},\ Z = \frac{\zeta\bar{\zeta}-1}{\zeta\bar{\zeta}+1}.$$

To avoid including ∞, I can replace ζ by two complex numbers, $\xi, \eta \in \mathbb{C}$ where $\zeta = \frac{\xi}{\eta}$. This leads to :

$$X = \frac{\xi\bar{\eta}+\eta\bar{\xi}}{\xi\bar{\xi}+\eta\bar{\eta}},\ Y = \frac{\xi\bar{\eta}-\eta\bar{\xi}}{i(\xi\bar{\xi}+\eta\bar{\eta})},\ Z = \frac{\xi\bar{\xi}-\eta\bar{\eta}}{\xi\bar{\xi}+\eta\bar{\eta}}.$$

The representation of the future null directions as an intersection of the null cone with $T = 1$ was an arbitrary choice. I could have chosen any other intersection of the null cone with hyperplane of constant T. This would translate to other points on the straight line between $(1, X, Y, Z)$ and the origin, such as $\frac{\xi\bar{\xi}+\eta\bar{\eta}}{2}(1, X, Y, Z)$. Since the factor before $(1, X, Y, Z)$ is real we can be sure this new vector is the same direction as the old. This gives the new coordinates:

$$T' = \frac{1}{2}(\xi\bar{\xi}+\eta\bar{\eta}),\ X' = \frac{1}{2}(\xi\bar{\eta}+\eta\bar{\xi}),\ Y' = \frac{1}{i2}(\xi\bar{\eta}-\eta\bar{\xi}),\ Z' = \frac{1}{2}(\xi\bar{\xi}-\eta\bar{\eta}).$$

These coordinates can be represented by a 2×2 hermitian matrix:

$$\begin{bmatrix} T'+Z' & X'+iY' \\ X'-iY' & T'-Z' \end{bmatrix} = \begin{bmatrix} \xi\bar{\xi} & \xi\bar{\eta} \\ \eta\bar{\xi} & \eta\bar{\eta} \end{bmatrix} = \begin{bmatrix} \xi \\ \eta \end{bmatrix} \begin{bmatrix} \bar{\xi} & \bar{\eta} \end{bmatrix},$$

Therefore, any null direction can be represented by $\begin{bmatrix} \xi \\ \eta \end{bmatrix}$, which will now be known as the *spinor representation* of the null direction.[2] Note that the determinant of this hermitian matrix is $T'^2 - X'^2 - Y'^2 - Z'^2$, which is 0 since (T', X', Y', Z') is a null vector.

In addition, for any spinor $\begin{bmatrix} \xi \\ \eta \end{bmatrix}$ it is true that

$$\det\left(\begin{bmatrix} \xi\bar{\xi} & \xi\bar{\eta} \\ \eta\bar{\xi} & \eta\bar{\eta} \end{bmatrix}\right) = \xi\bar{\xi}\eta\bar{\eta} - \xi\bar{\xi}\eta\bar{\eta} = 0.$$

Therefor, any spinor gives a hermitian 2×2 matrix with determinant 0, and therefore represents a null direction.

The spinor representation of null directions is important because it generalizes light-like vectors in a nice way. In general, adding two light-like vectors will not result in another light-like vector. For example, both $(1, 1, 0, 0)$ and $(1, 0, 1, 0)$ are null vectors but $(1, 1, 0, 0) + (1, 0, 1, 0) = (2, 1, 1, 0)$ is not a null vector since $2^2 + 1^2 + 1^2 + 0^2 = 2 \neq 0$. However, adding any two spinors will give another spinor, which is another light-like vector.

[2]This derivation appears in reference [3] chapter 1.2.

1.3. **Spinors.** As it stands, the spinor representation of a null vector is the coordinate pair (ξ, η). This definition is redundant up to a phase, since $\xi \mapsto e^{i\theta}\xi$ and $\eta \mapsto e^{i\theta}\eta$ results in the same overall null direction. It is possible to give a spinor greater geometric structure to reduce the redundancy to a single sign ambiguity.

To reduce the redundancy I will consider both a *null flag*, $\mathbf{K}$, defined to be the coodinates (ξ, η), and a *flag plane*, which represents the phase of the vector. The goal of adding the flag plane is to explain more of the geometry of the spinor, therefore the choice of flag plane cannot depend on the coordinatization chosen and will be invariant under a change in coordinates.

Consider a future null direction on S^+ labeled P, which projects to $\zeta = \frac{\xi}{\eta}$. It is possible to represent ξ and η individually from this picture, in addition to the ratio, ζ. To do this we need to define a real tangent vector, $\mathbf{L}$, to S^+ at P. To span the tangent space of S^+ at P, which is an image of $\mathbb{C}$, both the real and imaginary parts of $\frac{\partial}{\partial\zeta}$ must be considered. Therefore, we want to consider any real tangent vector, $\mathbf{L}$, as

$$\mathbf{L} = \lambda\frac{\partial}{\partial\zeta} + \bar{\lambda}\frac{\partial}{\partial\zeta}$$

for some $\lambda \in \mathbb{C}$ to make $\mathbf{L}$ real. Note that at the north pole of the stereographic projection it is necessary to use another coordinization, such as $\frac{1}{\zeta}$ instead of ζ, to avoid $\zeta = \infty$.

I want to choose λ in terms of ξ and η so that a recoordinization of S^+ will not effect the image of $\mathbf{L}$. This will guarantee that $\mathbf{L}$ represents something physical about ξ and η. In general, a recoordinization can be defined as:

$$\tilde{\zeta} = \frac{\alpha\zeta + \beta}{\gamma\zeta + \delta},\ \tilde{\xi} = \alpha\xi + \beta\eta,\ \tilde{\eta} = \gamma\xi + \delta\eta$$

where $\alpha\delta - \beta\gamma = 1$. So I want to force

$$\tilde{\lambda}\frac{\partial}{\partial\tilde{\zeta}} + \bar{\tilde{\lambda}}\frac{\partial}{\partial\tilde{\zeta}} = \lambda\frac{\partial}{\partial\zeta} + \bar{\lambda}\frac{\partial}{\partial\zeta} \tag{1.9}$$

Through the product rule, it follows that

$$\frac{\partial}{\partial\zeta} = \eta^2\tilde{\eta}^{-2}\frac{\partial}{\partial\zeta}$$

Substituting this back into (1.9) gives

$$\tilde{\lambda}\tilde{\eta}^2 = \lambda\eta^2$$

Therefore, λ should be a real multiple of η^{-2}. For simplicity choose $\lambda = \eta^{-2}$ to get

$$\mathbf{L} = \eta^{-2}\frac{\partial}{\partial\zeta} + \bar{\eta}^{-2}\frac{\partial}{\partial\zeta}$$

Therefore, knowing ζ and η (and therefore ξ) will give $\mathbf{L}$ and knowing $\mathbf{L}$ and ζ will give ξ and η up to a sign. This means that it is possible to reduce the ambiguity of the phase of ξ and η with $\mathbf{L}$.

The official terminology is that the set $\{\mathbf{K}, a\mathbf{K} + b\mathbf{L}\}$, where $a, b \in \mathbb{R}$, $b > 0$, and $\mathbf{K}$ is (ξ, η), is the *null flag*, $\mathbf{K}$ is the *flagpole*, the direction of $\mathbf{K}$ is the *flagpole direction*, and $a\mathbf{K} + b\mathbf{L}$ as included in the above set is the *flag plane*.[3]

1.4. **Cross-Ratio.**

Definition 1.10. The *Cross-Ratio* of four points $\zeta_1, \zeta_2, \zeta_3, \zeta_4 \in \mathbb{C}$ is:

$$\chi = \{\zeta_1, \zeta_2, \zeta_3, \zeta_4\} = \frac{(\zeta_1 - \zeta_2)(\zeta_3 - \zeta_4)}{(\zeta_1 - \zeta_4)(\zeta_3 - \zeta_2)}$$

Theorem 1.11. *This cross-ratio is invariant under bilinear transformations.*

Before I prove this I will have to define a *bilinear transformation* and prove one fact about it.

Definition 1.12. A *bilinear transformation*, T, is a transformation defined by

$$w = T(z) = \frac{az + b}{cz + d}$$

where $a, b, c, d \in \mathbb{C}$ and $ad \neq bd$.

Lemma 1.13.

$$T(z) - T(y) = \frac{(ad - bc)(z - y)}{(cz + d)(cy + d)}$$

Proof. The lemma is proven as follows:

$$\begin{aligned} T(z) - T(y) &= \frac{az + b}{cz + d} - \frac{ay + b}{cy + d} \\ &= \frac{(az + b)(cy + d) - (ay + b)(cz + d)}{(cz + d)(cy + d)} \\ &= \frac{azcy + bcy + azd + bd - aycz - bcz - ayd - bd}{(cz + d)(cy + d)} \\ &= \frac{bcy + azd - bcz - ayd}{(cz + d)(cy + d)} \\ &= \frac{(ad - bc)(z - y)}{(cz + d)(cy + d)} \end{aligned}$$

□

Now here is the proof of the original theorem:

Proof.

$$\begin{aligned} \{T(\zeta_1), T(\zeta_2), T(\zeta_3), T(\zeta_4)\} &= \frac{(T(\zeta_1) - T(\zeta_2))(T(\zeta_3) - T(\zeta_4))}{(T(\zeta_1) - T(\zeta_4))(T(\zeta_3) - T(\zeta_2))} \\ &= \frac{\frac{(ad-bc)(\zeta_1-\zeta_2)}{(c\zeta_1+d)(c\zeta_2+d)} \frac{(ad-bc)(\zeta_3-\zeta_4)}{(c\zeta_3+d)(c\zeta_4+d)}}{\frac{(ad-bc)(\zeta_1-\zeta_4)}{(c\zeta_1+d)(c\zeta_4+d)} \frac{(ad-bc)(\zeta_3-\zeta_2)}{(c\zeta_3+d)(c\zeta_2+d)}} \\ &= \frac{(\zeta_1 - \zeta_2)(\zeta_3 - \zeta_4)}{(\zeta_1 - \zeta_4)(\zeta_3 - \zeta_2)} \\ &= \{\zeta_1, \zeta_2, \zeta_3, \zeta_4\} \end{aligned}$$

[3]Refer to [3] chapter 1.4. for the derivation of the $\mathbf{L}$ and a more detailed development of the final definitions.

□

There are at most six different values for the cross-ratio of the same four points taken in all possible permutations, which can be easily verified[4]:

$$\chi,\ \frac{1}{\chi},\ 1-\chi,\ \frac{1}{1-\chi},\ \frac{\chi-1}{\chi},\ \frac{\chi}{\chi-1}$$

If the value of the cross ratio, χ, is real, then the four points are collinear. It is this real value of collinear points that is commonly known as the cross-ratio.

A possibly more natural way to consider the six values of the cross ratio of four points, without preferring any one value, is through six variables a, b, c, A, B, C and the following six algebraic relations:

$$a + A = 1, \quad b + B = 1, \quad c + C = 1,$$
$$aB = 1, \quad bC = 1, \quad cA = 1.$$

It is easily verified that this is equivalent to the six values above.

From here I want to resolve the ambiguity in the six values by looking for an equation that does not depend on the labeling. One interesting choice to consider is the polynomial defined by

$$f(x) = (x-a)(x-A)(x-b)(x-B)(x-c)(x-C)$$

This is a degree six polynomial and it is also easily seen as a palindromic polynomial from the consideration of elementary symmetric polynomials. Palindromic comes from the fact that for every root of the polynomial, the inverse of this root is another root. Consider, for example, the coefficients on the x^5 and x terms. The first is the sum of all the roots while the second is the sum of all possible combinations of five roots. Picking any five of these six roots will result in two pairs of inverses, so only the odd root out will remain. This means the coefficient on x will be the sum of all the roots and the same as the coefficient on x^5. This leaves a polynomial of the form

$$f(x) = x^6 + Sx^5 + Tx^4 + Ux^3 + Tx^2 + Sx + 1$$

As stated above S is the sum of all the roots, so from the equations above:

$$S = a + A + b + B + c + C = 1 + 1 + 1 = 3.$$

T is the sum of all the different combinations of multiples of two roots. There are $\binom{6}{2} = 15$ of these multiples which can be written as:

$$\begin{aligned} T &= (a+A)(b+B) + (b+B)(c+C) + (c+C)(a+A) + aA + bB + cC \\ &= (1)(1) + (1)(1) + (1)(1) + aA + bB + cC \\ &= 3 + (aA + bB + cC) \end{aligned}$$

[4]Reference [3] chapter 1.3. contains this definition of the cross ratio with further discussion unrelated to what follows.

Finally U is the sum of all the different combinations of three roots. There are $\binom{6}{3} = 20$ of these multiples which can be written as:

$$\begin{aligned} U &= (a+A)(b+B)(c+C) + (a+A+b+B)(cC) + ... \\ &... + (b+B+c+C)(aA) + (c+C+a+A)(bB) \\ &= (1)(1)(1) + (1+1)(cC) + (1+1)(aA) + (1+1)(bB) \\ &= 1 + 2(aA + bB + cC) \end{aligned}$$

Therefore, this polynomial of all the possible values for the cross ratio of four points only depends on $(aA + bB + cC)$. This value does not distinguish any of the roots from each other. There is therefore reason to believe that this value is special, so it is of interest to solve for it in terms of only one of the possible values of cross ratio, a. I will omit the tedious algebraic details, the final equation is:

$$aA + bB + cC = 3 - \frac{(1 - a + a^2)^3}{(a(1-a))^2}$$

Note that the labeling of a, A, b, B, c, C was arbitrary so this equation will look the same no matter which variable is solved for. Now I want to call attention to the *Klein j-invariant* and the similarities to $aA + bB + cC$.

Definition 1.14. The *Klein j-invariant* is typically used to parametrize elliptic curves over $\mathbb{C}$. It is often defined as:

$$j(\lambda) = \frac{256(1 - \lambda + \lambda^2)^3}{(\lambda(1-\lambda))^2}$$

It is easily verified that j will give the same value for any of the six possible values of the cross ratio. In addition:

$$(aA + bB + cC) = 3 - \frac{1}{256} j(a)$$

Therefore, $(aA + bB + cC)$ is the same for any of the six possible values of the cross-ratio and is linked to a known invariant. This means it is unambiguously defined for a set of four points and is therefore an interesting property to study.

1.5. **Abstract Index Notation.** Abstract index notation is a mathematical notation used to describe the types of tensors and spinors. The indices are placeholders and not related to any basis. In general, a tensor is an element of a tensor product of spaces. Abstract index notation is used to simplify the notation of tensors and operations on tensors by indicating the spaces involved.

Consider a complex vector space V. Associated with this vector space is the usual dual of the vector space, V^*, which is the space of all linear maps from V to $\mathbb{C}$. There is also the space of complex, antilinear maps that is associated with V, denoted $\bar{V}^*$. These functions take λv to $\bar{\lambda} v$ where $\lambda \in \mathbb{C}$ and $v \in V$. By taking the dual of $\bar{V}^*$ we get a space $\bar{V}$, called the complex conjugate vector space of V.

To get an idea of what $\bar{V}$ is, consider some $w \in \bar{V}$. Then it must be true that there exists $v \in V$ and $\phi \in \bar{V}^*$ such that $\phi(v) = w \in \mathbb{C}$. It follows that V and $\bar{V}$ are the same as sets and real vector spaces. However, they are different as complex vector spaces and, since all $\phi \in \bar{V}$ are anti-linear, they are not isomorphic.

Therefore, there are four vector spaces associated with V: V, V^*, $\bar{V}^*$, and $\bar{V}$. To denote the makeup of the spaces in a tensor product, consider the tensor h. The

following notation is used:

$$V \longleftrightarrow h^A$$
$$V^* \longleftrightarrow h_A$$
$$\bar{V}^* \longleftrightarrow \bar{h}_{A'}$$
$$\bar{V} \longleftrightarrow \bar{h}^{A'}$$

This notation helps to illustrate that you should pair the upper indices with the lower indices and the primed indices with each other.

For example, consider the tensor $h \in V \otimes V^* \otimes V^* \otimes V \otimes V^*$. Using abstract index notation this is denoted as

$$h^A{}_{BC}{}^D{}_E \in V^A{}_{BC}{}^D{}_E$$

In addition, abstract index notation makes it easier to denote the contraction, symmetrization, and antisymmetrization of tensors and spinors. When two of the vector spaces are being contracted through the trace, it is typically denoted through a map. For example, consider $h^A{}_{BC}{}^D{}_E$ and Tr_{12} which takes $V \otimes V^* \otimes V^* \otimes V \otimes V^*$ to $V^* \otimes V \otimes V^*$. Using abstract index notation this would be denoted $h^A{}_{AC}{}^D{}_E$.

For symmetrization or antisymmetrization, there is a formal summation typically used. Abstract index formalism simplifies this process. Here it will be shown for the tensor $h_{ABC} \in V_{ABC}$. Let Σ_3 be the symmetric group on three elements. The antisymmetrization is denoted as

$$h_{[ABC]} = \frac{1}{3!} \sum_{\sigma \in \Sigma_3} (-1)^{\mathrm{sgn}(\sigma)} h_{\sigma(A)\sigma(B)\sigma(C)}$$

and the symmetrization is denoted as

$$h_{(ABC)} = \frac{1}{3!} \sum_{\sigma \in \Sigma_3} h_{\sigma(A)\sigma(B)\sigma(C)}$$

Something of interest to the discussion in this paper is the distinction between uppercase and lowercase indices. There is an isomorphism between $V \otimes \bar{V}$ and $\mathbb{V}$, the Minkowski vector space. Elements of the former are denoted with a pair of uppercase Latin letters as described above, AA'. Elements of the later are denoted with the one lowercase Latin letter, m. The isomorphism has been conventionally denoted by $\sigma_m{}^{AA'}$, since Pauli. This isomorphism will take any m vector and return a tensor with upper indices AA'.

1.6. **Curvature.** The Riemann Curvature, R_{abcd} describes the curvature of spacetime in general relativity. This is typically a tensor but has a spinor form

$$R_{abcd} = X_{ABCD}\epsilon_{A'B'}\epsilon_{C'D'} + \Phi_{ABC'D'}\epsilon_{A'B'}\epsilon_{CD} + ...$$
$$... + \bar{\Phi}_{A'B'CD}\epsilon_{AB}\epsilon_{C'D'} + \bar{X}_{A'B'C'D'}\epsilon_{AB}\epsilon_{CD}$$

The details of this expression will not be important for this paper.[5] I am interested in the symmetric part of this tensor, known as the *Weyl* curvature, which denotes the change in shape of an object:

$$\Psi_{ABCD} = R_{(ABCD)}.$$

[5]Though discussion can be found in reference [3] chapter 4.6.

Note that the tensor form of the Weyl Curvature is denoted C. The rest of the Rieman curvature is known as the *Ricci* curvature, which keeps track of the change in volume of an object.

For any totally symmetric spinor not equal to zero, such as Ψ_{ABCD}, it is possible to write the spinor as a symmetrized product of one-index spinors $\alpha_A, \beta_B, \gamma_C, \delta_D$, so $\Psi_{ABCD} = \alpha_{(A}\beta_B\gamma_C\delta_{D)}$. This decomposition is unique up to proportionality or reordering of factors and is known as the *canonical decomposition.*[6] A *principle spinor* is any spinor that appears in the canonical decomposition.

Definition 1.15. The *principle null directions* (PNDs) of a spinor are the flagpole directions corresponding to the principle spinors.

It is possible to solve for the PNDs of the Weyl curvature through the knowledge that

$$\Psi_{ABCD}\xi^A\xi^B\xi^C\xi^D = 0.$$

Using further knowledge of spinors[7] this is equivalent to finding the roots of the polynomial

$$\begin{aligned}\Psi_{ABCD}\xi^A\xi^B\xi^C\xi^D &= \Psi_0 + 4\Psi_1 z + 6\Psi_2 z^2 + 4\Psi_3 z^3 + \Psi_4 z^4 \\ &= (\alpha_0 + z\alpha_1)(\beta_0 + z\beta_1)(\gamma_0 + z\gamma_1)(\delta_0 + z\delta_1).\end{aligned}$$

In four dimensions, like in space-time, it takes 20 dimensions to specify the curvature at a point. Half of these dimensions are the Ricci curvature and the other half are the Weyl curvature. Therefore, there are 10 dimensions needed to specify Weyl curvature.[8] Eight of the dimensions of the Weyl curvature are specified by the PNDs, since each flagpole makes up two dimensions. The last two dimensions are determined by a complex coefficient on the PNDs. This means it is possible for two different Weyl curvature spinors to have the same PNDs.

2. Weyl Curvature as a Ridge System

Gravity can bend light in a phenomena known as *Gravitational Lensing.* In general, a point mass deflects light at an angle, α of:

$$\alpha = \frac{4GM}{c^2 b}$$

where G is the gravitational constant, M is the mass, c is the speed of light, and b is the impact parameter, the perpendicular distance between the path of the light ray and the point mass.

The Weyl curvature at a point in the universe can be understood via gravitational lensing between an observer at that point and a distant light source. Consider the field of vision of the observer, which is a celestial sphere or a 2-sphere. Suppose there is a sphere of light centered at each point on this 2-sphere. As the light approaches the observer, it will be deflected by the gravitational field. In general, gravitational lensing does not have to smoothly distort a sphere of light. An interesting example of the distortion of light by strong gravitational lensing is the Einstein Cross. This is a quasar sitting behind a galaxy, and the final image has five distinct spherical

[6]Uniqueness is proven in reference [3] chapter 3.5.

[7]This property and other discussion is found in reference [3] chapter 3.5.

[8]Reference [1] contains an intuitive discussion of curvature.

images. However, through a weak gravitational field or over short distances, the image of a sphere is expected to be a sphere or an ellipse. Therefore, suppose the spherical source of light is approaching the observer with a distance of $\epsilon \to 0$.

In this situation, the image is an ellipse almost everywhere. There are two characteristic measurements of an ellipse, the major and minor axis. At every point, assign the major axis direction of this sphere to the point. By smoothly varying the point the light is centered on, the semi-major axis is expected to also change smoothly, except around a point that produces a circle of light.

This assignment of major-axis directions to points will define a ridge system on the 2-sphere with zeros where the image was a circle. The zeros of the ridge system are the PNDs of the Weyl curvature and therefore are what I would like to further study.[9]

3. Ridge Systems on a 2-Sphere

In order to study the Weyl curvature as a ridge system, it is necessary to understand characteristics of a ridge system on a 2-sphere. To do this, I will start by focusing on the characteristics of a vector field on a 2-sphere. In particular, I want to know about the zeros, as these were highlighted at the end of the previous section.

To begin with I want to define the index of a vector field, which looks at the change in direction around the zeros. There are some preliminary ideas necessary to rigorously understanding this concept mathematically, which I will briefly discuss. Consider two manifolds X and Y, where Z is a submanifold of Y, and a map f, where $f : X \to Y$ and $\dim(X) + \dim(Z) = \dim(Y)$.

Definition 3.1. A map f is considered to be *transversal* to Z if

$$\text{Image}(df_x) + T_y(Z) = T_y(Y)$$

at every point $x \in f^{-1}(Z)$.

For example, consider $Y = \mathbb{R}^2$ and Z a curve in $\mathbb{R}^2$. If $f(X)$ is a line that only intersects Z once amd is tangent to Z at that point, then f is not transversal to Z. This is because the point of intersection has multiplicity two and the full tangent space of Y, which is $\mathbb{R}^2$, is not spanned. If $f : X \to Y$ is transversal to Z, then $f^{-1}(Z)$ is a finite number of points, each with an orientation number ± 1 provided by the preimage orientation.[10] It is necessary to have a finite number of points in the preimage in order to have a well defined sum.

Definition 3.2. If $f : X \to Y$ is transversal to Z, the *intersection number* of f and Z, $I(f, Z)$, is the sum of the orientation numbers of $f^{-1}(Z)$.

Definition 3.3. The *degree* of an arbitrary smooth map $f : X \to Y$ is the intersection number of f with any point y, $\deg(f) = I(f, \{y\})$.

[9] Reference [4] chapter 8.2 contains a more rigorous derivation of this ridge system, known as the *fingerprint* of the Weyl tensor.

[10] Reference [2] contains a more detailed discussion of transversality as well as the definitions up to and including degree.

The index of a vector field describes the behavior of the directional change of $\vec{v}$ at it's zeros, since the zeros are the only place where $\vec{v}$ can abruptly change direction. To study this change, consider a vector field, $\vec{v}$, on $\mathbb{R}^k$ and with an isolated zero at the origin. This means there is some small ball, S_ϵ, such that the origin is the only zero in the ball. The direction of $\vec{v}$ at $x \in S^\epsilon$ is the unit vector $\frac{\vec{v}(x)}{|\vec{v}(x)|}$. Therefore, we can look at the directional variation around the origin by considering the map $x \mapsto \frac{\vec{v}(x)}{|\vec{v}(x)|}$ which takes S_ϵ to S^{k-1}.

Definition 3.4. The *index* of $\vec{v}$ at a zero is the degree of the map described above.

In the two dimensional case the index has a visual explanation. Consider a small closed circle around the zero such that there is only one zero in the circle. As you move around this circle in the counterclockwise direction, the index counts the number of times $\vec{v}$ rotates completely. A full (360°) rotation of $\vec{v}$ in the counterclockwise direction contributes +1 to the sum while a full rotation in the clockwise direction contributes -1 to the sum.

Theorem 3.5. *(Poincaré-Hopf Index Theorem) If $\vec{v}$ is a smooth vector field on the compact, oriented manifold X with only finitely many zeros, then the global sum of the indices of $\vec{v}$ equals the Euler characteristic of X.*[11]

From the Poincaré-Hopf Theorem it follows that the global sum of all the indices on the 2-sphere will always be the same and equal to the Euler Characteristic for the 2-sphere. It is well known that the Euler characteristic of an n-sphere is $1 + (-1)^n$ so for a 2-sphere the Euler characteristic is 2. Consider Figure 1, which shows an example of a vector field on a 2-sphere. Here we see the vector field has two zeros, and each zero has an index of +1. Therefore, the sum of the indices here is +2, as is expected for a 2-sphere.

The index defined above is for vector fields, which have an orientation. I want to define the same concept for ridge systems, which are not oriented. This means that what previously was a vector pointing right and a vector pointing left would be identical as a horizontal lines. Since there is no difference between these vectors in a ridge system, going from one to the other would be a full loop in the direction. This is only half of what made up a full loop in a vector field and only represents a 180° rotation there. Therefore, a vector field index is interested in 360° rotations while a ridge system index is interested in 180° rotations.

To make this concept more clear, I will discuss a couple of the basic types of singularities for a vector field and ridge system.[12] Figure 2 shows two singularities for a vector field while Figure 3 shows two singularities for a ridge system. To verify the values of each singularity in Figure 2, consider the vectors in a counterclockwise fashion. The one on the left has vectors that complete a 360° counterclockwise rotation, so it has an index of +1. The one on the right has vectors that complete a 360° clockwise rotation, so it has an index of -1. To verify the values of each singularity in Figure 3, consider the lines in a counterclockwise fashion. The lines on the left trace out a 180° arc in the counterclockwise direction, so it has an index of +1. The lines on the right trace out a 180° arc in the clockwise direction, so it has an index of -1.

[11] A proof of this theorem is outlines in reference [2].

[12] Reference [5] contains more examples of singularities and indices.

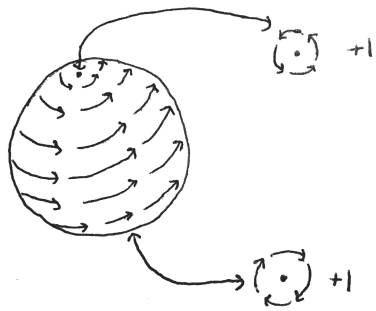

FIGURE 1. An example of a vector field on a 2-sphere. Here the two singularities are redrawn to the side of the 2-sphere for a cleaner image. Following the vector field around both of the singularities in a counterclockwise fashion shows the vectors in both cases make a 360° counterclockwise rotation. This means each singularity contributes +1 to the index and the total index of the field is +2.

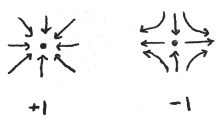

FIGURE 2. Two of the most basic types of singularities found in a vector field.

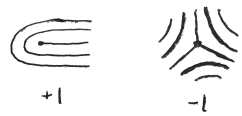

FIGURE 3. Two of the most basic types of singularities found in a ridge system.

Figure 3 shows that it is impossible to smoothly label the lines with orientations to make these singularities exist in vector fields. This hints that there is a difference between what is allowed for the two systems, including the amount of change in direction possible in a region. As shown above the simplest vector fields must still complete at least a 360° rotation to be a smooth field. Ridge systems on the other hand only need to complete a 180° rotation to be smooth. This is the same conclusion I came to earlier through the left-right ambiguity.

Therefore, the index from a vector field must be doubled for the same field represented as a ridge system. The index of a ridge system is still a characteristic

of the manifold it is on. Therefore, the sum of all the indices of a ridge system on a 2-sphere will always be 4.

4. Conlusion and Future Direction

Overall, I have found that it is possible to express the Weyl curvature at a point in space-time as a ridge system on a 2-sphere. Typically there will be four zeros of this field corresponding to the four PNDs of the Weyl curvature. Through stereographic projection, these four PNDs correspond to four $\zeta_i \in \mathbb{C}$. Therefore, it is possible to consider the cross-ratio of the PNDs, which define the Weyl curvature up to a complex phase. More interesting is the value of $aA + bB + cC$ as discussed in the cross-ratio subsection, and the closely related j-invariant. It should be possible to compute this invariant in terms of the Weyl curvature, where the complex coefficient on the Weyl curvature should not affect the result, since this does not affect the PNDs. We conjecture that the j-invariant in terms of the Weyl curvature tensor C is $\frac{tr(C^3)^2}{tr(C^2)^3}$, where tr is the trace. Each is raised to the necessary power to give the resulting expressions the same scaling behavior so that their ratio should not depend upon the overall scale of C. However, further investigation is needed to justify this result.

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