MOTIVATING SMOOTH MANIFOLDS

MIRA WATTAL

Abstract. In this paper, we build an intuitive and rigorous understanding of shapes that look "locally" like Euclidean space, with a little help from multivariable calculus. After introducing the necessary definitions and theorems, we will use our newfound understanding to show an interesting result about n-spheres.

Contents

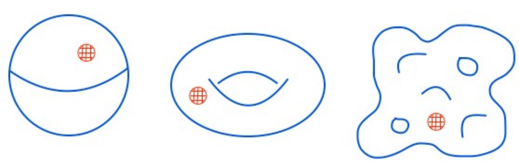


Figure 1. Examples of Smooth Manifolds

1. Introduction

As humans, we inhabit a world consisting of incredibly complex geometries. Everyday objects often have insightful structures and properties. Take, for example, donuts and basketballs. Though both of these objects are curved, donuts have holes while basketballs do not. But what does it mean for an object to be "curved" or to have "holes"? As mathematicians, we want to make our visual descriptions precise. To do this, we need to define a **coordinate system**.

Coordinate systems uniquely determine the positions of points in a geometric space using coordinates. In our study, coordinates encode location in Euclidean space. Thus, we can describe familiar shapes, like donuts and basketballs, locally as coordinates in $\mathbb{R}^n$ with one caveat. Our shapes must look *locally* like Euclidean space. Donuts and basketballs satisfy this condition. Every point has an open neighborhood that can be deformed (or flattened, in this case) to look like the plane.

Shapes that are locally Euclidean are called **smooth manifolds**. In fact, we think of coordinate systems as coordinate maps between open subsets of smooth manifolds and open subsets of $\mathbb{R}^n$. The "smooth" condition of smooth manifolds implies that our coordinate maps are **smooth**.

Definition 1.1. Let $U \subset \mathbb{R}^n$ and $V \subset \mathbb{R}^m$ be open sets. Then, a map $f : U \to V$ is called **smooth** if it has continuous partial derivatives of all orders.

Formally, a coordinate map is an n-tuple of smooth functions $(x_1, \ldots, x_n)$, which assigns points on a manifold coordinates in $\mathbb{R}^n$. But each x_i should be more than smooth. It should also preserve the intrinsic structure of the original space, otherwise our coordinate system is useless! In differential topology, we call a structure preserving map a **diffeomorphism**.

Definition 1.2. A **diffeomorphism** is a bijection, $f : U \to V$, such that f and f^{-1} are smooth. U and V are "diffeomorphic" if they can be related by a diffeomorphism.

At last, we have all the tools that we need to define a smooth manifold.

Definition 1.3. Let X be a topological space. Then, X is an n-dimensional **smooth manifold** if it is Hausdorff[1] and it has a countable cover of open subsets, $U_\alpha \subset X$, corresponding to smooth maps, ψ_α. These maps carry U_α homeomorphically onto an open subset of $\mathbb{R}^n$ and "agree" on the overlaps. More precisely, if U_1 and U_2 overlap, then the transition map:

$$\psi_2 \circ \psi_1^{-1} : \psi_1(U_1 \bigcap U_2) \to \psi_2(U_1 \bigcap U_2)$$

is a diffeomorphism. Each 2-tuple, $\{U_\alpha, \psi_\alpha\}$, is a coordinate chart, where U_α is a coordinate patch paired with a coordinate map, ψ_α. The collection of coordinate charts is called an atlas.

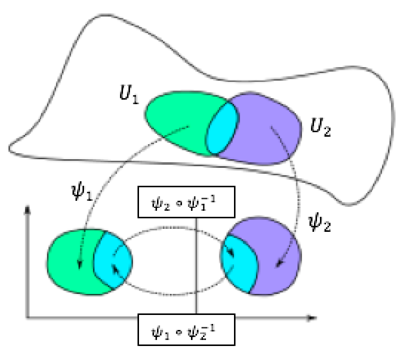


Figure 2. Charts "Agreeing" on the Overlaps (adapted from [4])

[1]Hausdorff-ness allows us to separate points from open sets. We can use this property in the next section when we apply calculus to smooth manifolds.

Remark 1.4. We will refer to $\psi_\alpha : U_\alpha \to \mathbb{R}^n$ as a coordinate system on U_α, and $\psi_\alpha^{-1} : \mathbb{R}^n \to U_\alpha$, a parametrization of U_α.

Using atlases, we can describe smooth manifolds as patches of Euclidean space. Often times, more than one patch is necessary[2]. Remember that smooth manifolds satisfy a weaker condition of being "locally Euclidean". In particular, neither basketballs nor donuts can be described by any one coordinate chart because they are topologically distinct from the plane.

Consider the circle, another compact, smooth manifold. At the very least, we need two coordinate patches, or local parameterizations, to "see" the circle as $\mathbb{R}$. Define the first parametrization as:

$$\phi_1 : \mathbb{R} \to S^1$$
$$\theta \in (0, 2\pi) \mapsto (\cos\theta, \sin\theta)$$

And define the second parametrization, ϕ_2, similarly, except restrict θ to $(-\pi, \pi)$. These patches cover S^1 and their corresponding mappings are homeomorphisms, which agree on the overlaps.

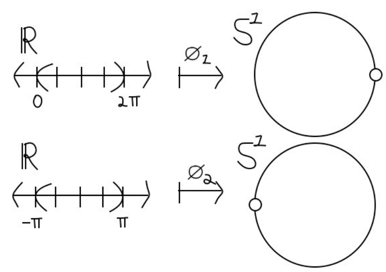


FIGURE 3. Coordinate Patches for S^1

Because coordinate patches are neighborhoods in Euclidean space, we can use calculus to study the properties of smooth manifolds in greater depth.

2. Calculus on Manifolds

Recall that derivatives are the closest linear approximations in Euclidean space. If $\psi : \mathbb{R}^n \to \mathbb{R}^m$ is a smooth map, then the **directional derivative** is defined by taking the limit:

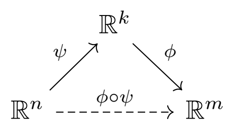
$$d\psi_x(v) = \lim_{t\to 0} \frac{f(x+tv) - f(x)}{t}$$

This limit describes the behavior of $x \in \mathbb{R}^n$ along $v \in \mathbb{R}^n$ with the vector $d\psi_x(v) \in \mathbb{R}^m$. With x fixed, we define the **derivative**, $d\psi_x$, to be the linear map, which assigns to each $v \in \mathbb{R}^n$ its directional derivative. We call the **Jacobian** of ψ at x the matrix representation of $d\psi_x$ with respect to the standard bases.

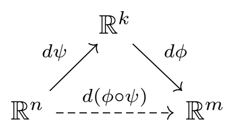
[2]We think of $\mathbb{R}^n$ as a vacuous example.

As linear mappings, derivatives have nice properties. Consider the *Chain Rule.*

Suppose $\phi \circ \psi : \mathbb{R}^n \to \mathbb{R}^m$ is a composition of smooth maps, where $\psi : \mathbb{R}^n \to \mathbb{R}^k$ and $\phi : \mathbb{R}^k \to \mathbb{R}^n$. We can draw this as the diagram:



Taking derivatives, the above becomes a commutative triangle of derivative maps:



So, the Chain Rule asserts that the derivative of composition is the composition of the derivatives. That is, $d(\phi \circ \psi) = d\phi \circ d\psi$.

To coarsely summarize, we can use the derivative to assign to a point a collection of vectors, which describe its behavior in some open neighborhood of Euclidean space. Not surprisingly, the image of a derivative mapping is a vector subspace, which we will call a **tangent space**.

Tangent spaces are useful. They allow us to "flatly" approximate how an object changes in the neighborhood of one of its points. But we should note that we define tangent spaces by taking derivatives in Euclidean space. To generalize our definition for smooth manifolds, we exploit local parametrizations.

Definition 2.1. Let X be an n-dimensional smooth manifold. Then, the **tangent space** at a point x in X, $T_x(X)$, is the image of $d\psi_x$, where $\psi_x : \mathbb{R}^n \to U$ is a local parametrization of X about x.

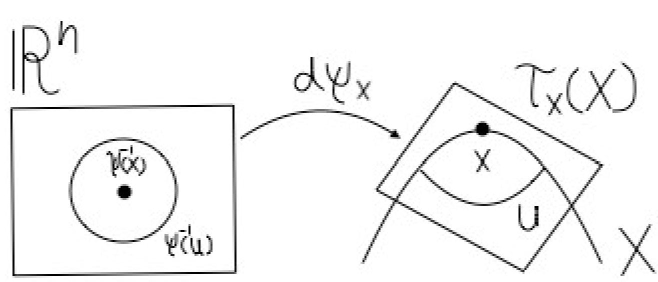


FIGURE 4. Tangent "Plane"

Intuitively, we attach to x a "plane"[3] consisting of **tangent vectors**, $d\psi_x(v)$. Just like our ordinary multivariable functions, $d\psi_x(v)$ describes how x changes along v. In particular,

Proposition 2.2. *$T_x(X)$ is n-dimensional.*

Proof. Because ψ_x is a local parametrization, ψ_x is a diffeomorphism on some open neighborhood U of x. Then, ψ_x^{-1} exists and is smooth. According to the Chain Rule:

$$d\psi_x^{-1} \circ d\psi_x = d(\psi_x^{-1} \circ \psi_x) = d(id) = id$$

So, $d\psi_x$ is an isomorphism because it has an inverse, $d\psi_x^{-1}$. As such, $d\psi_x$ is surjective, implying that $dim(T_x(X)) = n$.

□

Thus far, our definitions rely on our choice of parametrization, ψ_x. But does this "choice" uniquely determine how $T_x(X)$ behaves?

Proposition 2.3. *Tangent spaces are independent of the choice of parametrization.*

Proof. Let $\phi_x : \mathbb{R}^m \to V$ be another local parametrization for X about x. If we restrict the domains of ϕ_x and ψ_x so that $U = V$, $\phi_x^{-1} \circ \psi_x : \mathbb{R}^n \to \mathbb{R}^m$ is a diffeomorphism. From the Chain Rule, we have that:

$$d(\phi_x \circ \phi_x^{-1} \circ \psi_x) = d\phi_x \circ (d\phi_x^{-1} \circ d\psi_x) = d\psi_x$$

Clearly, $Im(d\psi_x) \subseteq Im(d\phi_x)$. To show inclusion from the other direction, simply switch ϕ_x and ψ_x in the preceding arguments. Thus, $Im(d\psi_x) = Im(d\phi_x)$, which is equivalent to stating that the tangent space at x is independent of ϕ_x or ψ_x.

□

As expected, the choice of parametrization does not matter. If we shrink domains appropriately and our parametrizations are "good" (that is, ψ_x and ϕ_x are diffeomorphisms), then $d\psi_x$ and $d\phi_x$ are isomorphic. Their tangent spaces should be indistinguishable.

With local parametrizations, defining differentiability between manifolds and Euclidean space is easy. How do we define differentiability *between* manifolds? First, remember that derivatives assign points on smooth manifolds tangent spaces. For maps between smooth manifolds, derivatives are maps *between* tangent spaces.

Let $f : X \to Y$ be a map, where X and Y are n-dimensional and m-dimensional manifolds, respectively. Also, suppose that $\psi_x : \mathbb{R}^n \to U$ is a parametrization of X about x and $\phi_y : \mathbb{R}^m \to V$ is a parametrization of Y about y. After shrinking U so that $f(U) \subseteq V$, we define the derivative of f by first drawing the square:

[3] A tangent "plane" only makes sense in $\mathbb{R}^3$. More generally, we can think of these spaces as tangent "bundles".

$$\begin{array}{ccc} \mathbb{R}^n & \xrightarrow{g} & \mathbb{R}^m \\ \psi_x \big\downarrow & & \big\downarrow \phi_y \\ U & \xrightarrow{f} & V \end{array}$$

Note that g, ϕ, and ψ are maps between Euclidean spaces. Thus, their derivatives are defined as usual, and by taking derivatives of the preceding diagram, we obtain:

$$\begin{array}{ccc} \mathbb{R}^n & \xrightarrow{dg} & \mathbb{R}^m \\ d\psi_x \big\downarrow & & \big\downarrow d\phi_y \\ T_x(U) & \overset{df}{\dashrightarrow} & T_y(V) \end{array}$$

Because the derivative of f is an isomorphism, there is only one map that makes this diagram commute. Naturally, it is the one that is independent of the choice of parametrization and satisfies the Chain Rule! With these conditions, the only acceptable definition[4] for df is the composition:

$$d(\phi_y \circ g \circ \psi_x^{-1}) = d\phi_y \circ dg \circ d\psi_x^{-1}$$

Now that we have defined differentiability between manifolds, we should consider when f is smooth. Our definition suggests that f is smooth only when g is smooth (that is, $\phi_y^{-1} \circ f \circ \psi_x$ is smooth).

With these generalized notions of a derivative, we can make a powerful assertion, namely, the *Inverse Function Theorem*.

Theorem 2.4. *Let $f : X \to Y$ be a smooth map between manifolds of the same dimension. If the derivative of f at a point $x \in X$ is an isomorphism, then f maps an open neighborhood of x diffeomorphically onto an open neighborhood of $f(x) = y$.*

From linear algebra, we know that a linear map is an isomorphism if any matrix representation is non-singular. With the Inverse Function Theorem, we can use this result to determine if an open neighborhood U of X is diffeomorphic to an open neighborhood V of Y. We just need to find a map between both neighborhoods (that is, a local parametrization or a coordinate system), whose Jacobian determinant everywhere in U is non-zero.

3. Stereographic Projection

Some of the simpler examples of smooth manifolds are spheres. We hinted at this idea in the introduction when we constructed an atlas for S^1. To show that S^2 is a smooth manifold is not particularly difficult either. An interesting question to

[4]One should verify that our definition is indeed independent of the choice of parametrization and satisfies the Chain Rule.

ask is: how can we show, more generally, that S^n is a smooth manifold?

We will show that this is the case in the "usual" way, by constructing a collection of coordinate charts. Like our method for S^1, we will create two.

Suppose that S^n lives[5] in $\mathbb{R}^{n+1}$ and define the first coordinate chart as $\{U_1, \psi_1\}$. U_1 is an open set in $\mathbb{R}^n$ that covers the entire sphere except the north pole, N. ψ_1 maps each point $p \in U_1$ to the unique point Q on the $x_{n+1} = 0$ plane such that $\overline{NQ}$ contains p.

Without loss of generality, we will illustrate this process for S^2 living in $\mathbb{R}^3$:

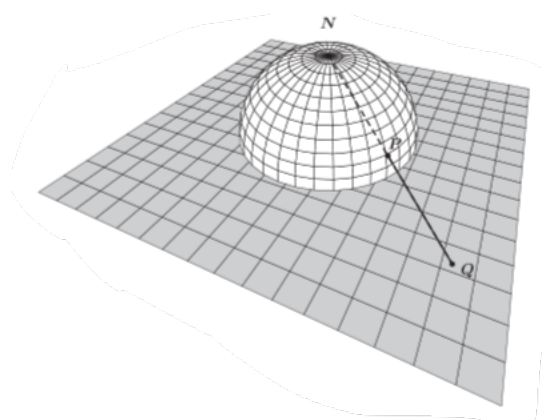


FIGURE 5. Stereographic Projection in $\mathbb{R}^3$ (adapted from [5])

Let $p = (u_1, \dots, u_{n+1})$ and $Q = (u'_1, \dots, u'_n, 0)$. Using a method of similar triangles, we find:

$$\|(u'_1, \dots, u'_n, 0)\| = \frac{1}{1 - u_{n+1}} \|(u_1, \dots, u_n, 0)\|$$

[5]What I really mean is "embedded". As it turns out, all n-spheres can be smoothly embedded in $\mathbb{R}^{n+1}$. You will see a proof of this in the next section.

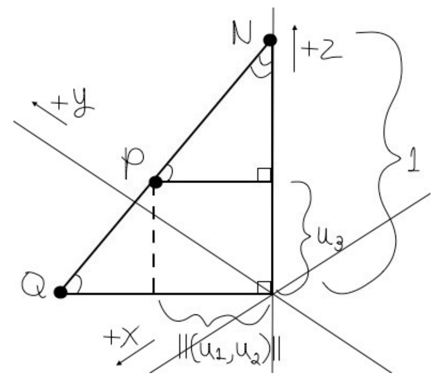


FIGURE 6. Method of Similar Triangles

Thus, ψ_1 is the coordinate transformation:

$$(u_1, \ldots, u_{n+1}) \mapsto \frac{1}{1-u_{n+1}}(u_1, \ldots, u_n)$$

Define the second coordinate chart $\{U_2, \psi_2\}$ similarly, except U_2 is an open set in $\mathbb{R}^n$ that covers the entire sphere except the south pole. From the lower hemisphere, our projection point will be the south pole, S. As before, for each point $a \in U_2$, $\psi_2(a)$ is the unique point B on the $x_{n+1} = 0$ plane such that $\overline{SB}$ contains a.

Let $a = (v_1, \ldots, v_{n+1})$ and $B = (v'_1, \ldots, v'_n, 0)$. Because we project from the south pole, v_{n+1} switches signs. Consequently, we define ψ_2 as:

$$(v_1, \ldots, v_{n+1}) \mapsto \frac{1}{1+v_{n+1}}(v_1, \ldots, v_n)$$

By construction, ψ_1 and ψ_2 are homeomorphisms. We will check that $\psi_2 \circ \psi_1^{-1}$ is a diffeomorphism by computing the Jacobian. To determine ψ_1^{-1}, we write $\overline{NQ}$ parametrically as $N + t(Q - N)$. Since $\overline{NQ}$ contains p, we have:

$$(u_1, \ldots, u_{n+1}) = (0, 0, 1) + t(u'_1, \ldots, u'_n, -1)$$

Because p lies on the unit sphere, $\|p\| = 1$. We solve for t by equating the norm squared of the RHS to 1:

$$(tu'_1)^2 + \cdots + (tu'_n)^2 + (1-t)^2 = 1$$
$$t^2(u'^2_1 + \cdots + u'^2_n + 1) = 2t$$
$$t = \frac{2}{1 + \|Q\|^2}$$
$$\implies u_1 = \frac{2u'_1}{1 + \|Q\|^2}, \ldots, u_n = \frac{2u'_2}{1 + \|Q\|^2}, u_{n+1} = \frac{\|Q\|^2 - 1}{\|Q\|^2 + 1}$$

So, ψ_1^{-1} is defined as:

$$(u'_1, \dots, u'_n) \mapsto (\frac{2u'_1}{1 + \|Q\|^2}, \dots, \frac{2u'_2}{1 + \|Q\|^2}, \frac{\|Q\|^2 - 1}{\|Q\|^2 + 1})$$

Note that $\|Q\| = \|B\|$. Composing ψ_2 and ψ_1^{-1} yields $\frac{(u'_1,\dots,u'_n)}{\|Q\|^2} = (v'_1, \dots, v'_n)$. This relationship helps us determine the matrix of partial derivatives, which has entries:

$$[\frac{\partial v'_i}{\partial u'_j}]_{i,j} = \begin{cases} \frac{\|Q\|^2 - 2u_i'^2}{\|Q\|^4} & i = j \\ \\ \frac{-2u'_i u'_j}{\|Q\|^4} & i \neq j \end{cases}$$

We will show that this matrix is non-singular. Write:

$$\det([\frac{\partial v'_i}{\partial u'_j}]_{i,j}) = \det(\begin{bmatrix} \frac{\|Q\|^2 - 2u_1'^2}{\|Q\|^{4n}} & \cdots & \frac{-2u'_1 u'_n}{\|Q\|^{4n}} \\ \vdots & & \vdots \\ \frac{-2u'_n u'_1}{\|Q\|^{4n}} & \cdots & \frac{\|Q\|^2 - 2u_n'^2}{\|Q\|^{4n}} \end{bmatrix})$$

$$= \frac{1}{\|Q\|^{4n}} \det(\begin{bmatrix} \|Q\|^2 - 2u_1'^2 & \cdots & -2u'_1 u'_n \\ \vdots & & \vdots \\ -2u'_n u'_1 & \cdots & \|Q\|^2 - 2u_n'^2 \end{bmatrix})$$

$$= \frac{1}{\|Q\|^{4n}} \det(\|Q\|^2 \cdot I_n - \begin{bmatrix} 2u_1'^2 & \cdots & 2u'_1 u'_n \\ \vdots & & \vdots \\ 2u'_n u'_1 & \cdots & 2u_n'^2 \end{bmatrix})$$

$$= \frac{1}{\|Q\|^{4n}} \det(\|Q\|^2 \cdot I_n - (*))$$

The eigenvalues of $(*)$ are precisely 0 with a multiplicity of $(n-1)$ and $2\|Q\|^2$ with a multiplicity of 1. Therefore, the characteristic polynomial of $(*)$ evaluated at $\|Q\|^2$, or equivalently, $\det([\frac{\partial v'_i}{\partial u'_j}]_{i,j})$, is:

$$f_*(\|Q\|^2) = (\|Q\|^2)^{n-1}(\|Q\|^2 - 2\|Q\|^2) = -\frac{1}{\|Q\|^{2n}}$$

As desired, the determinant is non-zero everywhere. However, it appears to blow up at $\|Q\| = 0$, that is, the north pole. This is no cause for alarm; recall that the poles are not included in intersection of U_1 and U_2! By the inverse function theorem, $\psi_2 \circ \psi_1^{-1}$ is a diffeomorphism.

We call this process **stereographic projection**. Stereographic projection mimics shining a flashlight from the poles of an n-sphere and creating coordinate covers via light projections onto some plane. As the flashlight's beam approaches the projection point, the end of the beam approaches infinity. By construction, our coordinate covers cannot include their projection points because stereographic projection sends them to infinity.

4. The Regular Value Theorem

Constructing n-spheres from coordinate patches is geometrically intuitive, but computationally hairy. Fortunately, a simple, but highly non-trivial result exists to show that S^n is a smooth manifold. Before we can explain this finding, we will need to expend some elbow grease.

We begin by rolling up our sleeves and defining a slew of terms.

Let $f : X \to Y$ be a smooth map between open neighborhoods of an m-dimensional smooth manifold, X, and an n-dimensional smooth manifold, Y.

Definition 4.1. If $m \leq n$ and at some point $x \in X$, df is injective, then f is called an **immersion** at x.

The **canonical immersion** is the standard inclusion map of $\mathbb{R}^m$ into $\mathbb{R}^n$. Note that every immersion between smooth manifolds is locally canonical, up to a diffeomorphism.

We call X an **immersed sub-manifold** of Y, if f is an immersion for all $x \in X$. And if f is injective, then X is an **embedded sub-manifold** of Y. We can equivalently state that f is a smooth embedding of X in Y. With this, we make a remarkable, but slightly unrelated claim for the scope of this paper. In fact, *all* smooth manifolds can be smoothly embedded in a subset of some big, ambient Euclidean space, $\mathbb{R}^N$. This follows from *Whitney's Embedding Theorem*, which proves that an n-dimensional, smooth manifold can be embedded in $\mathbb{R}^{2n}$.

To build our intuition for immersions and embeddings, we should consider their differences. For one, embeddings are injective. Because of this added structure, the image of an embedding cannot have self-intersections, whereas the image of an immersion can. Below are some examples:

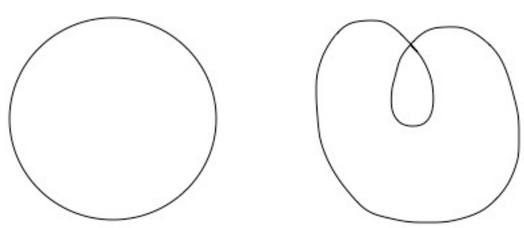


Figure 7. Embedded Circle (left) and Immersed Circle (right)

Now, suppose that $m \geq n$.

Definition 4.2. If df is surjective at some point $x \in X$, then f is called a **submersion** at x.

The **canonical submersion** is the standard projection map of $\mathbb{R}^m$ onto $\mathbb{R}^n$. As with immersions, every submersion between smooth manifolds is also locally canonical (up to a diffeomorphism, of course).

A point $q \in Y$ is called a **regular value** of f, if f is a submersion at every point in $f^{-1}(q)$. Otherwise, q is called a **critical value**. By the theorems of Brown and Sard, the regular values are dense in a smooth function, f. Roughly, this means that almost all of the points in Y are regular.

We finally have the necessary machinery to introduce the *Regular Value Theorem.*

Theorem 4.3. *Let $f : X \to Y$ be a smooth map between manifolds, where dim $X = m$ and dim $Y = n$. Suppose that $m \geq n$ and $q \in Y$ is a regular value of f. Then, $f^{-1}(q)$ is a smooth, embedded sub-manifold of X with dimension $m - n$.*

Proof. Let $\{y_1, \dots, y_n\}$ be a set of coordinate functions that carry an open neighborhood of a point y in Y diffeomorphically onto an open neighborhood of $\mathbb{R}^n$. Without loss of generality, let each $y_i(q) = 0$. Fix an open neighborhood U about q. By continuity, $f^{-1}(U) = U'$ is also open. For points in U', consider the n-tuple, $\{g_i = y_i \circ f : i \in [n]\}$. Our claim is that we can complete $\{g\}$[6] to an m-dimensional coordinate system on U'. And with the right restriction, $U' \bigcap f^{-1}(q)$[7], only $m - n$ coordinate functions are non-trivial.

To start, suppose that $\{x_1, \dots, x_m\}$ is already a valid set of coordinates for U'. Write the Jacobian of f with respect to this basis:

$$[\frac{\partial g_i}{\partial x_j}]_{1 \leq i \leq n, 1 \leq j \leq m}$$

Because q is a regular value of f, for all points in its pre-image, the Jacobian has rank n. For simplicity, suppose that the $n \times n$ non-singular sub-matrix is indexed by the first n coordinates. Discard these and with the remaining $m - n$, complete $\{g\}$ to the set, $\{g_1, \dots, g_n, x_{n+1}, \dots, x_m\}$. Could this new set be the coordinate system we were looking for? If this is the case, then $\{g, x\}$ satisfies two conditions.

First, for each $p \in f^{-1}(q)$, $\{g, x\}$ must be a homeomorphism of $U' \bigcap f^{-1}(q)$ onto its image in $\mathbb{R}^{m-n}$. To show this, it is sufficient to prove that the coordinate transformation, $\{x_1, \dots, x_m\} \mapsto \{g_1, \dots, g_n, x_{n+1}, \dots, x_m\}$, is a diffeomorphism.

Let $1 \leq i \leq n$, $n + 1 \leq k \leq m$, and $1 \leq j \leq m$. Denote the Jacobian of this transformation as:

$$\begin{bmatrix} \frac{\partial g_i}{\partial x_k} & \frac{\partial x_j}{\partial x_k} \end{bmatrix}^T$$

The above is equivalent to block matrix:

---

[6]Reader Beware: I have symbolically rewritten many of these coordinate systems out of notational ease (or perhaps, laziness).

[7]You might wonder why we restrict our coordinate system to $U' \bigcap f^{-1}(q)$, as opposed to $f^{-1}(q)$. Careful. Remember that we defined coordinate systems on *open* sets, and the set, $f^{-1}(q)$, is *closed.*

$$\begin{bmatrix} \frac{\partial g_i}{\partial x_i} & B \\ 0 & I_n \end{bmatrix}$$

By construction, $[\frac{\partial g_i}{\partial x_i}]$ is non-singular, implying that the Jacobian is also non-singular. According to the Inverse Function Theorem, this transformation is a diffeomorphism. In fact, only $m - n$ coordinates of $\{g, x\}$ are needed to make up a coordinate system on $U' \bigcap f^{-1}(q)$! Remember that each $y_i(q)$ is 0. This implies that for all points p in $f^{-1}(q)$, we have the map:

$$\begin{aligned} p &\mapsto \{g_1(p), \ldots, g_n(p), x_{n+1}(p), \ldots, x_m(p)\} \\ &= \{0, \ldots, 0, x_{n+1}(p), \ldots, x_m(p)\} \end{aligned}$$

Finally, transition maps of this form must also be diffeomorphisms.

Let $\{h_1, \ldots, h_n, z_{n+1}, \ldots z_m\}$ be another valid coordinate system on an open neighborhood V' containing $f^{-1}(q)$, constructed as above. From the first condition, the coordinate transformation, $\{g, x\} \mapsto \{h, y\}$ is a diffeomorphism, and therefore, the Jacobian determinant is non-zero. Represent this determinant as:

$$\begin{vmatrix} \frac{\partial h}{\partial g} & \frac{\partial h}{\partial x} \\ \frac{\partial z}{\partial g} & \frac{\partial z}{\partial x} \end{vmatrix}$$

We want to show that $[\frac{\partial y}{\partial x}]$ is non-singular (that is, the transition map, $\{0, x\} \mapsto \{0, y\}$, is a diffeomorphism). To do this, note that everywhere in $f^{-1}(q)$, $h_i(0, x)$ equals 0, implying that $|\frac{\partial h}{\partial x}|$ also equals 0. But if the Jacobian is non-singular, then $|\frac{\partial y}{\partial x}|$ is necessarily non-zero! So, the second condition is proved.

As we had hoped, $\{g, x\}$ is precisely the coordinate system on $U' \bigcap f^{-1}(q)$ that describes an embedded, $(m - n)$-dimensional, smooth sub-manifold of X.

□

Using this theorem, we can succinctly show that S^n is an embedded, n-dimensional, smooth sub-manifold of $\mathbb{R}^{n+1}$. Define the smooth map, $f : \mathbb{R}^{n+1} \to \mathbb{R}$, so that for each $x \in \mathbb{R}^n$:

$$f(x) = \|x\|^2 = x_1^2 + \cdots + x_{n+1}^2$$

The Jacobian of this transformation is:

$$\begin{bmatrix} 2x_1 & \cdots & 2x_{n+1} \end{bmatrix}$$

This matrix has maximal rank when $f(x)$ is non-zero. So, the regular values of f are every point in $\mathbb{R}/\{0\}$. Consider the entire set, $f^{-1}(1)$, which describes a coordinate system on S^n. Apply the Regular Value Theorem, and we are done.

Acknowledgments. To my friends in Max Pavelsky, your constant laughs and midnight cookies kept me sane. To my mentor, Randall Van Why, your advice and inspiration completed this paper. To Professor Babai, your lectures left me contemplating and hungry (for more mathematics and possibly, lunch). Finally, to Professor May, you made these experiences possible. Thank you all.

References

[1] Guillemin, Victor, and Alan Pollack. *Differential Topology.* Vol. 370, American Mathematical Soc., 2010.

[2] Milnor, John W. *Topology from the Differentiable Viewpoint.* Princeton University Press, 1997.

[3] Munkres, James R. *Topology.* 2nd ed., Prentice Hall Inc., 2000.

[4] Original work by Stomatapoll, distributed by the Wikipedia Creative Commons Attribution-Share Alike 3.0

[5] Rentlen, Paul. *Manifolds, Tensors, and Forms: An Introduction for Mathematicians and Physicists.* Cambridge University Press, 2013.