ULTRAPRODUCTS IN ALGEBRA

JOSH TURNER

Abstract. This paper will delve into the algebraic properties of ultraproducts and ultrapowers of various algebraic structures. Ultraproducts are relatively simple to define, but have deep implications, primarily due to Łós's Theorem, which for our purposes essentially states that statements that can be written in first order logic can be transferred back and forth between structures and their ultraproducts. We will primarily prove one theorem about a particular ultraproduct of fields, which, along with Łós's Theorem, can imply some quite notable algebraic facts. The use of the ultraproducts and Łós's Theorem in algebra is still relatively new, and there is still much to investigate on this topic.

Contents

1. Notation

In this paper, we will use $P(I)$ to denote the power set of a set I, and $|I|$ will refer to the cardinality of I. We will use $\omega = \aleph_0 = |\mathbb{N}|$ to refer to the cardinality of the natural numbers.

2. Background

2.1. Filters and Ultrafilters.

Definition 2.1. Given a set I, a set $D \subseteq P(I)$ of subsets of I is called a filter on I or over I if it satisfies the following conditions:
(a) $I \in D$, $\emptyset \notin D$,
(b) if $A \in D$ and $A \subseteq B$ for some set B, then $B \in D$,
(c) if $A, B \in D$, then $A \cap B \in D$.

Colloquially, we will say a subset A of I is *large* (with respect to a given filter D) if $A \in D$. So a filter is essentially a cohesive, formal way of labeling certain subsets of I as large.

We can quickly see that for any set I, there is a trivial filter, $D = \{I\}$, in which only I is large. This filter is rather uninteresting, so we will restrict our attention to *proper*, or non-trivial filters. From now on all filters will be assumed to be proper.

Remark 2.2. If $x \in I$, then $D = \{A \subseteq I \mid x \in A\}$ is a filter. We call a filter of this type the *principal* filter generated by x, and denote it D_x.

Definition 2.3. A non-principal (and proper) filter over I is called a $free$ filter.

Definition 2.4. The *Frechét Filter* on an infinite set I is the filter:

$$F_I = \{A \subseteq I \mid A \text{ is cofinite}\}.$$

Remark 2.5. If I is infinite, then F_I is clearly a filter. No singleton set is cofinite, so F_I is not principal, and is therefore a free filter.

We would like to be able to construct filters from a given collection of subsets of I, but to ensure that the result will in fact be a filter, we must require a certain condition of our collection.

Definition 2.6. A collection $S \subseteq P(I)$ of subsets of I has the *finite intersection property (FIP)* if any finite intersection of elements of S is nonempty.

Note that any filter has the FIP, because it is closed under finite intersection and does not contain the empty set.

Lemma 2.7. *Given a set $S \subseteq P(I)$ with the FIP, there is minimal filter containing it. We call this the filter generated by S.*

Proof. Close S under finite intersections and under taking of supersets until arriving at a filter. Since S has the FIP, taking finite intersections will never result in the empty set, so the this process will in fact end at a filter. $\square$

Now we will introduce a particular type of filter that will be instrumental in defining the ultraproduct.

Definition 2.8. A filter D over a set I is an $ultrafilter$ if for all $A \subseteq I$, either $A \in D$ or $I \backslash A \in D$.

Notice that if both of these conditions held for a filter D, then we would have that $A \cap (I \backslash A) = \emptyset \in D$. This case is excluded in the part (a) of the definition of a filter in 2.1, so these two conditions cannot simultaneously hold.. We can think of this as saying that an ultrafilter makes a decision about every subset $A \subseteq I$. Either A is large, or A is small (its complement is large).

Theorem 2.9. *A filter is an ultrafilter if and only if it is a maximal filter.*

Proof. ($\rightarrow$) If D is an ultrafilter over a set I, assume by contradiction that there is another filter C over I such that $D \subset C$ (strictly contained). If $A \subseteq I$ is such that $A \in C$ and $A \notin D$, then since D is an ultrafilter, $I \backslash A \in D$. But then we would have that $I \backslash A \in C$ and $A \in C$, which is impossible by Definition 2.1. So D is a maximal filter.

($\leftarrow$) Assume by contradiction that D is a maximal filter but not an ultrafilter. Then there is some $A \subseteq I$ such that $A \notin D$ and $I \backslash A \notin D$. Consider the set $D \cup \{A\}$. We want to show that this set has the FIP.

D already has the FIP because it is a filter, and since D is closed under finite intersections, to check if $D \cup \{A\}$ has the FIP we only need to check that $A \cap B$ is nonempty for any $B \in D$. If this intersection were empty, then we would have that $B \subseteq I \backslash A$, but that would imply that $I \backslash A \in D$, which is false. So $D \cup \{A\}$ does in fact have the FIP, and by Lemma 2.7, there is some filter containing it, contradicting that D was maximal. $\square$

Theorem 2.10. *An ultrafilter D over a set I is free if and only if $\bigcap_{A \in D} = \emptyset$.*

Proof. ($\Rightarrow$) We will show the contrapositive. If $\bigcap_{A \in D} = Y$ for some nonempty set $Y \subseteq I$, then choose some $x \in Y$. Since every set $A \in D$ contains x, it is clear that $D \subseteq D_x$. But by Theorem 2.9, D is maximal, so we have in fact that $D = D_x$, and D is not free.

($\Leftarrow$) If D were a principal ultrafilter generated by x, then we would have that $x \in A$ for all $A \in D$. Since the intersection of D is empty, this is not the case, so D is free. $\square$

Corollary 2.11. *Since filters are closed under finite intersections, Theorem 2.10 implies that there are no free ultrafilters on finite sets.*

Since there are no free ultrafilters on finite sets, we will from now on restrict our attention to infinite sets. However there is still no guarantee that any such free ultrafilters exist, even when I is infinite. To show that they do exist, we need to do some work, and eventually invoke the Axiom of Choice, in the form of Zorn's Lemma. The use of Choice is in fact necessary, and it is consistent with ZF set theory that free ultrafilters do not exist.

Theorem 2.12. *Every free ultrafilter contains the Frechét Filter.*

Proof. If D is a free ultrafilter over I, and F_I is the Frechét Filter, consider a set $B \in F_I$. By definition, $I \backslash B$ is finite. Since D is free, we have that $\bigcap_{A \in D} = \emptyset$ by Theorem 2.10. So given any $x \in I \backslash B$, x is not in the intersection of D, so there is some set $A_x \in D$ that does not contain x. Then the set $A = \bigcap_{x \in I \backslash B} A_x$ is disjoint from $I \backslash B$, so $A \subseteq B$. Further, since A is a finite intersection of members of D, $A \in D$. Since filters are upwardly closed, this shows that $B \in D$, and therefore that $F_I \subseteq D$. $\square$

Theorem 2.13. *Every filter E over I is contained in some ultrafilter D.*

Proof. We can consider the collection of filters over I containing E as a set partially ordered by containment. We want to show that any chain (totally ordered subset) C of such filters is bounded, so that we can apply Zorn's Lemma to get a maximal element. Given a chain C, consider $\bigcup C$. Clearly the chain is bounded by $\bigcup C$, and E is contained in it, but we still must show that it is a filter.

If $X \in \bigcup C$ and $X \subseteq Y$, then $X \in F$ for some $F \in C$. Then $Y \in F \subseteq \bigcup C$, which is therefore upwardly closed. If $X, Y \in \bigcup C$, then $X \in F$ and $Y \in F'$ for some $F, F' \in C$. Since C is totally ordered, without loss of generality, $F \subseteq F'$, so $X \in F'$. This gives us that $X \cap Y \in F' \subseteq \bigcup C$, so $\bigcup C$ is closed under finite intersection. Finally, since every $F \in C$ is a filter and does not contain the empty set, $\bigcup C$ does not contain the empty set either, and is therefore a filter.

Now we have shown that every chain of filters containing E is bounded, so we can apply Zorn's Lemma to give us a maximal such filter D. By Theorem 2.9, since D is a maximal filter, it is an ultrafilter, so we have proven the desired result. □

Corollary 2.14. *There exists a free ultrafilter on every infinite set I.*

Proof. The Frechét Filter F_I is a filter over the infinite set I. By Theorem 2.13, F_I is contained in some ultrafilter D. Then $\bigcap D \subseteq \bigcap F_I$, and it is clear from the definition of F_I that $\bigcap F_I = \emptyset$. So $\bigcap D = \emptyset$, and D is free by Theorem 2.10. □

From this point on, all ultrafilters will be assumed free unless otherwise stated.

2.2. **Ultraproducts.** This section will define the ultraproduct on sets. The actual model-theoretic definition of ultraproducts is defined on models, but will not be provided here. It can be found in Section 4.1 of [5].

Definition 2.15. If I is an infinite index set and $\{A_i\}_{i\in I}$ is a collection of nonempty sets indexed by I, then the arbitrary cartesian product is defined as:

$$\prod_{i\in I} A_i = \{f \mid f \text{ has domain } I \text{ and } f(i) \in A_i \text{ for all } i \in I\}$$

This arbitrary cartesian product is just a generalization of the finite one, and although it is defined as a set of functions, we can think of each element f in the product as a set of coordinates indexed by I, where f evaluated at i gives the ith coordinate.

We will now define the ultraproduct of sets as a set of equivalence classes on this cartesian product. The equivalence relation is defined as follows.

Definition 2.16. If D is an ultrafilter on an index set I, we say that two functions $f, g \in \{A_i\}_{i\in I}$ are *equivalent modulo D* if $\{i \in I \mid f(i) = g(i)\} \in D$. We will denote this as $f =_D g$.

In other words, f and g are equivalent if the set of coordinates on which they agree is large. We say that a property holds *almost everywhere* or *almost always* on I if the set of indices on which it holds is large, with respect to a given ultrafilter. So we could also say that f and g are equivalent if their coordinates agree almost everywhere on I. Similarly if $\{A_i\}_{i\in I}$ are sets indexed over I, we say that *almost all* of them have a certain property if the set of indices i for which A_i has that property is large.

Lemma 2.17. *The relation $=_D$ is an equivalence relation for any ultrafilter D.*

Proof. We will show that $=_D$ is reflexive, symmetric, and transitive.

- Reflexive: If $f \in \prod_{i\in I} A_i$, then $\{i \in I \mid f(i) = f(i)\} = I \in D$. So $f =_D f$.
- Symmetric: This follows from the fact that $\{i \in I \mid f(i) = g(i)\} = \{i \in I \mid g(i) = f(i)\}$.
- Transitive: If $f =_D g$ and $g =_D h$, then let $A = \{i \in I \mid f(i) = g(i)\} \in D$ and $B = \{i \in I \mid g(i) = h(i)\} \in D$. Since filters are closed under intersection, we have that $A \cap B = \{i \in I \mid f(i) = g(i) = h(i)\} \in D$, and since filters are upwardly closed, $\{i \in I \mid f(i) = g(i) = h(i)\} \subseteq \{i \in I \mid f(i) = h(i)\} \in D$, so $f =_D h$.

□

Definition 2.18. We will use $[f]_D$ to denote the equivalence class of a function $f \in \prod_{i\in I} A_i$ under the equivalence relation $=_D$. The *ultraproduct of sets* $\{A_i\}_{i\in I}$ *modulo* D is the set of equivalence classes under this equivalence relation. In symbols,

$$\prod_{i\in I} A_i/D = \{[f]_D \mid f \in \prod_{i\in I} A_i\}.$$

In the case where each $A_i = A$ for some set A, then we call the ultraproduct an *ultrapower of sets*, and denote it

$$\prod_{i\in I} A/D = \{[f]_D \mid f \in \prod_{i\in I} A\}.$$

2.3. **Łós's Theorem.** It may not be clear at this point what exactly we can do with this ultraproduct of sets we have just defined. What structure if any does it have in common with the structures we started with? The answer turns out to be quite a lot more than it might seem at first glance. To show this requires a lot of definitions and some proofs in the realm of model theory, but we will just state the results here. If the reader wants to see the details, Section 4.1 of [5] is a great place to go.

The following is what we will use as our statement of Łós's Theorem, also sometimes called the Fundamental Theorem of Ultraproducts. This statement is from Theorem 4.1.9 in [5], and although we have not defined the ultraproduct on models here, the definitions can all be found there.

Theorem 2.19 (Łós's Theorem)**.** *Let $\mathcal{L}$ be a language, D be an ultrafilter over a set I, and $M_i = (A_i, I_i, \beta_i)$ be a model for $\mathcal{L}$ for all $i \in I$. If $M_\natural = \prod_{i\in I} M_i/D$ is the ultraproduct, then for any sentence ϕ of $\mathcal{L}$,*

$$M_\natural \models \phi \textit{ if and only if } \{i \in I \mid M_i \models \phi\} \in D.$$

In other words, if ϕ is a statement in first order logic and $A_\natural$ is the ultraproduct of some A_i's, then ϕ holds for $A_\natural$ if and only if it holds for almost all of the A_i's.

3. Ultragroups, Ultrarings, and Ultrafields

Using Łós's Theorem, we can show that for many algebraic structures, an ultraproduct of objects with a given structure is again an object with the same structure. In fact we will require that almost all of the A_i have a given structure, but not necessarily all of them, since that is all that Łós's Theorem requires. To do this, we need to show that the axioms for our given structure are all expressible in first order logic, and then Łós's Theorem will tell us that these axioms still hold for the ultraproduct.

Theorem 3.1. *Let $\{A_i\}_{i\in I}$ almost all be groups (resp. rings, fields), D be a free ultrafilter on I (infinite), and $A_\natural = \prod_{i\in I} A_i/D$ be their ultraproduct. Then $A_\natural$ is also a group (ring, field).*

Proof. If $a, b, c \in A_\natural$, then we will define addition and multiplication coordinatewise. Let $a = [a_n]_D$, $b = [b_n]_D$, and $c = [c_n]_D$ for some sequences (or equivalently functions) (a_n), (b_n), and (c_n) in $\prod_{i\in I} A_i$. We define $a + b = [(a_n + b_n)]$, and when applicable, $ab = [(a_n b_n)]$.

It is sufficient to show that the axioms for a group (ring, field) are all expressible in first order logic. Then since these axioms are satisfied by almost all of the A_i, Łós's Theorem will tell us that they are satisfied by $A_\natural$ as well, with the operations as defined above. The axioms in first order logic are as follows:

- Commutativity of Addition: $(\forall a, b)(a + b = b + a)$
- Associativity of Addition: $(\forall a, b, c)(a + (b + c) = (a + b) + c)$
- Additive Identity: $(\exists 0)(\forall a)(0 + a = a + 0 = a)$
- Additive Inverse: $(\forall a)(\exists b)(a + b = b + a = 0)$

- Commutativity of Multiplication: $(\forall a, b)(ab = ba)$
- Associativity of Multiplication: $(\forall a, b, c)(a(bc) = (ab)c)$
- Multiplicative Identity: $(\exists 1)(\forall a)(a(1) = (1)a = a)$
- Multiplicative Inverse: $(\forall a)(\exists b)(a = 0 \vee ab = ba = 1)$

- Distributivity: $(\forall a, b, c)(a(b + c) = ab + ac)$
- Distinctness of Identities: $\neg(0 = 1)$

Since the axioms for groups, rings, and fields are each a subset of these axioms, we have shown the desired result. □

What we have essentially done here is shown that if almost all of the $\{A_i\}$s are groups, rings or fields, then we can view the ultraproduct of the *sets* $\{A_i\}$ as a group, ring or field as well, with operations as defined above.

Definition 3.2. If a group (resp. ring, field) is equal to an ultraproduct of groups (rings, fields), then it is called an *ultragroup* (*ultraring*, *ultrafield*).

Some of the most studied ultrarings are the *non-standard integers*, which are ultrapowers $\mathbb{Z}_\natural$ of $\mathbb{Z}$, and the *hyperreals*, ultrapowers $\mathbb{R}_\natural$ of $\mathbb{R}$, which is in particular an ultrafield. The hyperreals can be used to define non-standard analysis, where $\mathbb{R}_\natural$ is thought of as an ordered field containing a copy of $\mathbb{R}$ along with infinitesimal and unlimited elements (see [4]).

Theorem 3.3. *If A_i is a collection of fields such that for each prime p, only finitely many of the A_i have characteristic p, then $A_\natural$ has characteristic* 0.

Proof. If for a particular i and p, A_i does not have characteristic p, then p is non-zero in A_i and thus has a multiplicative inverse. In other words, the following statement is true for some $a \in A_i$:

$$(\exists a)(pa - 1 = 0)$$

Since only finitely many of the A_i have characteristic p, the above statement holds for all A_i such that i is in some cofinite subset of I, and therefore the statement holds for almost all A_i by Theorem 2.12. So we can apply Łós's Theorem to see that this statement also holds over $A_\natural$, which means that $A_\natural$ is not of characteristic p. Since we can apply this for every prime p, we see that $A_\natural$ is not of any positive characteristic, and thus must have characteristic 0. □

Lemma 3.4. *If almost all of the A_i are algebraically closed fields, then so is $A_\natural$.*

Proof. We can express the algebraic closure of a field by the following countable collection of first order logic statements (for $n \geq 2$):

$$(\forall a_0, a_1, ...a_n)(\exists x)(a_n = 0 \vee a_0 + a_1 x + ... + a_n x^n = 0)$$

So by Łós's Theorem if this holds for almost all of the A_i, it also holds for $A_\natural$. □

The following proof relies the Continuum Hypothesis for expediency, although it can be proven without this assumption. See Proposition 4.3.7 in [5] for such a proof. This means that any following results do not rely the Continuum Hypothesis either.

Theorem 3.5. *If* $|A_i| = |I| = \omega$*, then* $|A_\natural| = 2^\omega$*.*

Proof. Since $A_\natural$ is defined by an equivalence relation on $\prod_{i\in I} A_i$, it is clear that the map that sends an element of the product to its equivalence class in $A_\natural$ is a surjection, so we have that:

$$|A_\natural| \leq |\prod_{i\in I} A_i| = \omega^\omega = 2^\omega$$

by basic cardinal arithmetic.

We will now use a version of Cantor's diagonal argument to show that $|A_\natural| > w$ Consider a countable list of elements of $\bar{a}_k \in A_\natural$. For each $\bar{a}_k$, choose a representative $a_k \in \prod_{i\in I} A_i$ that is a countable sequence of elements from each of the A_i's. Our list will look like this:

$$\bar{a}_1 = [(a_{1,1}, a_{1,2}, ...a_{1,i}...)]$$

$$\bar{a}_2 = [(a_{2,1}, a_{2,2}, ...a_{2,i}...)]$$

$$\vdots$$

$$\bar{a}_k = [(a_{k,1}, a_{k,2}, ...a_{k,i}...)]$$

$$\vdots$$

Choose $b_1 \in A_1$ such that $b_1 \neq a_{1,1}$, choose $b_2 \in A_2$ such that $b_2 \neq a_{1,2}, a_{2,2}$, and in general $b_i \in A_i$ such that $b_i \neq a_{1,i}, ...a_{i,i}$. This can be done for every i because each A_i is countably infinite. Let

$$b = (b_1, b_2, ...b_i, ...).$$

By construction, b differs from each a_k on all but the first $k-1$ coordinates. So the set of indices on which they differ is cofinite and therefore large by Theorem 2.12. This means that b is in a distinct equivalence class from all of the a_k's, and therefore $\bar{b}$ is a distinct element of the ultraproduct from each $\bar{a}_k$. So we have shown that any countable list of elements of $A_\natural$ is not complete, and therefore that $|A_\natural| > \omega$.

Assuming the Continuum Hypothesis, which states that there are no other cardinals between ω and 2^ω, since $\omega < |A_\natural| \leq 2^\omega$ we must have that in fact $|A_\natural| = 2^\omega$. □

In fact we have shown that if I and A_i are all infinite, then the ultraproduct will be uncountably infinite (i.e. have cardinality larger than ω). It can also be shown that if almost all of the A_i's are finite and bounded in size, or if I itself were finite, then the ultraproduct would also be finite.

3.1. **Transcendence Bases.** In this section, our goal is ultimately to prove that algebraically closed fields with the same characteristic and uncountable cardinality are isomorphic. This result requires some set up, much of which comes from [3]. Throughout this section, we will be working with a field extension K/F.

Definition 3.6. A finite set $S = \{x_1, ...x_n\} \subseteq K$ is *algebraically independent* over F if there is no non-zero polynomial $P(t_1, ...t_n) \in F[t_1, ...t_n]$ such that $P(x_1, ...x_n) = 0$. An arbitrary set $S \subseteq K$ is *algebraically independent* if all of its finite subsets are such.

Definition 3.7. We say that an extension K/F is *purely transcendental* if $K = F(S)$ for some algebraically independent set S.

Theorem 3.8. *If $S = \{x_i\} \subseteq K$, then S is algebraically independent over F if and only if the map*

$$\phi : F(\{t_i\}) \to F(S), t_i \mapsto x_i$$

is an isomorphism of fields.

Proof. The image of a polynomial $P(\{t_i\})$ under ϕ is $P(\{x_i\})$. The definition of algebraic independence states that $P(\{x_i\}) = 0$ if and only if $P(\{t_i\}) = 0$, so ϕ is injective if and only if S is algebraically independent. It is clear that ϕ is always surjective, since the image of $F(\{t_i\})$ under ϕ is the field generated by S, $F(S)$. So S is algebraically independent if and only if ϕ is an isomorphism. □

Corollary 3.9. *If $S = \{x_i\}$ and $T = \{y_k\}$ are two algebraically independent sets over F, then $F(S)$ and $F(T)$ are isomorphic to $F(\{t_i\})$ and $F(\{t_k\})$ respectively. This means that*

$$F(S) \cong F(T)$$

if and only if S and T have the same cardinality.

Definition 3.10. A subset S of K/F is called a *transcendence basis* if it is algebraically independent over F, and $K/F(S)$ is an algebraic extension. The *transcendence degree* of an extension is the minimum cardinality of a transcendence basis.

In other words, a transcendence basis decomposes K/F into a purely transcendental extension followed by an algebraic one, $K/F(S)/F$. It can be shown that the transcendence degree is well-defined in a manner essentially identical to showing that the dimension of a vector space is well-defined, by proving something akin to the exchange lemma, so we will omit the proof here.

Now we shall see how the transcendence degree is related to the cardinality of K and F, which is crucial to the theorem we will prove.

Proposition 3.11. *If K/F is an infinite field with transcendence degree κ, then*

$$|K| = max(|F|, \kappa, \omega)$$

Proof. Let S be a transcendence base for K/F. Since K is infinite, clearly $|K| \geq \omega$. If $F(S)$ is also infinite, then since $K/F(S)$ is algebraic, we have that $|K| = |F(S)| = |F| \cdot \kappa = \max(|F|, \kappa) \geq \omega$. If $K(S)$ is finite, then again since $K/K(S)$ is algebraic, we would have $|K| = \omega$, and both $|F|$ and κ would be less than ω. So we in fact have that $|K| = \max(|F|, \kappa, \omega)$. □

3.2. **Conclusion.** Now we can finally put together the pieces.

Theorem 3.12 (Steinitz)**.** *If K and L are algebraic closed fields with the same characteristic and the same uncountable cardinality, then $K \cong L$.*

Proof. Let k be the common base field of K and L ($\mathbb{Q}$ if the common characteristic is 0 and $\mathbb{F}_p$ if it is p). Let κ, λ be the respective transcendence degrees of K and L, and S and T be their transcendence bases. Since both fields are uncountable, Proposition 3.11 gives us that $\max(|k|, \kappa) = |K| = |L| = \max(|k|, \lambda)$, which implies that $\kappa = \lambda$. By Corollary 3.9, this means that $k(S) \cong k(T)$. Finally, since K and L are the algebraic closures of $k(S)$ and $k(T)$ respectively, the preceding isomorphism can be extended to one between K and L. □

Corollary 3.13. *If $A_p = \bar{\mathbb{F}}_p$ are the algebraic closures of the prime fields, indexed over $I = P$, the set of prime numbers, then their ultraproduct $A_\natural$ is isomorphic to the complex numbers. That is to say:*

$$\prod_{p \in P} \bar{\mathbb{F}}_p / D \cong \mathbb{C}.$$

Proof. By Lemma 3.4, the ultraproduct $A_\natural$ is algebraically closed. Since each $\mathbb{F}_p$ has cardinality ω, as does P, $A_\natural$ has cardinality 2^ω by Theorem 3.5. Finally, $A_\natural$ has characteristic 0 by Theorem 3.3. Since $A_\natural$ and $\mathbb{C}$ are algebraically closed fields with the same characteristic and uncountable cardinality, they are isomorphic by Theorem 3.12. □

Ultraproducts of even very familiar structures are often quite large and unfamiliar, but in this case at least, the ultraproduct is a very familiar field. This result is quite interesting conceptually, but it also has some very direct applications because of Łós's Theorem. This result allows us to prove certain properties of $\mathbb{C}$ by proving them for positive characteristic fields, and vice versa, transferring between positive and zero characteristic. In fact there is at least one highly non-obvious fact about the complex numbers that we can now prove almost trivially, which we will now do so.

Theorem 3.14 (Ax-Grothendieck)**.** *If a polynomial map $\mathbb{C}^n \to \mathbb{C}^n$ is injective, then it is also surjective.*

Proof. The statement is clearly true for polynomial maps over finite fields, since for finite sets with the same cardinality, injections are always surjections. It can be shown to be true as well for the fields $\bar{\mathbb{F}}_p$, since they are a union of finite fields. This statement can be formulated in first order logic, although it is not pretty and will not be done here. This allows us to apply Łós's Theorem, and this completes the proof. □

Connections like this between fields of positive and of zero characteristic have been long noticed, but it was not at all clear where they came from. It is quite remarkable that the connections turn out to come directly from logic and model theory, in a way that at first seems totally unrelated. There is much more to be explored about ultraproducts and Łós's Theorem in algebra, and if the reader wants to learn more, [1] goes deep into this area.

Acknowledgments. I would like to thank my mentor Hana Jia Kong for the help in choosing a topic and writing this paper, all of the lecturers who made the REU such a great learning experience, and all of the mentors and students who made the REU feel like a community. I especially want to thank Professor Peter May for organizing the REU every year, and always being there to assist whoever needs it. I am very grateful for having had the opportunity to participate.

References

[1] Hans Schoutens, *The Use of Ultraproducts in Commutative Algebra*, Springer, 1999.

[2] H. Salzmann, *The Classical Fields: Structural Features of the Real and Rational Numbers*, Cambridge University Press, 2007.

[3] Pete L. Clark, *Field Theory*, http://math.uga.edu/ pete/FieldTheory.pdf.

[4] Diego Andres Bejarano Rayo, *Introduction to Non-Standard Analysis*, http://math.uchicago.edu/ may/REUDOCS/Rayo.pdf.

[5] C.C. Chang, H. Jerome Keisler *Model Theory*, Dover Publications, 1990.

[6] Ben Reid, *Ultraproducts of Finite Groups*, https://www.math.vt.edu/ugresearch/reid.pdf.