RECURRENCE IN COUNTABLE STATE MARKOV CHAINS

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Abstract. This paper investigates the recurrence and transience of countable state irreducible Markov chains. Recurrence is the property that a chain returns to its initial state within finite time with probability 1; a non-recurrent chain is said to be transient. After establishing these concepts, we suggest a technique for identifying recurrence or transience of some special Markov chains by building an analogy with electrical networks. We display a use of this technique by identifying recurrence/transience for the simple random walk on $\mathbb{Z}^d$.

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1. Introduction to Markov Chains

The central topic of this paper is an instance of the long-time behavior of Markov chains on discrete state spaces. Throughout this paper, we seek to answer the following question.

Question 1.1 (Recurrence)**.** *What is the probability that a Markov chain will return to its initial state in finite time? If the answer is 1, what is the expected time that the chain will return to its initial state?*

In this section, we introduce what Markov chains are and how to describe them.

Definition 1.2. Let $P : \Omega \times \Omega \to [0, 1]$ be a map such that $\sum_{y\in\Omega} P(x, y) = 1$ for every $x \in \Omega$. A sequence of random variables $(X_t)_{t=0}^{\infty}$ with values in the state space Ω is a **Markov chain** if for every $t \in \mathbb{N}$, $x_0, x_1, \cdots, x_t \in \Omega$, we have

$$\mathbb{P}\{X_t = x_t | C_{t-1}\} = \mathbb{P}\{X_t = x_t | X_{t-1} = x_{t-1}\} = P(x_{t-1}, x_t)$$

where C_{t-1} is the event $\{X_0 = x_0, X_1 = x_1, \cdots, X_{t-1} = x_{t-1}\}$.

Date: August 28, 2017.

That is, the probability distribution of the state at time t conditioned on the past depends only on the state at time $t-1$, X_{t-1}. During this entire paper, we assume that Ω is either finite or countably infinite.

The simple random walk on $\mathbb{Z}^d$ is a classic example of a countable state Markov chain.

Definition 1.3. A **simple random walk on** $\mathbb{Z}^d$ is a Markov chain with the n-dimensional integer lattice as the state space with the map P given by

$$P(x,y) = \begin{cases} 1/2d & \text{if } \|x-y\|_1 = 1 \\ 0 & \text{otherwise} \end{cases}$$

for any $x, y \in \mathbb{Z}^d$. Here, $\|\cdot\|_1$ denotes the 1-norm on $\mathbb{R}^d$ given by $\|x\|_1 = |x_1| + |x_2| + \cdots + |x_d|$ if the coordinates of $x \in \mathbb{R}^d$ are $(x_1, x_2, \cdots, x_d)$.

Since Ω is discrete, we may order the elements Ω in a sequence. Fix this ordering and arrange the values of the map P into a matrix, such that if $x, y \in \Omega$ are respectively the ith and jth entries of the sequence, then $P(x,y)$ is located at the ith row and jth column. It is particularly useful to consider the map P in this matrix form, and from now on we call P the **transition matrix** of the Markov chain.

There are three types of "matrix multiplication" performed in this paper involving transition matrices. Let $(X_t)_{t=0}^{\infty}$ be a Markov chain on state space Ω with transition matrix P.

(1) Multiplying P by itself:
Let $x, y \in \Omega$ be any states and $t \geq 0$ be any time. Then for any $s \geq 1$,

$$\begin{aligned} P^s(x,y) &= \sum_{z_1 \in \Omega} \cdots \sum_{z_{s-1} \in \Omega} P(x,z_1)P(z_2,z_3)\cdots P(z_{s-1},y) \\ &= \sum_{z_1,\cdots,z_{s-1} \in \Omega} \mathbb{P}\{X_{t+1} = z_1, \cdots X_{t+s-1} = x_{s-1}, X_{t+s} = y | X_t = x\} \\ &= \mathbb{P}\{X_{t+s} = y | X_t = x\}. \end{aligned}$$

Hence, the sth power of P give the transition probability over time interval of s. We define P^0 to be the identity matrix I.

(2) Multiplying a row vector on the left:

Definition 1.4. We say that a function $\mu : \Omega \to [0,1]$ is a **probability distribution** on Ω if $\sum_{x \in \Omega} \mu(x) = 1$.

Suppose that μ_t is the probability distribution at time $t \geq 0$. That is, $\mu_t(x) = \mathbb{P}\{X_t = x\}$ for all $x \in \Omega$. Arrange the values of μ_t into a row vector in the same order that we arranged the entries of the transition matrix. Then for any $y \in \Omega$ and $t \geq 0$,

$$\begin{aligned} (\mu_t P)(y) &= \sum_{x \in \Omega} \mu_t(x)P(x,y) = \sum_{x \in \Omega} \mathbb{P}\{X_t = x\}P(x,y) = \mathbb{P}\{X_{t+1} = y\} \\ &= \mu_{t+1}(y). \end{aligned}$$

Hence, multiplying the probability distribution at time t to the left of the transition matrix P gives the probability distribution at time $t+1$.

(3) Multiplying a column vector to the right:
Let $f : \Omega \to \mathbb{R}$ be a real-valued function. Arrange the values of f into a column vector in the same order that we arranged the entries of the transition matrix. Then for any $x \in \Omega$ and $t \geq 0$,

$$(Pf)(x) = \sum_{y\in\Omega} P(x,y)f(y) = \sum_{y\in\Omega} \mathbb{P}\{X_{t+1} = y|X_t = x\}f(y) = \mathbb{E}\left[f(X_{t+1})|X_t = x\right].$$

Hence, multiplying a real-valued function on Ω to the right of the transition matrix P gives the expected value of f at time $t+1$ conditioned to $X_t = x$.

If we combine these three operations, we obtain

$$(\mu_t P^s f)(x) = \mathbb{E}\left[f(X_{t+s})|X_t \sim \mu_t\right]$$

where $X_t \sim \mu_t$ means that μ_t is the probability distribution of X_t.

One key property shared by many interesting Markov chains is irreducibility, which means that starting from any state $x \in \Omega$, every state $y \in \Omega$ is eventually reachable.

Definition 1.5. A transition matrix P of a Markov chain is **irreducible** if for every $x, y \in \Omega$, there exists some $t \in \mathbb{N}$ such that

$$P^t(x,y) = \sum_{z_1,\cdots,z_{t-1}\in\Omega} P(x,z_1)P(z_1,z_2)\cdots P(z_{t-1},y) > 0.$$

If a Markov chain has an irreducible transition matrix, we also call the chain itself irreducible.

2. Recurrent Markov Chains

As we introduced in Question 1.1, one of the main interests regarding countable state Markov chains is to analyze whether the chain would return to its initial state. Such notion is defined precisely as recurrence, which in turn is defined by the notion of a hitting time.

Definition 2.1. Let $(X_t)_{t=0}^{\infty}$ be a Markov chain on a state space Ω, and $A \subset \Omega$ be a subset of the state space. The **hitting time** for A is defined as

$$\tau_A := \inf\{t \geq 0 : X_t \in A\}.$$

If $A = \{x\}$ is a singleton set, then we denote the hitting time for x as τ_x. For situations where $X_0 \in A$, we also consider the **first hitting time** for A, defined as

$$\tau_A^+ := \inf\{t \geq 1 : X_t \in A\}.$$

For any event E, we use the notation $\mathbb{P}_x(E)$ to denote the probability of E occuring for a Markov chain with the initial state $X_0 = x$. We also use $\mathbb{E}_x[Y]$ to denote the expected value of a real-valued random variable Y for a Markov chain with the initial state $X_0 = x$.

Definition 2.2. We say that a state $x \in \Omega$ is **recurrent** if $\mathbb{P}_x\{\tau_x^+ < \infty\} = 1$. Otherwise, x is said to be **transient**.

Proposition 2.3. *Suppose that for an irreducible Markov chain defined on Ω, $\mathbb{P}_x\{\tau_x^+ < \infty\} = 1$ for some $x \in \Omega$. Then $\mathbb{P}_y\{\tau_z^+ < \infty\} = 1$ for all $y, z \in \Omega$.*

This proposition implies that for an irreducible Markov chain, either all states are recurrent or they are all transient. Hence, we can classify an irreducible chain as either a recurrent chain or a transient chain. We introduce the following useful lemma from Section 21.2 of [1], which captures the idea for the proof of Proposition 2.3. The proof below is based on the proof in [1], but fills in the details omitted in it.

Lemma 2.4. *Let $(X_t)_{t=0}^{\infty}$ be a Markov chain with an irreducible transition matrix P. The* ***Green's function****, defined as $G(x,y) := \mathbb{E}_x\left[\sum_{t=0}^{\infty} \mathbb{1}_{\{X_t=y\}}\right] = \sum_{t=0}^{\infty} P^t(x,y)$, where $\mathbb{1}_{\{X_t=y\}}$ is the indicator function for $\{X_t = y\}$, is the expected number of visits to y starting from x. Then the following statements are equivalent:*

(i) $\mathbb{P}_x\{\tau_x^+ < \infty\} = 1$ *for some* $x \in \Omega$.
(ii) $G(x,x) = \infty$ *for some* $x \in \Omega$.
(iii) $G(y,z) = \infty$ *for all* $y, z \in \Omega$.

Proof. $(i) \Leftrightarrow (ii)$ Suppose $X_t = x$ at some time $t \geq 0$. Because of the Markov property, the probability that the chain will never visit x again (i.e. $X_s \neq x$ for any $s > t$) is given by $\mathbb{P}_x\{\tau_x^+ = \infty\} = 1 - \mathbb{P}_x\{\tau_x^+ < \infty\}$ independent of time t.

We claim that $G(x,x)$ is the expectation of a geometric random variable with success probability $p = 1 - \mathbb{P}_x\{\tau_x^+ < \infty\}$. We prove by induction on $N = \sum_{t=0}^{\infty} \mathbb{1}_{\{X_t=y\}}$, the number of visits to x for the chain with initial state $X_0 = x$.

(1) Since $X_0 = x$, the smallest possible value of N is 1. $N = 1$ if and only if $X_t \neq x$ for all $t \geq 1$, so $\mathbb{P}\{N = 1\} = \mathbb{P}_x\{\tau_x^+ = \infty\} = p$.
(2) Suppose for some $K \in \mathbb{N}$, we have $\mathbb{P}\{N = k\} = p(1-p)^{k-1}$ for every integer $1 \leq k \leq K$. Then

$$\mathbb{P}\{N \geq K+1\} = 1 - \sum_{k=1}^{K} p(1-p)^{k-1} = (1-p)^K.$$

$N = K+1$ if and only if the chain does not visit x after visiting x for the $(K+1)$st time. That is, if τ is the time such that $\sum_{t=0}^{\tau} \mathbb{1}_{\{X_t=x\}} = K+1$, then $X_s \neq x$ for all $s > \tau$. Hence,

$$\mathbb{P}\{N = K+1\} = \mathbb{P}\{N \geq K+1\}\mathbb{P}_x\{\tau_x^+ = \infty\} = p(1-p)^K.$$

Note that $\mathbb{P}\{N < \infty\} = \sum_{n=1}^{\infty} \mathbb{P}\{N = n\} = \sum_{n=1}^{\infty} p(1-p)^{n-1} = 1$ and hence $\mathbb{P}\{N = \infty\} = 0$ for all $0 < p \leq 1$. Therefore,

$$G(x,x) = \mathbb{E}_x[N] = \sum_{n=1}^{\infty} np(1-p)^{n-1} = \sum_{n=1}^{\infty}\sum_{m=n}^{\infty} p(1-p)^{m-1} = \sum_{n=1}^{\infty}(1-p)^{n-1} = \frac{1}{p}.$$

This implies $G(x,x) = \infty$ if and only if $p = 0$ if and only if $\mathbb{P}_x\{\tau_x^+ < \infty\} = 1$.

$(ii) \Leftrightarrow (iii)$ Let $y, z \in \Omega$ be any states. Since the Markov chain is irreducible, there exists some positive integers m, n such that $P^m(y,x) > 0$ and $P^n(x,z) > 0$. Since $P^{m+t+n}(y,z) \geq P^m(y,x)P^t(x,x)P^n(x,z)$,

$$\begin{aligned} G(y,z) = \sum_{s=0}^{\infty} P^s(y,z) &\geq \sum_{t=0}^{\infty} P^m(y,x)P^t(x,x)P^n(x,z) \\ &= P^m(y,x)P^n(x,z)\sum_{t=0}^{\infty} P^t(x,x) \\ &= P^m(y,x)P^n(x,z)G(x,x). \end{aligned}$$

Hence, $G(x,x)=\infty$ implies $G(y,z)=\infty$. The other direction is trivial. □

Looking at the first part of the proof of Lemma 2.4, we see that the probability that the chain starting from x revisits x infinitely often is

$$\mathbb{P}_x\{N=\infty\} = 1-\sum_{n=0}^{\infty} p(1-p)^{n-1} = \begin{cases} 1 & \text{if } p=0 \\ 0 & \text{if } p\neq 0 \end{cases}.$$

That is, if $\mathbb{P}_x\{\tau_x^+<\infty\}=1$, then the probability that the chain visits x infinitely often is 1.

Proof of Proposition 2.3. $\{\tau_y<\tau_x^+\}\cap\{\tau_x^+=\infty\}\subset\{\tau_x^+=\infty\}$, so

$$0\le \mathbb{P}_x\{\tau_y<\tau_x^+\}\mathbb{P}_y\{\tau_x^+=\infty\}\le \mathbb{P}_x\{\tau_x^+=\infty\}=0.$$

We claim that $\mathbb{P}_x\{\tau_y<\tau_x^+\}>0$. If so, $\mathbb{P}_y\{\tau_x^+=\infty\}=0$ and $\mathbb{P}_y\{\tau_x^+<\infty\}=1$.

Let us now show that $\mathbb{P}_x\{\tau_y<\tau_x^+\}>0$. If $x=y$, then $\tau_y=0<\tau_x^+$, so $\mathbb{P}_x\{\tau_y<\tau_x^+\}=1$. Otherwise, let $\tau_{y,\min}=\inf\{t\ge 1: P^t(x,y)>0\}$. Since P is irreducible, $\tau_{y,\min}$ is finite. If $\tau_x^+<\tau_y=\tau_{y,\min}$, then $P^{(\tau_{y,\min}-\tau_x^+)}(x,y)>0$, so we have a contradiction. Hence, $\mathbb{P}_x\{\tau_y<\tau_x^+\}\ge\mathbb{P}_x\{\tau_y=\tau_{y,\min}\}=P^{\tau_{y,\min}}(x,y)>0$.

Similarly, $\mathbb{P}_x\{\tau_z<\tau_x^+\}>0$. If a chain starting from x never visits z, then for every time that the chain visits x, it must be true that $\tau_z\ge\tau_x^+$. Since the probability that the chain starting from x revisits x infinitely often is 1, the probability that the chain starting from x will never visit z is $\lim_{n\to\infty}(1-\mathbb{P}_x\{\tau_z<\tau_x^+\})^n=0$. That is, $\mathbb{P}_x\{\tau_z^+<\infty\}=1$.

Therefore,

$$1\ge \mathbb{P}_y\{\tau_z^+<\infty\}\ge\mathbb{P}_y\{\tau_x^+<\infty\}\mathbb{P}_x\{\tau_z^+<\infty\}=1$$

and $\mathbb{P}_y\{\tau_z^+<\infty\}=1$ for every $y,z\in\Omega$. □

Given that an irreducible Markov chain is recurrent, the next step is of course to calculate $\mathbb{E}_x\left[\tau_y^+\right]$. However, it turns out that $\mathbb{P}_x\{\tau_y^+<\infty\}=1$ does not always imply $\mathbb{E}_x\left[\tau_y^+\right]<\infty$. Examples of such chains are simple random walks on $\mathbb{Z}$ and $\mathbb{Z}^2$ (Corollary 3.23).

We say that a state $x\in\Omega$ is **positive recurrent** if $\mathbb{E}_x\left[\tau_x^+\right]$ is finite. The following proposition (Proposition 21.11 in [1]) says that positive recurrence is a property uniform to all states in an irreducible chain. Again, the proof is based on that in the reference, but it has been supplemented using the concept of a geometric random variable.

Proposition 2.5. *Let $(X_t)_{t=0}^{\infty}$ be an irreducible Markov chain which is recurrent. If $\mathbb{E}_x\left[\tau_x^+\right]<\infty$ for some $x\in\Omega$, then $\mathbb{E}_y\left[\tau_z^+\right]<\infty$ for all $y,z\in\Omega$.*

Proof. Let us denote the transition matrix as P. The first part of the proof is to show that for every $y\in\Omega$, $\mathbb{E}_y\left[\tau_x\right]<\infty$.

Define $m=\inf\{n\in\mathbb{N}: P^n(x,y)>0\}$; the set is nonempty due to irreducibility, so m is finite. Then for every integer t such that $0<t<m$, $P^{m-t}(x,y)=0$. So,

$$0\le\mathbb{P}_x\{\tau_x^+=t,\tau_y=m\}\le\mathbb{P}_x\{X_t=x,X_m=y\}=P^t(x,x)P^{m-t}(x,y)=0.$$

Then, by our choice of m, $\mathbb{P}\{\tau_x^+\ge\tau_y=m\}=P^m(x,y)>0$ and

$$P^m(x,y)\big(m+\mathbb{E}_y\left[\tau_x\right]\big)=\mathbb{P}_x\{\tau_x^+\ge\tau_y=m\}\mathbb{E}_x\left[\tau_x^+|\{\tau_x^+\ge\tau_y=m\}\right]\le\mathbb{E}_x\left[\tau_x^+\right]<\infty.$$

Hence, $\mathbb{E}_y\left[\tau_x\right]<\infty$.

The next part is to show $\mathbb{E}_x[\tau_z] < \infty$. Again due to irreducibility, we can choose $m' = \inf\{n \in \mathbb{N} : P^n(x,z) > 0\} < \infty$, so that if $X_t = x$ and $X_{t+m'} = z$, then $X_s \notin \{x, z\}$ for every $t < s < t + m'$. Define $\tau'_z = \inf\{t > m' : X_{t-m'} = x, X_t = z\}$ as the hitting time for z with the condition that stops only if the chain has taken a minimal-time path from x to z. By Wald's identity, $\mathbb{E}_x[\tau'_z - m']$ is equal to $\mathbb{E}_x[\tau_x^+]$ multiplied by the expectation for a geometric random variable of success probability $P^{m'}(x,z) > 0$. Clearly, $\tau_z \le \tau'_z$. Since $\mathbb{E}_x[\tau_x^+] < \infty$,

$$\begin{aligned}\mathbb{E}_x[\tau_z] \le \mathbb{E}_x[\tau'_z] &= m' + \mathbb{E}_x[\tau'_z - m'] \\ &= m' + \mathbb{E}_x\left[\tau_x^+\right] \sum_{k=0}^{\infty} kP^{m'}(x,z)\left(1 - P^{m'}(x,z)\right)^k \\ &= m' + \mathbb{E}_x\left[\tau_x^+\right]\left(\frac{1 - P^{m'}(x,z)}{P^{m'}(x,z)}\right) < \infty\end{aligned}$$

Thus, we have proved the proposition for the cases where $x = y$ or $x = z$.

Finally, note that τ_z for a chain starting from y is less than or equal to the time it takes for the same chain to visit z after visiting x. That is,

$$\mathbb{E}_y\left[\tau_z^+\right] \le \mathbb{E}_y[\tau_x] + \mathbb{E}_x[\tau_z] < \infty$$

assuming $x \ne y$, $x \ne z$. □

Definition 2.6. Let $(X_t)_{t=0}^{\infty}$ be an irreducible, recurrent Markov chain. We say the chain is **positive recurrent** if for every $x, y \in \Omega$, $\mathbb{E}_x\left[\tau_y^+\right] < \infty$. Otherwise, no states are positively recurrent, and we say that the chain is **null recurrent**.

Example 2.7 (Renewal shift)**.** Let $\Omega = \{0\} \cup \mathbb{N}$ and the transition matrix P be given by

$$\begin{array}{lll} P(0,0) = 0, & P(0,n) = 2^{-n} & \text{for every } n \in \mathbb{N} \\ P(n,m) = 0, & P(n,n-1) = 1 & \text{for every } n \in \mathbb{N} \text{ and } m \ne n-1. \end{array}$$

$\sum_{n=0}^{\infty} P(0,n) = \sum_{n=1}^{\infty} 2^{-n} = 1$ and $\sum_{n=0}^{\infty} P(m,n) = P(m,m-1) = 1$ for every positive integer m, so P is a transition matrix. Then for every $m, n \in \mathbb{N}$, $P^m(m,0) = 1$ and $P^{m+1}(m,n) = P(0,n) = 2^{-n} > 0$, so P is irreducible. $\mathbb{E}_0\left[\tau_0^+\right] = \sum_{n=1}^{\infty} n \cdot 2^{-n} = 2$, so the chain is positive recurrent.

Beware that positive recurrence does not imply that the expected time to return to the state at time $t = 0$ is finite for arbitrary initial distributions. Let μ be a probability distribution on $\{0\} \cup \mathbb{N}$ with $\mu(0) = 0$ and $\mu(n) = 6/\pi^2 n^2$ for $n \in \mathbb{N}$. Then

$$\mathbb{E}_\mu\left[\tau_0^+\right] = \sum_{n=0}^{\infty} \mu(n) \cdot \mathbb{E}_n\left[\tau_0^+\right] = \frac{6}{\pi^2} \sum_{n=1}^{\infty} \frac{1}{n} = \infty.$$

2.1. **Why is it important to distinguish between positive recurrence and null recurrence?** Positive recurrence is a necessary condition for a Markov chain to display convergence, which is another important topic regarding Markov chains introduced in the following question. As this is a topic that merits extensive research, we state two theorems that highlight the noteworthiness of positive recurrence and direct the reader to references for further information.

Question 2.8 (Convergence)**.** *In the limit $t \to \infty$, does the distribution of X_t converge to some fixed distribution on Ω?*

The fixed distribution here is the **stationary distribution**, defined as a probability distribution π on Ω such that $\pi = \pi P$, where P is the transition matrix of a given Markov chain. It turns out that the stationary distribution of an irreducible Markov chain exists if and only if the chain is positive recurrent. Refer to Theorem 21.12 in [1] for the proof of the following theorem.

Theorem 2.9. *An irreducible Markov chain on Ω with the transition matrix P is positive recurrent if and only if there exists a stationary distribution π on Ω such that $\pi = \pi P$.*

Hence, it is natural that positive recurrence is a necessary condition in Theorem 2.12, called the convergence theorem, which provides an answer to Question 1.1. The definitions are necessary for the precise statement of the theorem.

Definition 2.10. Let μ and ν be two probability distributions on Ω. The **total variation distance** between μ and ν is defined as

$$\|\mu - \nu\|_{TV} = \sup_{A \subseteq \Omega} |\mu(A) - \nu(A)| .$$

Definition 2.11. Let $(X_t)_{t=0}^{\infty}$ be a Markov chain on Ω with transition matrix P (not necessarily irreducible). For a state $x \in \Omega$, define $\mathcal{T}(x) = \{t \geq 1 : P^t(x,x) > 0\}$ be the set of positive times when a chain starting from x can return to x. We define the **period** of x to be the greatest common divisor of $\mathcal{T}(x)$. If all states in Ω have period 1, then we call the chain **aperiodic**. Otherwise, the chain is **periodic**.

Theorem 2.12 (Convergence)**.** *Let $(X_t)_{t=0}^{\infty}$ be an irreducible, aperiodic, and positive recurrent Markov chain on Ω with transition matrix P. Then there exists a unique stationary distribution π with respect to P such that for any probability distribution μ on Ω,*

$$\lim_{t \to \infty} \|\mu P^t - \pi\|_{TV} = 0.$$

Therefore, the distribution of X_t converges to π in terms of total variation distance regardless of the initial distribution of X_0.

Proof. Refer to Chapter 4 of [1] (Exercise 4.3 in particular). □

Remark 2.13. Aperiodicity is crucial for Theorem 2.12 to hold. A simple counterexample is a Markov chain on $\Omega = \{0, 1\}$ such that the transition matrix is given by $P(0,0) = P(1,1) = 0$ and $P(0,1) = P(1,0) = 1$. The stationary distribution is $\pi(0) = \pi(1) = 1/2$, but given the initial distribution μ_0, the probability distribution of the two states at time t is $(\mu_t(0), \mu_t(1)) = (\mu_0(0), \mu_0(1))$ whenever t is even and $(\mu_t(0), \mu_t(1)) = (\mu_0(1), \mu_0(0))$. Hence, $\|\mu_t - \pi\|_{TV} = \|\mu_0 - \pi\|_{TV}$ for all $t \geq 0$.

3. Identifying a Recurrent Markov Chain

In this section, we develop a technique for classifying an irreducible Markov chain into a transient, null recurrent, or positive recurrent chain. An analogy is made between random walks on networks and electrical networks by identifying the voltage as a harmonic function.

Our technique applies to special Markov chains that are called weighted random walks on networks. We consider an undirected, positively weighted graph G. We assume that there is at most one edge between any pair of vertices. If x and y share an edge, we denote $x \sim y$. Because we consider undirected graphs, $x \sim y$ and $y \sim x$ are equivalent.

The weights on edges are called **conductances**. For the edge between vertices $x \sim y$, we denote its conductance as $c(x,y) = c(y,x) > 0$; the equality follows since the graph is undirected. We also call the reciprocal $r(x,y) = 1/c(x,y)$ of conductance the **resistance**. Let $c(x) = \sum_{y\sim x} c(x,y)$.

Definition 3.1. A **weighted random walk on** G is a Markov chain on the vertex set Ω as the state space with the transition matrix given by $P(x,y) = \frac{c(x,y)}{c(x)}$.

The simple random walk on $\mathbb{Z}^d$ is an example of weighted random walks on networks. We will exploit it as the model with which we display the usefulness of the technique. It will turn out that the simple random network on $\mathbb{Z}^d$ is null recurrent if $d = 1$ or 2 and transient if $d \geq 3$.

Definition 3.2 (Restatement of Definition 1.3)**.** A **simple random walk on** $\mathbb{Z}^d$ is a Markov chain with the n-dimensional integer lattice as the state space with the map P given by

$$P(x,y) = \begin{cases} 1/2d & \text{if } \|x-y\|_1 = 1 \\ 0 & \text{otherwise} \end{cases}$$

for any $x, y \in \mathbb{Z}^d$. Here, $\|\cdot\|_1$ denotes the 1-norm on $\mathbb{R}^d$ given by $\|x\|_1 = |x_1| + |x_2| + \cdots + |x_d|$ where the coordinates of $x \in \mathbb{R}^d$ are $(x_1, x_2, \cdots, x_d)$.

We see that a simple random walk on $\mathbb{Z}^d$ can be thought of as a weighted random walk on the graph G where the vertex set is $\mathbb{Z}^d$ and the edge set consists only of those connecting the nearest neighbors (points with difference of 1 in the 1-norm) with uniform conductance $c > 0$ (and uniform resistance $r = 1/c > 0$). The transition matrix of the chain does not depend on the specific value of c as long as it is positive. There is a uniform probability of advancing exactly one unit in the positive or negative axial directions. Any connected path of length l traced out on the d-dimensional integer lattice can be realized with probability $(2d)^{-l}$, so the simple random walk is clearly irreducible.

3.1. **Harmonic Functions.** The virtue of considering the simple random walk in terms of an electrical network is that any harmonic function on $\mathbb{Z}^d$ has a natural analogy to the electric voltage.

Definition 3.3. Let P be the transition matrix of a Markov chain on Ω (not necessarily irreducible). A function $f : \Omega \to \mathbb{R}$ is **harmonic** at $x \in \Omega$ if

$$f(x) = \sum_{y\in\Omega} P(x,y)f(y).$$

If f is harmonic at every $x \in A \subseteq \Omega$, then we say that it is harmonic on A.

Let $\partial\Omega \subset \Omega$ be a subset (called the "boundary" of Ω), and $g : \partial\Omega \to \mathbb{R}$ be a given real-valued function. Our goal is to find an extension $f : \Omega \to \mathbb{R}$ of g that is harmonic on $\Omega \setminus \partial\Omega$. That is, f is a solution to the equation

$$f(x) = \begin{cases} \sum\limits_{y\in\Omega} P(x,y)f(y) & \text{if } x \in \Omega \setminus \partial\Omega \\ g(x) & \text{if } x \in \partial\Omega \end{cases}. \tag{3.4}$$

Proposition 3.5. *Let $(X_t)_{t=0}^{\infty}$ be an irreducible and recurrent Markov chain with transition matrix P. Let $\tau_{\partial\Omega} := \inf\{t \geq 0 : X_t \in \partial\Omega\}$ be the stopping time for $\partial\Omega$. Then $f(x) = \mathbb{E}_x\left[g(X_{\tau_{\partial\Omega}})\right]$, i.e. the expected value of g for a chain starting in*

$X_0 = x$ at the first time it lands on $\partial\Omega$, is a solution to Equation 3.4. If $\Omega \setminus \partial\Omega$ is a finite set, then this is the unique solution.

Proof. First, note that $\mathbb{P}_x\{\tau_{\partial\Omega} < \infty\} = 1$ because the chain is irreducible and recurrent (Proposition 2.3). Let us show that $f(x) = \mathbb{E}_x\left[g(X_{\tau_{\partial\Omega}})\right]$ satisfies Equation 3.4.

If $x \in \partial\Omega$, then $\tau_{\partial\Omega} = 0$. Hence, $\mathbb{E}_x\left[g(X_{\tau_{\partial\Omega}})\right] = g(X_0) = g(x)$.

If $x \in \Omega \setminus \partial\Omega$, then $\tau_{\partial\Omega} \geq 1$. Expressing the Markov property in the form

$$\begin{aligned}&\mathbb{P}\left(\{X_0 = x, X_1 = y, \cdots, X_n = z\} \cap \{\tau_{\partial\Omega} = n\}\right)\\ =&P(x,y)\,\mathbb{P}\left(\{X_0 = y, \cdots, X_{n-1} = z\} \cap \{\tau_{\partial\Omega} = n-1\}\right)\end{aligned}$$

shows that $\mathbb{E}_x\left[g(X_{\tau_{\partial\Omega}})\right]$ is a weighted average of $\mathbb{E}_{X_1}\left[g(X_{\tau_{\partial\Omega}})\right]$. Precisely,

$$\mathbb{E}_x\left[g(X_{\tau_{\partial\Omega}})\right] = \sum_{y\in\Omega} P(x,y)\mathbb{E}_y\left[g(X_{\tau_{\partial\Omega}})\right],$$

which is exactly in the form of Equation 3.4.

Now, let us prove that if $\Omega \setminus \partial\Omega$ is a finite set, then $f(x) = \mathbb{E}_x\left[g(X_{\tau_{\partial\Omega}})\right]$ is the unique solution. Suppose that there exists another solution $\tilde{f}$ to Equation 3.4. Let $h = f - \tilde{f}$. Then h satisfies

$$h(x) = \begin{cases} \sum\limits_{y\in\Omega} P(x,y)h(y) & \text{if } x \in \Omega \setminus \partial\Omega \\ 0 & \text{if } x \in \partial\Omega \end{cases}. \tag{3.6}$$

Since $\Omega \setminus \partial\Omega$ is finite, the set $\{h(x) : x \in \Omega \setminus \partial\Omega\}$ has a maximum and a minimum. Let $z \in \Omega \setminus \partial\Omega$ be a maximizer of h in $\Omega \setminus \partial\Omega$, and suppose that $h(z) > 0$. If there exists $\tilde{z} \in \Omega \setminus \partial\Omega$ such that $P(z,\tilde{z}) > 0$ and $h(\tilde{z}) < h(z)$, then

$$0 = h(z) - \sum_{y\in\Omega} P(z,y)h(y) = P(z,\tilde{z})\big(h(z) - h(\tilde{z})\big) + \sum_{y\in\Omega\setminus\{\tilde{z}\}} P(z,y)\big(h(z) - h(y)\big) > 0$$

since $h(z) \geq h(y)$ for every $y \in \Omega$. Hence, for every $y \in \Omega$, if $P(z,y) > 0$ then $h(y) = h(z)$.

But since P is irreducible, for every $x \in \Omega$ there exists $t \in \mathbb{N}$ such that $P^t(z,x) > 0$. In particular, there exists $y_1, y_2, \cdots, y_{t-1} \in \Omega$ such that

$$P(z,y_1) > 0, P(y_1,y_2) > 0, \cdots, P(y_{t-1},x) > 0.$$

Hence, $h(z) = h(x_1) = \cdots = h(x_{t-1}) = h(y)$, which means that $h(y) = h(z) > 0$ for every $y \in \Omega$, even if $y \in \partial\Omega$. Since this contradicts Equation 3.6, we must have $\max_{y\in\Omega} h(y) = 0$. Similarly, $\min_{y\in\Omega} h(y) = 0$. Then $h = 0$ everywhere on Ω, which implies that $f = \tilde{f}$. (This is the discrete version of the maximum principle for harmonic functions.) □

Remark 3.7. Suppose $V \subset \Omega$ is a finite subset, and let its boundary satisfy $\partial V \supseteq \{x \in V : P(x,y) > 0 \text{ for some } y \in \Omega \setminus V\}$. Consider a Markov chain on V with the transition matrix $\tilde{P}$ such that

$$\tilde{P}(x,y) = \begin{cases} P(x,y) & \text{if } x \in V \setminus \partial V \\ P(x,y) & \text{if } x \in \partial V \text{ and } y \in V \setminus \{x\} \\ 1 - \sum_{z\in V} P(x,z) & \text{if } x = y \in \partial V \end{cases}.$$

Let $(X_t)_{t=0}^{\infty}$ be the chain on Ω with transition matrix P and $(\widetilde{X}_t)_{t=0}^{\infty}$ be the chain on V with transition matrix $\tilde{P}$. Then for any $t \geq 1$, $x_0, x_1, \cdots, x_{t-1} \in V \setminus \partial V$, and

$x_t \in \partial V$,

$$\mathbb{P}_{x_0}\{X_1 = x_1, \cdots, X_t = x_t\} = \mathbb{P}_{x_0}\{\widetilde{X}_1 = x_1, \cdots, \widetilde{X}_t = x_t\}.$$

Let $\tilde{f} : V \to \mathbb{R}$ be a function which solves

$$\tilde{f}(x) = \begin{cases} \sum_{y \in \Omega} \tilde{P}(x,y)\tilde{f}(y) & \text{if } x \in D \\ g(x) & \text{if } x \in B \end{cases}. \tag{3.8}$$

Then f also satisfies Equation 3.4 for points in V. Since $\tilde{f}$ is the unique solution for Equation 3.8, any f which solves Equation 3.4 must satisfy $f(x) = \tilde{f}(x)$ for every $x \in V$.

3.2. **Voltages and Current Flows.** We now return to our discussion of random walks on networks. By Proposition 3.5, the harmonic extension of a function g defined on the boundary states is uniquely determined if there are finitely many non-boundary points. Therefore, in the rest of this section, we consider a subgraph G' of the graph G, where the vertex subset V of G' is finite and the edge subset E of G' consists of all edges in G that connect vertices in V. By Remark 3.7, we may restrict our attention to the unique harmonic extension of g defined on ∂V to vertices in V only.

Definition 3.9. Given $g : \partial V \to \mathbb{R}$, we call the unique harmonic extension of g onto V as the **voltage**, denoted $W : V \to \mathbb{R}$.

Because our interest in voltage is related to identifying the recurrence of a Markov chain, it is especially useful to consider the case where the boundary values are

$$g(x) = \begin{cases} v & \text{if } x = a \\ 0 & \text{if } x \in Z \end{cases} \tag{3.10}$$

for some $v \in \mathbb{R}$. Here, $a \in \partial V$ is some vertex which we call the **source** of the network; it is considered as the initial state of the Markov chain. $Z = \partial V \setminus \{a\}$ is called the **sink** of the network, which is considered as the set on which we terminate the chain. The sink may contain more than one vertex, just as we often "ground" the electric circuit at more than one point.

Definition 3.11. A **flow** from a to Z is a function $\theta : V \times V \to \mathbb{R}$ such that for every $x, y \in V$,

(i) $\theta(x,x) = 0$
(ii) $\theta(x,y) = 0$ if $x \nsim y$ or $x, y \in Z$
(iii) $\theta(x,y) = -\theta(y,x)$
(iv) $\operatorname{div}\theta(x) = 0$ if $x \in V \setminus \partial V$

where $\operatorname{div}\theta(x) := \sum_{y \in V} \theta(x,y)$, the net flow coming out of x, is called the **divergence** of θ at x. The fourth condition is known as *Kirchhoff's node law.* Finally, if $\operatorname{div}\theta(a) = 1$, then we call θ a **unit flow** from a to Z.

Remark 3.12. Equivalently, we can define the flow from a to Z as a real-valued function over E. Since the flow must be antisymmetric, we assign a fixed direction to each edge. If $e \in E$ is an edge between $x, y \in V$ in which the positive direction is from x to y, then $\theta_E : E \to \mathbb{R}$ is defined so that $\theta_E(e) = \theta(x,y) = -\theta(y,x)$. In this perspective, the only two conditions necessary for θ_E to be a flow from a to Z are (i) $\theta_E(e) = 0$ if $e \in E$ connects two vertices in Z and (ii) $\operatorname{div}\theta_E(x) = 0$ for every $x \in V \setminus \partial V$.

Definition 3.13. Suppose W is a voltage on V. The **current flow** I associated with W is the flow from a to Z given by

$$I(x,y) = \frac{W(x) - W(y)}{r(x,y)} = c(x,y)\big[W(x) - W(y)\big]$$

for all $x, y \in V$ such that $x \sim y$ and $I(x,y) = 0$ for all $x, y \in V$ such that $x \nsim y$.

Let us check that the current flow is a flow from a to Z. For all $x, y \in V$:

(i) If $x = y$, then $I(x,y) = c(x,y)\big[W(x) - W(y)\big] = c(x,x) \cdot 0 = 0$.
(ii) If $x \nsim y$, then $I(x,y) = 0$ by definition. If $x, y \in Z$ and $x \sim y$, then $W(x) = W(y) = 0$ implies $I(x,y) = 0$.
(iii) $I(x,y) = c(x,y)\big[W(x) - W(y)\big] = -c(x,y)\big[W(y) - W(x)\big] = -I(y,x)$.
(iv) If $x \in V \setminus \partial V$,

$$\sum_{y \in V} I(x,y) = \sum_{y \in V} c(x,y)\,[W(x) - W(y)] = c(x)\left[W(x) - \sum_{y \in V} P(x,y)W(y)\right] = 0$$

since W is harmonic at x.

The following proposition shows that our definition of voltage as the unique harmonic extension is equivalent to how voltage is defined in physics in terms of Kirchhoff's laws and Ohm's law.

Proposition 3.14 (Cycle law)**.** *Let θ be a flow from a to Z. Then θ is a current flow if and only if*

$$\sum_{i=1}^{k} r(x_{i-1}, x_i)\,\theta(x_{i-1}, x_i) = 0 \tag{3.15}$$

for every $k \geq 1$, $x_0, x_1, \cdots, x_k \in V$ such that (i) $x_0 = x_k$ or $x_0, x_k \in Z$, and (ii) $x_0 \sim x_1, x_1 \sim x_2, \cdots, x_{k-1} \sim x_k$.

Proof. Let $x_0, \cdots, x_k$ be vertices that satisfy the above two conditions.

For the current flow I,

$$\begin{aligned}\sum_{i=1}^{k} r(x_{i-1}, x_i)\,I(x_{i-1}, x_i) &= \sum_{i=1}^{k} r(x_{i-1}, x_i)\left(\frac{W(x_{i-1}) - W(x_i)}{r(x_{i-1}, x_i)}\right) \\ &= \sum_{i=1}^{k} \big(W(x_{i-1}) - W(x_i)\big) = 0.\end{aligned}$$

On the other hand, suppose that θ satisfies Equation 3.15 but is not a current flow. We claim that there exists a current flow I from a to Z such that $\operatorname{div}\theta(a) = \operatorname{div} I(a)$. For all $x \in V$, $W(x) = \mathbb{E}_x[g(X_{\tau_{\partial V}})] = v \cdot \mathbb{P}_x\{\tau_a < \tau_Z\}$ by Proposition 3.5. Then $W(x) = \kappa(x) \cdot v$, where $\kappa(x) = \mathbb{P}_x\{\tau_a < \tau_Z\}$ is constant for each vertex $x \in V$. So,

$$\operatorname{div} I(a) = \sum_{x \sim a} I(a,x) = \sum_{x \sim a} c(a,x)[W(a) - W(x)] = v \cdot \left(c(x) - \sum_{x \sim a} c(a,x)\kappa(x)\right)$$

and we may obtain the (unique) current flow I with $\operatorname{div}\theta(a) = \operatorname{div} I(a)$ by choosing the right value of v.

Now, it is easy to check that $\eta = \theta - I$ is a flow from a to Z that satisfies Equation 3.15. Furthermore, $\operatorname{div}\eta(a) = 0$ and $\sum_{z \in Z} \operatorname{div}\eta(z) = 0$.

Let us now construct a sequence of nodes which violates Equation 3.15. Since $\eta \neq 0$ and η is antisymmetric, there exists some $x_0, x_1 \in V$ such that $x_0 \sim x_1$ and $\eta(x_0, x_1) > 0$. Now, there are two possibilities for x_1.

(1) $x_1 \notin Z$: since $\operatorname{div} \eta(x_1) = 0$, there exists $x_2 \in V$ such that $x_1 \sim x_2$ and $\eta(x_1, x_2) > 0$.
(2) $x_1 \in Z$: since $\sum_{z \in Z} \operatorname{div} \eta(z) = 0$, there exists $x_2 \in Z$ and $x_3 \in V \setminus Z$ such that $x_2 \sim x_3$ and $\eta(x_2, x_3) > 0$.

Let us continue to choose $x_3, x_4, \cdots$ through this process. Since V is finite, by the pigeonhole principle there exist some $m, n \in \mathbb{N}$ $(m < n)$ such that n is the smallest integer greater than m where $x_m = x_n$ or $x_m, x_n \in Z$. Then $\eta(x_{i-1}, x_i) \geq 0$ for every $m < i \leq n$. Due to our method of construction, $x_m, x_{m+1}, \cdots, x_n$ satisfies the two required conditions and $\eta(x_i, x_{i+1}) > 0$ for all $m \leq i < n$. Therefore, we achieve the following contradiction:

$$\begin{aligned} 0 &< \sum_{i=m}^{n-1} r(x_i, x_{i+1}) \eta(x_i, x_{i+1}) \\ &= \sum_{i=m}^{n-1} r(x_i, x_{i+1}) \theta(x_i, x_{i+1}) - \sum_{i=m}^{n-1} r(x_i, x_{i+1}) I(x_i, x_{i+1}) = 0 - 0 = 0. \end{aligned}$$

□

3.3. **Effective Resistance and Escape Probability.** We are now ready to utilize techniques from network theory to identify recurrence or transience of the simple random walk on $\mathbb{Z}^d$. Proposition 3.17 is the key; it identifies the relationship between effective resistance and escape probability. Effective resistance can be calculated using ideas from network theory, namely Thomson's Principle, Proposition 3.19, and Nash-Williams Inequality, Proposition 3.22. Escape probability is related to recurrence/transience of the simple random walk on $\mathbb{Z}^d$, as explained in Remark 3.18.

Definition 3.16. Let I be the current flow from a to Z. Define the **effective resistance** between a and Z by

$$\mathcal{R}(a \leftrightarrow Z) := W(a)/\operatorname{div} I(a)$$

and the **effective conductance** $\mathcal{C}(a \leftrightarrow Z)$ between a and Z to be the reciprocal of effective resistance.

Proposition 3.17. *Define the* ***escape probability*** *from a to be $\mathbb{P}_a\{\tau_Z < \tau_a^+\}$. The escape probability from a is given by*

$$\mathbb{P}_a\{\tau_Z < \tau_a^+\} = \frac{1}{c(a)\mathcal{R}(a \leftrightarrow Z)} = \frac{\mathcal{C}(a \leftrightarrow Z)}{c(a)}.$$

Proof. The proof follows from Proposition 3.5 and the succeeding remark, that $f(x) = \mathbb{E}_x\left[g(X_{\tau_B})\right]$ is the unique harmonic extension to the boundary condition given by Equation 3.10. Hence, the voltage is determined uniquely by $W(x) = \mathbb{E}_x\left[g(X_{\tau_B})\right]$ for every $x \in A$. Since $g(y) = 0$ for every $y \in Z$,

$$W(x) = \mathbb{E}_x\left[g(X_{\tau_B})\right] = \mathbb{P}_x\{X_{\tau_B} = a\} \cdot g(a) + \mathbb{P}_x\{X_{\tau_B} \in Z\} \cdot 0 = \mathbb{P}_x\{\tau_Z > \tau_a\} \cdot W(a).$$

Therefore,

$$\begin{aligned}\mathbb{P}_a\{\tau_Z < \tau_a^+\} &= \sum_{x\sim a} P(a,x)\mathbb{P}_x\{\tau_Z < \tau_a\} = \sum_{x\sim a} P(a,x)\big(1-\mathbb{P}_x\{\tau_Z > \tau_a\}\big)\\ &= \sum_{x\sim a}\frac{c(a,x)}{c(a)}\left(1-\frac{W(x)}{W(a)}\right) = \frac{1}{c(a)W(a)}\sum_{x\sim a} c(a,x)\big(W(a)-W(x)\big)\\ &= \frac{1}{c(a)W(a)}\sum_{x\sim a} I(a,x) = \frac{\mathrm{div}I(a)}{c(a)W(a)} = \frac{\mathcal{C}(a\leftrightarrow Z)}{c(a)}.\end{aligned}$$

□

Remark 3.18. We can now reduce the problem of identifying recurrence and transience of the simple random walk on $\mathbb{Z}^d$ into calculating the effective resistance between the origin as the source and points infinitely far away as the sink. In our discussion of a simple random walks on $\mathbb{Z}^d$, we would like Z to be the boundary $\{x \in V : x \sim y \text{ for some } y \in \mathbb{Z}^d \setminus V\}$ of a large but finite subset $V \subset \mathbb{Z}^d$. In this way, we can approximate a random walker who proceeds infinitely far away from the origin in the following manner: move Z farther away from the origin and consider whether the walker starting from the origin hits Z in finite time.

Our formulation is to set $Z_n = \{x \in \mathbb{Z}^d : \|x\|_1 = n\}$ for $n \in \mathbb{N}$, and let n grow to infinity. Since a random walker starting from the origin must first visit Z_n to visit Z_{n+1}, $\{\tau_{Z_{n+1}} < \tau_0^+\} \subset \{\tau_{Z_n} < \tau_0^+\}$ for every positive integer n. Hence, $\big(\mathbb{P}_0\{\tau_{Z_n} < \tau_0^+\}\big)_{n\in\mathbb{N}}$ is a nonnegative decreasing sequence and $\lim_{n\to\infty}\mathbb{P}_0\{\tau_{Z_n} < \tau_0^+\}$ exists. We consider the following two possibilities:

(1) If $\lim_{n\to\infty}\mathbb{P}_0\{\tau_{Z_n} < \tau_0^+\} = 0$, then the random walk is recurrent.

First, we claim that $\mathbb{P}_0\{\tau_{Z_n} = \infty\} = 0$. Since the region bounded by Z_n contains finitely many points, an application of the pigeonhole principle tells us that for the simple random walk starting from the origin, there always exists some $y \in \mathbb{Z}^d$ with $\|y\|_1 < n$ such that the random walker visits y infinitely often. But for any such y, since the minimum distance between y and Z_n is $n - \|y\|_1$,

$$\mathbb{P}_y\{\tau_y^+ > \tau_{Z_n}\} \geq \mathbb{P}_y\{\tau_{Z_n} = n - \|y\|_1\} \geq (2d)^{-(n-\|y\|_1)} > 0.$$

Hence,

$$\mathbb{P}_y\{\tau_{Z_n} = \infty\} = \lim_{k\to\infty}\big(\mathbb{P}_y\{\tau_y^+ < \tau_{Z_n}\}\big)^k = 0.$$

This implies $\mathbb{P}_0\{\tau_{Z_n} = \infty\} = 0$.

That is, conditioned on the event $\tau_0^+ = \infty$, $\tau_{Z_n} < \tau_0^+$ with probability 1 for every $n \in \mathbb{N}$. Therefore,

$$\mathbb{P}_0\{\tau_0^+ = \infty\} \leq \mathbb{P}_0\left(\bigcap_{n=1}^{\infty}\{\tau_{Z_n} < \tau_0^+\}\right) = \lim_{n\to\infty}\mathbb{P}_0\{\tau_{Z_n} < \tau_0^+\} = 0$$

and the random walk is recurrent.

(2) If $\lim_{n\to\infty}\mathbb{P}_0\{\tau_{Z_n} < \tau_0^+\} > 0$, then the random walk is transient.

For a random walker starting at the origin, $\tau_{Z_n} \to \infty$ as $n \to \infty$ because $\tau_{Z_n} \geq n$. Hence, $\cap_{n=1}^{\infty}\{\tau_{Z_n} < \tau_0^+\} \subseteq \{\tau_0^+ = \infty\}$. This implies that

$$\mathbb{P}_0\{\tau_0^+ = \infty\} \geq \mathbb{P}_0\left(\bigcap_{n=1}^{\infty}\{\tau_{Z_n} < \tau_0^+\}\right) = \lim_{n\to\infty}\mathbb{P}_0\{\tau_{Z_n} < \tau_0^+\} > 0.$$

The following proposition is useful for finding the upper bound of $\lim_{n\to\infty}\mathcal{R}(0\leftrightarrow Z_n)$, hence the lower bound of $\lim_{n\to\infty}\mathbb{P}_0\{\tau_{Z_n}<\tau_0^+\}$.

Proposition 3.19 (Thomson's Principle)**.** *Define the* ***energy*** *of a flow θ from a to Z by*

$$\mathcal{E}(\theta):=\sum_{e\in E}r(e)\theta(e)^2=\frac{1}{2}\sum_{\substack{x,y\in V\\ x\sim y}}r(x,y)\theta(x,y)^2.$$

(The ambiguity in the sign of $\theta(e)$ does not cause problem because it is squared.) Then

$$\mathcal{R}(a\leftrightarrow Z)=\mathcal{E}(I)=\inf\{\mathcal{E}(\theta):\theta \text{ a unit flow from } a \text{ to } z\}$$

where I is the unit current flow from a to Z.

Proof. Fix a direction on each edge, so that we may consider a flow from a to Z as a real-valued function on E. Let Φ be the real Hilbert space of functions from E to $\mathbb{R}$ with $<\varphi_1,\varphi_2>=\sum_{e\in E}r(e)\varphi_1(e)\varphi_2(e)$ as the inner product. For every $x\in V\setminus\partial V$, define $f_x:\Phi\to\mathbb{R}$ be a linear map defined by $f_x(\varphi)=\operatorname{div}\varphi(x)$. If we denote $\Theta\subset\Phi$ to be the subset of flows from a to Z, then $\Theta=\cap_{x\in V\setminus\partial V}\ker f_x$. This shows that Θ is a subspace of Φ. Moreover, for any $\theta\in\Theta$, $\mathcal{E}(\theta)=<\theta,\theta>=\|\theta\|^2$.

Let $I\in\Theta$ be the unit current flow from a to Z. For any $\varphi\in\Theta$,

$$\begin{aligned}
<I,\varphi> &= \frac{1}{2}\sum_{\substack{x,y\in V\\ x\sim y}}r(x,y)I(x,y)\varphi(x,y)\\
&= \frac{1}{2}\sum_{\substack{x,y\in V\\ x\sim y}}r(x,y)\left(\frac{W(x)-W(y)}{r(x,y)}\right)\varphi(x,y)\\
&= \frac{1}{2}\sum_{\substack{x,y\in V\\ x\sim y}}\left(W(x)-W(y)\right)\varphi(x,y)\\
&= \frac{1}{2}\sum_{x\in V}W(x)\sum_{\substack{y\in V\\ x\sim y}}\varphi(x,y)-\frac{1}{2}\sum_{y\in V}W(y)\sum_{\substack{x\in V\\ x\sim y}}\varphi(x,y)\\
&= \sum_{x\in V}W(x)\cdot\operatorname{div}\varphi(x)=W(a)\cdot\operatorname{div}\varphi(a).
\end{aligned}$$

For any unit flow $\theta\in\Theta$, $\theta-I\in\Theta$ and $\operatorname{div}(\theta-I)(a)=0$. So, $<I,\theta-I>=0$ and

$$\mathcal{E}(\theta)=\|I\|^2+\|\theta-I\|^2+2<I,\theta-I>=\mathcal{E}(I)+\|\theta-I\|^2\geq\mathcal{E}(I).$$

In particular, $\mathcal{E}(\theta)=\mathcal{E}(I)$ if and only if $\theta=I$.

Finally,

$$\mathcal{E}(I)=<I,I>=W(a)\cdot\operatorname{div}I(a)=W(a)=\frac{W(a)}{\operatorname{div}I(a)}=\mathcal{R}(a\leftrightarrow Z).$$

□

Corollary 3.20. *The simple random walk on $\mathbb{Z}^d$ is transient for $d\geq 3$.*

Proof. Since Z_n is finite for every $n\in\mathbb{N}$, we can apply the Thomson's Principle to find the upper bound for $\mathcal{R}(0\leftrightarrow Z_n)$. We construct a unit flow θ from 0 to Z_n based on a stochastic process known as *Polya's urn.* The idea is that the

random walker starts from the origin, and at time t, the random walker is on $Z_t^+ = \{x \in \mathbb{Z}^d : \|x\|_1 = t \text{ and all coordinates of } x \text{ are nonnegative}\}$. The walker moves from $X_t = x \in Z_t^+$ to $X_{t+1} \in Z_{t+1}^+$ by taking the edge in the direction of the ith unit vector with probability proportional to the ith coordinate of x. Finally, the flow on the edge is defined to be the probability that the random walker will pass through it.

Let us now construct the flow θ formally. It shall be constructed so that the flow is nonzero only between points whose coordinates are all nonnegative. Let x, y be points in $\mathbb{Z}^d$ such that $\|x\|_1, \|y\|_1 \leq n$. We define $\theta(x, y) = 0$ if either x or y contains a negative coordinate. Clearly, $\operatorname{div}\theta(x) = 0$ whenever $x \in \mathbb{Z}^d$ with $\|x\|_1 < n$ and x has a negative coordinate.

This leaves edges between points of the form $x = (x_1, \cdots, x_d)$ (where $x_1, \cdots, x_d \geq 0$ and $x_1 + \cdots + x_d \leq n$) and $y = (x_1, \cdots, x_{i-1}, x_i + 1, x_{i+1}, \cdots, x_d) = x + u_i$ for some $1 \leq i \leq d$, where u_i is the unit vector in the ith coordinate. Let $Z_{\leq n}^+ = \cup_{0 \leq m \leq n} Z_m^+$. Define a function $\phi : Z_{\leq n}^+ \to [0, 1]$ whose values satisfy

$$\phi(x) = \begin{cases} 1 & \text{if } x = 0 \\ \sum\limits_{\substack{i=1 \\ x_i \neq 0}}^{d} \big(x_i \phi(x - u_i)\big) / \big(\|x\|_1 + d - 1\big) & \text{otherwise} \end{cases} .$$

Now, we construct the flow on $Z_{\leq n}^+$ such that for every $x \in Z_{\leq n-1}^+$ and $1 \leq i \leq d$,

$$\theta(x, x + u_i) = \phi(x) \cdot \frac{x_i + 1}{\|x\|_1 + d}.$$

This determines all values of θ for the edges between points in $\{x \in \mathbb{Z}^d : \|x\|_1 \leq n\}$. θ is a unit flow from 0 to Z_n because

$$\operatorname{div}\theta(0) = \sum_{i=1}^{d} \theta(0, u_i) + \sum_{i=1}^{d} \theta(0, -u_i) = \sum_{i=1}^{d} \phi(0) \cdot \frac{1}{d} + 0 = 1$$

and for every $x \in Z_m^+$ where $1 \leq m < n$,

$$\begin{aligned} \operatorname{div}\theta(x) &= \sum_{i=1}^{d} \theta(x, x + u_i) - \sum_{\substack{i=1 \\ x_i \neq 0}}^{d} \theta(x - u_i, x) \\ &= \sum_{i=1}^{d} \phi(x) \cdot \frac{x_i + 1}{\|x\|_1 + d} - \sum_{\substack{i=1 \\ x_i \neq 0}}^{d} \phi(x - u_i) \cdot \frac{x_i}{\|x\|_1 + d - 1} \\ &= \phi(x) - \sum_{\substack{i=1 \\ x_i \neq 0}}^{d} \phi(x - u_i) \cdot \frac{x_i}{\|x\|_1 + d - 1} \\ &= \sum_{\substack{i=1 \\ x_i \neq 0}}^{d} \frac{x_i \phi(x - u_i)}{\|x\|_1 + d - 1} - \sum_{\substack{i=1 \\ x_i \neq 0}}^{d} \phi(x - u_i) \cdot \frac{x_i}{\|x\|_1 + d - 1} = 0. \end{aligned}$$

We claim that for every $0 \leq m \leq n$, ϕ is uniform on Z_m^+. The proof is by induction on m. The case $m = 0$ is trivial, because Z_0^+ is a singleton set. Suppose the claim is true when $m = k$ for some integer $k \geq 0$, so that for every $y \in Z_k^+$,

$\phi(y) = c_k$. Then for every $x \in Z_{k+1}^+$,

$$\phi(x) = \sum_{\substack{i=1 \\ x_i \neq 0}}^{d} \phi(x - u_i) \cdot \frac{x_i}{\|x\|_1 + d - 1} = c_k \sum_{\substack{i=1 \\ x_i \neq 0}}^{d} \frac{x_i}{\|x\|_1 + d - 1} = c_k \cdot \frac{k+1}{k+d} = c_{k+1}.$$

In fact, we see that

$$c_m = \prod_{k=0}^{m-1} \frac{k+1}{k+d} = \binom{d+m-1}{m}^{-1}.$$

Suppose that the uniform resistance between immediate neighbors of $\mathbb{Z}^d$ is $r > 0$ (the specific value of r does not matter as long as it is positive). Let $V = \cup_{m=0}^{n} Z_m$ and $\partial V = Z_n$. Since $|Z_m^+| = \binom{m+(d-1)}{m}$, we obtain

$$\begin{aligned}
\mathcal{R}(0 \leftrightarrow Z_n) \leq \mathcal{E}(\theta) &= \sum_{e \in E} r(e)\, \theta(e)^2 = r \sum_{e \in E} \theta(e)^2 \\
&= r \sum_{m=0}^{n-1} \sum_{x \in Z_m^+} \sum_{i=1}^{d} \theta(x, x+u_i)^2 \\
&= r \sum_{m=0}^{n-1} \sum_{x \in Z_m^+} \phi(x)^2 \sum_{i=1}^{d} \left(\frac{x_i + 1}{\|x\|_1 + d} \right)^2 \\
&\leq rd \sum_{m=0}^{n-1} \sum_{x \in Z_m^+} \phi(x)^2 = rd \sum_{m=0}^{n-1} \binom{d+m-1}{m}^{-1}.
\end{aligned}$$

Since

$$\begin{aligned}
&\lim_{m \to \infty} \left(\binom{d+m-1}{m}^{-1} \Big/ \frac{1}{m^{d-1}} \right) = \lim_{m \to \infty} \left(\binom{d+m-1}{d-1}^{-1} \Big/ \frac{1}{m^{d-1}} \right) \\
&= \lim_{m \to \infty} \prod_{k=1}^{d-1} \frac{k}{m+k} \cdot m = (d-1)! \prod_{k=1}^{d-1} \left(\lim_{m \to \infty} \frac{m}{m+k} \right) = (d-1)!,
\end{aligned}$$

$\sum_{m=0}^{\infty} \binom{d+m-1}{m}^{-1}$ is convergent for $d \geq 3$ by the limit comparison test with $\sum_{m=0}^{\infty} m^{-(d-1)}$. Therefore,

$$0 \leq \lim_{n \to \infty} \mathcal{R}(0 \leftrightarrow Z_n) \leq rd \sum_{m=0}^{\infty} \binom{d+m-1}{m}^{-1} < \infty.$$

By Proposition 3.17,

$$\lim_{n \to \infty} \mathbb{P}_0\{\tau_{Z_n} < \tau_0^+\} = \lim_{n \to \infty} \frac{1}{c(0)\mathcal{R}(0 \leftrightarrow Z_n)} = \frac{r}{2d} \lim_{n \to \infty} \frac{1}{\mathcal{R}(0 \leftrightarrow Z_n)} > 0.$$

We refer to Remark 3.18 to conclude that the simple random walk on $\mathbb{Z}^d$ is transient for $d \geq 3$. □

Remark 3.21. The above proof provides the upper bound for the growth rate of $\mathcal{R}(0 \leftrightarrow Z_n)$ even when $d = 1$ or 2. It shows that

$$0 \leq \mathcal{R}(0 \leftrightarrow Z_n) \leq rd \sum_{m=0}^{n-1} \binom{d+m-1}{m}^{-1}.$$

For $d = 1$, $\binom{d+m-1}{m} = 1$, so $\mathcal{R}(0 \leftrightarrow Z_n) \leq nr$.
For $d = 2$, $\binom{d+m-1}{m} = \frac{1}{m+1}$, so $\mathcal{R}(0 \leftrightarrow Z_n) \leq 2r \sum_{m=0}^{n-1} \frac{1}{m+1} \leq 2(1 + \log n)r$.
Hence,

$$\mathbb{P}_0\{\tau_{Z_n} < \tau_0^+\} = \frac{r}{2d\,\mathcal{R}(0 \leftrightarrow Z_n)} \geq \begin{cases} 1/2n & \text{if } d = 1 \\ 1/8(1 + \log n) & \text{if } d = 2 \end{cases}.$$

To find the lower bound for the growth rate of $\mathcal{R}(0 \leftrightarrow Z_n)$, the following Proposition is useful.

Proposition 3.22 (Nash-Williams Inequality)**.** *Call a collection of edges $\Pi \subset E$ an **edge-cutset separating** a **from** Z if every path from a to Z includes some edge in Π. If $\{\Pi_k\}$ is a family of disjoint edge-cutsets separating a from Z such that $|\Pi_k|$ is finite for every index k, then*

$$\mathcal{R}(a \leftrightarrow Z) \geq \sum_k \left(\sum_{e \in \Pi_k} c(e) \right)^{-1}.$$

Proof. Let I be the unit current flow from a to Z. For any index k, due to the Cauchy-Schwarz inequality,

$$\left(\sum_{e \in \Pi_k} c(e) \right) \left(\sum_{e \in \Pi_k} r(e) I(e)^2 \right) \geq \left(\sum_{e \in \Pi_k} |I(e)| \right)^2.$$

Let V_k be the set of nodes that can be reached from a without passing through any edge in Π_k (including a). Also let $\partial V_k \subset V_k$ be the subset of nodes that are on one side of some edge in Π_k: if $x \in \partial V_k$, then $(x, y) \in \Pi_k$ for some $y \in V$. Then due to the antisymmetry of I, a discrete version of the divergence theorem applies:

$$\begin{aligned} \sum_{e \in \Pi_k} |I(e)| &\geq \left| \sum_{x \in \partial V_k} \sum_{\substack{y \in V \\ (x,y) \in \Pi_k}} I(x, y) \right| \\ &= \left| \sum_{x \in \partial V_k} \sum_{\substack{y \in V \\ (x,y) \in \Pi_k}} I(x, y) + \sum_{x_1 \in V_k} \sum_{\substack{x_2 \in V_k \\ x_1 \sim x_2}} I(x_1, x_2) \right| \\ &= \left| \sum_{x \in V_k} \operatorname{div} I(x) \right| = |\operatorname{div} I(a)| = 1. \end{aligned}$$

By Thomson's Principle (Proposition 3.19),

$$\mathcal{R}(a \leftrightarrow Z) = \mathcal{E}(I) = \sum_{e \in E} r(e) I(e)^2 \geq \sum_k \sum_{e \in \Pi_k} r(e) I(e)^2 \geq \sum_k \left(\sum_{e \in \Pi_k} c(e) \right)^{-1}.$$

□

Corollary 3.23. *The simple random walks on $\mathbb{Z}$ and $\mathbb{Z}^2$ are null recurrent.*

Proof. Let n be an arbitrary positive integer. For every $0 \leq k < n$, we choose the edge-cutset Π_k separating 0 from Z_n to be all edges between points in Z_k and points in Z_{k+1}. Again, suppose that the uniform resistance of the edges are $r > 0$.

$d = 1$: For the one-dimensional case, $|\Pi_k| = 2$. Then by the Nash-Williams Inequality,

$$\mathcal{R}(0 \leftrightarrow Z_n) \geq \sum_{k=0}^{n-1} \left(\sum_{e \in \Pi_k} \frac{1}{r} \right)^{-1} = \sum_{k=0}^{n-1} \frac{r}{2} = n \cdot \frac{r}{2}.$$

Combining the result from Remark 3.21,

$$\frac{nr}{2} \leq \mathcal{R}(0 \leftrightarrow Z_n) \leq nr \text{ and } \mathbb{P}_0\{\tau_{Z_n} < \tau_0^+\} = \Theta\left(\frac{1}{n}\right).$$

$d = 2$: For the two-dimensional case, $|\Pi_k| = 4(2k+1)$. By the Nash-Williams Inequality,

$$\mathcal{R}(0 \leftrightarrow Z_n) \geq \sum_{k=0}^{n-1} \left(\sum_{e \in \Pi_k} \frac{1}{r} \right)^{-1} = \sum_{k=0}^{n-1} \frac{r}{4(2k+1)} \geq \frac{r}{4} \cdot \frac{1}{2} \log(2n+1).$$

Combining the result from Remark 3.21,

$$\frac{r \log(2n+1)}{8} \leq \mathcal{R}(0 \leftrightarrow Z_n) \leq 2r(1 + \log n).$$

An application of L'Hôpital's rule shows $\lim_{n \to \infty} ((1 + \log n)/\log(2n+1)) = 1$, so

$$\mathbb{P}_0\{\tau_{Z_n} < \tau_0^+\} = \Theta\left(\frac{1}{\log n}\right).$$

We refer to Remark 3.18 to conclude that the simple random walks on $\mathbb{Z}$ and $\mathbb{Z}^2$ are both recurrent.

Let us now prove that the simple random walk on $\mathbb{Z}$ and $\mathbb{Z}^2$ are not positive recurrent. For every $n \in \mathbb{N}$, the random walker starting from the origin must visit Z_n before visiting Z_{n+1}, so $\tau_{Z_n} < \tau_{Z_{n+1}}$. Thus $\{\tau_{Z_1} < \tau_0^+\} \supset \{\tau_{Z_2} < \tau_0^+\} \supset \{\tau_{Z_3} < \tau_0^+\} \supset \cdots$, and

$$\mathbb{P}_0\{\tau_{Z_n} < \tau_0^+ < \tau_{Z_{n+1}}\} = \mathbb{P}_0\{\tau_{Z_n} < \tau_0^+\} - \mathbb{P}_0\{\tau_{Z_{n+1}} < \tau_0^+\}.$$

Also, $\mathbb{E}_0\left[\tau_0^+ \middle| \{\tau_{Z_n} < \tau_0^+\}\right] \geq 2n$, since for a random walker starting from the origin, it takes at least n steps to reach Z_n and at least n further steps to return to the origin. Since $\left(\{\tau_{Z_n} < \tau_0^+ < \tau_{Z_{n+1}}\}\right)_{n \in \mathbb{N}}$ are disjoint events,

$$\begin{aligned}
\mathbb{E}_0\left[\tau_0^+\right] &= \sum_{n=1}^{\infty} \mathbb{P}_0\{\tau_{Z_n} < \tau_0^+ < \tau_{Z_{n+1}}\} \cdot \mathbb{E}_0\left[\tau_0^+ \middle| \{\tau_{Z_n} < \tau_0^+ < \tau_{Z_{n+1}}\}\right] \\
&\geq \sum_{n=1}^{\infty} \left(\mathbb{P}_0\{\tau_{Z_n} < \tau_0^+\} - \mathbb{P}_0\{\tau_{Z_{n+1}} < \tau_0^+\}\right) \cdot 2n \\
&= \lim_{N \to \infty} 2 \left(\sum_{n=1}^{N} \mathbb{P}_0\{\tau_{Z_n} < \tau_0^+\} - N \cdot \mathbb{P}_0\{\tau_{Z_{N+1}} < \tau_0^+\} \right) \\
&= \lim_{N \to \infty} \sum_{n=1}^{N} 2 \left(\mathbb{P}_0\{\tau_{Z_n} < \tau_0^+\} - \mathbb{P}_0\{\tau_{Z_{N+1}} < \tau_0^+\}\right)
\end{aligned}$$

For each N, choose M_N to be the largest integer such that $\mathbb{P}_0\{\tau_{Z_{M_N}} < \tau_0^+\} \geq 2\mathbb{P}_0\{\tau_{Z_N+1} < \tau_0^+\}$. Since $\mathbb{P}_0\{\tau_{Z_n} < \tau_0^+\} \to 0$ as $n \to \infty$ for both dimensions,

$M_N \to \infty$ as $n \to \infty$. Then

$$\lim_{N\to\infty} \sum_{n=1}^{N} 2\left(\mathbb{P}_0\{\tau_{Z_n} < \tau_0^+\} - \mathbb{P}_0\{\tau_{Z_{N+1}} < \tau_0^+\}\right)$$
$$\geq \lim_{N\to\infty} \sum_{n=1}^{M_N} \mathbb{P}_0\{\tau_{Z_n} < \tau_0^+\} = \sum_{n=1}^{\infty} \mathbb{P}_0\{\tau_{Z_n} < \tau_0^+\}.$$

Hence, $\mathbb{E}_0\left[\tau_0^+\right] = \infty$ for both dimensions. □

Remark 3.24. In fact, we can use Nash-Williams inequality to find the lower bound of $\mathcal{R}(0 \leftrightarrow Z_n)$ for $d \geq 3$ as well. The vertices in Z_k with nonzero coordinates each share edges with exactly d vertices in Z_{k+1}. The vertices in Z_k that have at least one zero coordinates share edges with at most $2d$ vertices in Z_{k+1}. But we count the latter kind of vertices more than once if we multiply $|Z_k^+| = \binom{k+d-1}{k}$ by 2^d, so $|\Pi_k| \leq d \cdot 2^d \cdot \binom{k+d-1}{k}$ and

$$\mathcal{R}(0 \leftrightarrow Z_n) \geq \sum_{k=0}^{n-1} \left(\sum_{e\in\Pi_k} \frac{1}{r}\right)^{-1} \geq \frac{r}{d\cdot 2^d} \sum_{k=0}^{n-1} \binom{k+d-1}{k-1}^{-1}.$$

Combining this result from Corollary 3.20,

$$\frac{r}{d\cdot 2^d} \sum_{k=0}^{n-1} \binom{k+d-1}{k-1}^{-1} \leq \mathcal{R}(0 \leftrightarrow Z_n) \leq rd \sum_{k=0}^{n-1} \binom{k+d-1}{k-1}^{-1}.$$

Since we calculated $\binom{k+d-1}{k-1} = \Theta\left(1/k^{d-1}\right)$ in Corollary 3.20, we conclude that the growth rate of $\mathcal{R}(0 \leftrightarrow Z_n)$ is on the order of $\sum_{k=1}^{n} k^{1-d} = \Theta(n^{2-d})$.

Remark 3.25. Since the simple random walks on $\mathbb{Z}^d$ are either null recurrent or transient for every dimension d, there do not exist any stationary distributions for these chains.

Acknowledgments. I wish to thank my mentor Peter Morfe for his guidance and support throughout the REU. His help was crucial during every stage of writing this paper. I would also like to thank Professor Peter May for his dedication to this program.

References

[1] David A. Levin, Yuval Peres, and Elizabeth L. Wilmer. *Markov Chains and Mixing Times.* American Mathematical Society, 2008.
[2] Peter G. Doyle and J. Laurie Snell. *Random Walks and Electrical Networks.* 2006.
[3] S. R. S. Varadhan. *Probability Theory.* American Mathematical Society, 2001.
[4] Greg Lawler. Notes on Probability. http://www.math.uchicago.edu/~lawler/probnotes.pdf
[5] Avner Friedman. *Foundations of Modern Analysis.* Dover, 1982.