HAMILTONICITY IN CAYLEY GRAPHS AND DIGRAPHS OF FINITE ABELIAN GROUPS.

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ABSTRACT. Cayley graphs and digraphs are introduced, and their importance and utility in group theory is formally shown. Several results are then presented: firstly, it is shown that if G is an abelian group, then G has a Cayley digraph with a Hamiltonian cycle. It is then proven that every Cayley digraph of a Dedekind group has a Hamiltonian path. Finally, we show that all Cayley graphs of Abelian groups have Hamiltonian cycles. Further results, applications, and significance of the study of Hamiltonicity of Cayley graphs and digraphs are then discussed.

CONTENTS

1. INTRODUCTION.

The topic of Cayley digraphs and graphs exhibits an interesting and important intersection between the world of groups, group theory, and abstract algebra and the world of graph theory and combinatorics. In this paper, I aim to highlight this intersection and to introduce an area in the field for which many basic problems remain open.The theorems presented are taken from various discrete math journals. Here, these theorems are analyzed and given lengthier treatment in order to be more accessible to those without much background in group or graph theory. I end by briefly explaining the importance between this area of study and a significant related problem, the Lovasz conjecture.

This paper assumes an elementary knowledge of group theory, although relevant group-theoretic terms used heavily in proofs will be defined or otherwise given context. An introduction to the graph theory concepts utilized in this paper is presented in order to prevent any confusion regarding the terminology later employed.

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2. Graph Theory: Introductory Definitions.

A graph is comprised of a set of vertices, denoted $\{v\}$, and a set of edges between these vertices, denoted $\{e\}$. If edge e runs between v_1 and v_2 in $\{v\}$, we call v_1 and v_2 the *ends* of e. Additionally, we call any two vertices that are ends of the same edge *adjacent*, and denote this $v_1 \sim v_2$.

Definition 2.1 (Directed Graph)**.** A graph is *directed* if each edge in $\{e\}$ is given an orientation from one of its ends to the other. A directed graph is called a *digraph.*

Definition 2.2 (Connected Graph)**.** In this paper, we will call an undirected graph *connected* if there is a path along edges between any two vertices in $\{v\}$. A directed graph is connected if its underlying undirected graph is connected. That is, the connectedness of a directed graph does *not* depend on the orientation of directed edges.

Definition 2.3 (Graph Isomorphism)**.** For any graphs Γ and Γ', we define an isomorphism $\phi : \Gamma \to \Gamma'$ as a map between vertices in Γ and vertices in Γ' such that, for v_1 and v_2 in $\{v\}$, if $v_1 \sim v_2$ then $\phi(v_1) \sim \phi(v_2)$. We denote the group of isomorphisms of a graph $Aut(\Gamma)$, and call Γ *isomorphic* to Γ'.

Definition 2.4 (Labeled Graph)**.** If Γ is a graph, and all edges in $\{e\}$ are assigned labels from a set $\{s\}$, then we call Γ a *labeled graph.*

Definition 2.5 (Orientation and Label Preserving)**.** For any labeled, directed graph Γ, if v and v' are connected by some directed edge e running from v to v' with label s, then ϕ is *orientation and label preserving* if $\phi(v)$ is adjacent to $\phi(v')$ by an edge labeled s directed from $\phi(v)$ to $\phi(v')$. We denote the group of isomorphisms that also preserve orientation and labelling of the graph $Aut^+(\Gamma)$.

Definition 2.6 (Hamiltonian Path and Cycle)**.** An undirected graph is called *Hamiltonian* if there is a path that visits each vertex exactly once. Such a path is called Hamiltonian. Similarly, a directed graph is called Hamiltonian if there is a path that visits every vertex in the graph exactly once while observing edge directions, and this path is similarly called a Hamiltonian path. A *Hamiltonian cycle* is a Hamiltonian path that begins and ends at the same vertex (so one vertex is visited twice: once at the beginning, and once at the end).

3. Cayley Graphs and Digraphs.

For any group G with some generating set S of finite order $|S|$, we can define a labeled, directed graph with vertex set $\{v\}$ such that each vertex corresponds to a specific g in G. Let there be an edge with label s from g to g' whenever $g' = gs$ for some s in S. Note that this graph is regular, with each vertex having degree $2|S|$. Such a graph Γ is called a *Cayley digraph.*

It is perhaps not immediately obvious that a Cayley digraph would reveal anything meaningful about its associated group and generating set. However, a group G and graph Γ are actually intimately related through the concept of a group action.

Definition 3.1 (Group Action)**.** Let X be a set, and G be a group. A homomorphism from a group G to the group of permutations of a set X is called a *group action* of G on X, denoted $G \curvearrowright X$.

Definition 3.2 (Free Group Action)**.** A group action G on a space X is *free* if, for any x in X, and any g and g' in G, $gx = g'x$ implies $g = g'$.

An important example of a group action is the action of G on the set of elements in G, denoted $G \curvearrowright \{g \in G\}$, by group multiplication on the left. Such consideration leads to a fundamental result, formalized by Arthur Cayley, about the relationship between the group G and its symmetry group $Aut(G)$. We present this elementary result because it leads to an elegant formalization of the utility of Cayley graphs and digraphs.

Theorem 3.3 (Cayley's Theorem)**.** *Every group G is isomorphic to a subgroup of the symmetry group of G.*

Proof. We want to find a homomorphism from G to the permutation group $Aut(G)$. Let $g \mapsto \pi_g$, where, for all $h \in G$, we define $\pi_g(h)$ as gh. First we must show that π_g permutes elements of G. Since G is closed under the group operation, $\pi_g(h)$ is in G for all $h \in G$. Since $gh = gh'$ implies $h = h'$, $\pi_g(h) = \pi_g(h')$ implies $h = h'$. Thus, π_g permutes the elements of G.

To confirm this is a homomorphism, we check that the map preserves the group structure. Indeed, $\pi_{gh}(j) = gh(j) = g(h(j)) = g(\pi_h(j)) = \pi_g(\pi_h(j))$. We note that the homomorphism is injective, because if $\pi_g(h) = \pi_{g'}(h)$, for any h in G, then $g = g'$ by cancellation by right inverses. Thus, we note that it is indeed an isomorphism between G and some subgroup of $Aut(G)$, namely, the image of G under the map $g \mapsto \pi_g$. This subgroup may certainly contain fewer elements than $Aut(G)$; its elements are just the permutations described by π_g for each g in G. □

Cayley's theorem can quickly be extended to precisely describe the directed, locally finite Cayley digraph defined in the introduction.

Theorem 3.4 (Cayley's Theorem Extended)**.** *Any Cayley digraph is connected and locally finite, and G acts freely on the vertices of a Cayley digraph.*

Proof. Let G be generated by S, and let graph Γ be constructed as above. Again, note that each vertex v in Γ corresponds to an element g in G. We consider a group action of $G \curvearrowright \{v\}$ so that, for h, g in G and vertex h in $\{v\}$, $g \cdot h = gh$. Note that this is indeed a group action since our vertices are correspond to $h \in G$, so each g will permute $\{v\}$. Associativity holds because $g(g'(h)) = (gg')h$.

As introduced previously, each vertex has $2|S|$ edges extending from it, so it is indeed locally finite. Moreover, connectedness comes from the fact that since we have our generating set S, each g in G is connected to our identity element by a word composed of generators and their inverses, and there is thus a path between any two vertices in the underlying undirected graph.

To check that G acts freely on $\{v\}$, we note that if v is in $\{v\}$, $g \cdot v = h \cdot v \implies g = h$. Moreover, the Cayley digraph structure is preserved under the group action $G \curvearrowright \{v\}$, because we defined the directed edges in terms of right multiplication. Thus, if we have an edge labeled s in $\{e\}$ running from vertex g to vertex gs , then in $h \cdot \Gamma$ the edge from hg to hgs is labeled s as well. □

We've shown that the action of G on our graph Γ is an orientation and label preserving isomorphism. Furthermore, the following theorem shows us that for any λ in $Aut^+(\Gamma)$, $\lambda(\Gamma) = g \cdot \Gamma$, for some g in G.

Theorem 3.5 (Frucht's Theorem)**.** *The elements of a group G biject to $Aut^+(\Gamma)$, where Γ is a Cayley digraph of G.*

Proof. We know already that for every g in G, $g \cdot \Gamma = \phi(\Gamma)$, for some ϕ in $Aut^+(\Gamma)$. So it suffices to show that every λ in $Aut^+(\Gamma)$, $\lambda(\Gamma) = g \cdot \Gamma$, for some g in G. Let $\lambda \in Aut^+(\Gamma)$. Consider a vertex h and its image $\lambda(h)$. Then there exists some g in G so that $\lambda(h) = gh$, (i.e., take $g = \lambda(h) * h^{-1}$). Now $g^{-1} \cdot \lambda(\Gamma)$ moves h to h, so all of h's adjacencies remain fixed, and all of their adjacencies, so ultimately, $g^{-1} \cdot \lambda$ maps every labeled vertex in Γ to itself. Also, g^{-1} is an element of $Aut^+(G)$. Thus $g^{-1} \cdot \lambda$ is the identity element of $Aut^+(G)$, and $g \cdot \Gamma = \lambda(\Gamma)$. So our map is surjective: every symmetry corresponds to some g in G. □

Remark 3.6. This is a proof of Frucht's theorem, but it is not Frucht's proof. In his own solution, Frucht did not consider directly what an isomorphism of a labeled, directed graph might be – instead, he modified our current description of a Cayley digraph into an unlabeled, undirected graph.

Remark 3.7. Of course, the particular generating set selected for our Cayley digraph of group G determines the digraph's construction and behavior. While every Cayley digraph of a group will obviously have the same number of vertices, the degree and adjacencies of vertices can vary widely based on the generating set.

Remark 3.8. As a point of historical context, we note that Arthur Cayley only considered finite groups in his study of Cayley graphs. It is important to observe that the theorems shown previously made no requirement that G be finite. Thusfar, our only stipulation has been that our Cayley graphs be locally finite, i.e. that there is a finite generating set for G. For this reason, it is perfectly natural and certainly interesting to consider the Cayley graphs of infinite groups like $(\mathbb{Z}, +)$, or the free group on n generators.

Example 3.9 (Cayley Digraphs).

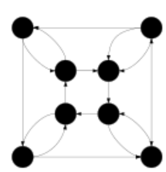

FIGURE 1. Dihedral group D_4, with generating set $\langle s, r | s^2 = r^4 = e, sr = r^3 s \rangle$

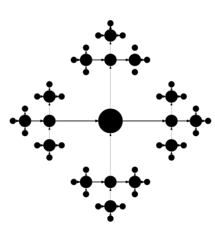

FIGURE 2. Free group on two generators

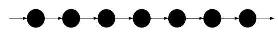

Figure 3. Integers under addition

Definition 3.10 (Cayley graph)**.** A *Cayley graph* is the underlying undirected graph of a Cayley digraph.

4. Hamiltonian Cycles in Cayley Digraphs of Finite Abelian Groups

As the previous section shows, there is a close relationship between groups and their Cayley graphs. Therefore, it is certainly worthwhile to study their properties. An interesting topic of investigation in the study of Cayley graphs is their Hamiltonicity. One might wonder, for instance, if Hamiltonicity in either type of graph would depend on the chosen set of generators. It is instructive to show that in certain elementary cases – for example, in the case of finite Abelian groups – it is indeed possible to find a Hamiltonian Cayley digraph. In fact, in this case, we can always find a generating set for which the digraph has a Hamiltonian cycle.

Lemma 4.1. *If* $G = C_m \times C_n$*, and* Γ *is a Cayley digraph of* G *on the generators* $(1, 0)$ *and* $(0, 1)$*,* Γ *has a Hamiltonian cycle if* $m|n$*.*

Proof. Let $m = kn$. First consider the case when $k = 1$. Then $G = C_n \times C_n$. Drawing the Cayley graph with respect to generators $(1, 0)$ and $(0, 1)$, it is enough to explicitly find a Hamiltonian cycle: Let $e = (0, 1)$ and $f = (1, 0)$. We begin at $(0, 0)$ and move along generators e, f with the sequence $e, e, \ldots, e, f, e, \ldots, e, f, \ldots, e \ldots, e, f$, where e occurs $n - 1$ times per f and there are a total of n f's. To move from the jth row of the Cayley digraph the $j + 1$th row we move along the $n + 1 - j$th column. (This is illustrated in the case of $G = C_4 \times C_4$ in Figure 5).

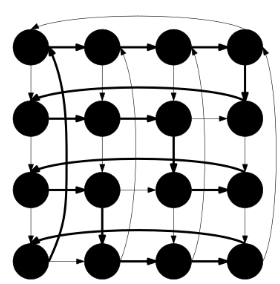

Figure 4. Hamiltonian cycle: $C_4 \times C_4$

To apply this to the general case when $m = kn, k > 1$, we note that we can consider Γ to be comprised of k blocks, each of size $n \times n$. We begin our path in the first $n \times n$ block as described above, and then proceed from Block 1 to Block 2 by moving from the vertex in the first column and last row of Block 1 to the vertex in the first column and first row of Block 2 (see Figure 6). When the path reaches the end of the kth block, it can be made a cycle by connecting the vertex in the first column and the nth row of this final block to the first row and column of the entire graph.

□

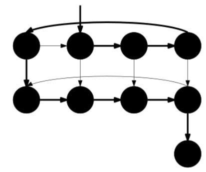

FIGURE 5. Transition from one $n \times n$ block to the next

Definition 4.2 (Nonredundant Generator, Minimal Generating Set)**.** A generator s in some generating set S of group G is called *nonredundant* if it cannot be expressed as a word using the other generators. If no generator in S is redundant, we call S a *minimal generating set*.

Theorem 4.3. *The removal of the edges corresponding to a nonredundant generator h from our Cayley digraph yields isomorphic disjoint subgraphs.*

Proof. Let G be a group, with Cayley digraph Γ corresponding to some generating set S. If h is a nonredundant generator in S, then its removal will cause any two vertices g and gh to become disconnected. Thus disjoint subgraphs are created. To construct an bijection between any two disjoint subgraphs, note that in each subgraph, given some vertex g, every other connected vertex can be expressed as $g\alpha$ for some word α composed of generators in $\{S \setminus h\} \cup \{S \setminus h\}^{-1}$. So by taking some vertex g in subgraph Γ_1, disconnected from gh in a different subgraph Γ_2, and simply sending every $g\alpha \to gh\alpha$, we create an bijection between Γ_1 and Γ_2. This isomorphism is orientation and label preserving. □

To obtain our desired result, we will use our Lemma and employ the Fundamental Theorem of Finite Abelian Groups. We make use of the form that states:
Every finite Abelian group is isomorphic to $C_{m_1} \times C_{m_2} \times \ldots \times C_{m_n}$, where $m_i | m_{i-1}$.

Theorem 4.4. *If G is a finite abelian group, the Cayley digraph of G with the standard generators has a Hamiltonian cycle.*

Proof. By the Fundamental Theorem of Finite Abelian Groups, we note that $G \cong C_{m_1} \times C_{m_2} \times \ldots C_{m_n}$, where $m_i | m_{i-1}$. Let Γ be the Cayley digraph of $C_{m_1} \times C_{m_2} \times \ldots C_{m_n}$, on the standard generators. Our proof proceeds inductively: Consider the case when $G \cong C_{m_1}$. Then G is cyclic, and therefore has a Hamiltonian cycle. Now consider $G \equiv C_{m_1} \times C_{m_2} \times \ldots C_{m_n}$. Our inductive hypothesis assumes that for all $k \leq n-1$, the Cayley digraph of $C_{m_1} \times \ldots \times C_{m_k}$ has a Hamiltonian cycle on the standard generators.

Remove the edges corresponding to the generator $h = (0, 0, \ldots, 1)$ from the Cayley digraph of G, leaving disjoint isomorphic subgraphs. These subgraphs are each copies of the Cayley digraph of $C_{m_1} \times C_{m_2} \times \ldots C_{m_{n-1}}$ on the standard generators. By the inductive hypothesis, these each have a Hamiltonian cycle with q vertices, where $q = \prod_{i=1}^{n-1} m_i$. The subgraphs spanned by these Hamiltonian cycles are isomorphic to the Cayley digraph of C_q on the generator 1. Hence, the Cayley digraph of $C_q \times C_{m_n}$ is a subgraph of Γ, which contains all the vertices of Γ. Since this has a Hamiltonian cycle, so does Γ. □

Remark. This result is instructive and nontrivial because it certainly remains possible to find Abelian groups with digraphs of generating set M such that there is no Hamiltonian cycle. This example comes from Holztynski and Strube (1977):

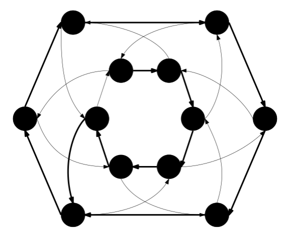

FIGURE 6. $C_2 \times C_6$ with generators $m_1 = (0,1)$ and $m_2 = (1,2)$. A Hamiltonian path is displayed, but it clearly cannot be completed into a cycle. In fact, it can be shown that no Hamiltonian cycle exists in this digraph.

5. Hamiltonian Paths in Cayley Digraphs of Dedekind Groups.

Suppose we loosen the condition, and consider the existence of Hamiltonian paths, rather than cycles, in Cayley digraphs. One could ask whether specific types of groups are guaranteed Hamiltonian paths in all of their Cayley digraphs. In fact, it can be shown that any Cayley digraph of an Abelian or Dedekind group has a Hamiltonian path. This following result is from Holsztynski and Nathanson (1977):

Definition 5.1 (Hamiltonian A-Path on G)**.** Let G be a group of finite order n, with minimal generating set A. A sequence S of elements in G, $[s_1, s_2 \ldots s_n]$, is called a *Hamiltonian A-path on G* if $s_1, s_2, \ldots s_n$ are elements of A, and if the partial products of sequential elements, $s_1, s_1 s_2, \ldots, \prod_{i=1}^{n} s_i = 1$, are all unique elements of G.

Note that if there exists a Hamiltonian A-path on G, then the Cayley digraph of G with generating set A has a Hamiltonian path, where the sequence S is obtained by reading labels of the edges in the path.

Definition 5.2 (Dedekind Group)**.** A group is *Dedekind* if each subgroup is normal.

Notice that every Abelian group is a Dedekind group.

Theorem 5.3. *For any minimal generating set A of a finite Dedekind group G, there exists a Hamiltonian A-path on G. Thus all Cayley digraphs of G have a Hamiltonian path.*

Proof. Let G be a finite Dedekind group, and suppose A is a generating set of G. We proceed by induction on the order of generating set A. Now we know that if $|A| = 1$, and a in A has order n, then G has elements $\{1, a, a^2, \ldots a^{n-1}\}$ and clearly has an A-path. Then suppose that for all A with $|A| < k$ there is an A-path in G. Let $|A| = k$, and take $B = A - a$, for some a in A.

Let the quotient group $H = G/\langle a \rangle$, where H is also Dedekind and $\langle a \rangle$ is the cyclic subgroup with elements $\{1, a, a^2, \ldots a^{n-1}\}$. Consider the image of B in H under the map that takes every b in B to its left coset $b\langle a \rangle$. Denoting this image B', note that it generates H, since $\langle a \rangle$ is a normal subgroup.

Since $|B'| < k$ and it is a generating set for H, we know by the inductive hypothesis that there is Hamiltonian B'-path in H.
Let this B'-path be $b_1, \ldots b_k$, where all b_i are left cosets of $\langle a \rangle$. We can pick distinct

representatives of these cosets $c_1, \dots c_k$ in B. Thus if a has order n, we have that $(a, a, \dots, a, c_1, a, \dots, a, c_2, a, \dots a, c_{k-1}, a, \dots, a, c_k, a, \dots a)$, where each $(a, a, \dots a)$ has $n-1$ elements, is a Hamiltonian A-path in G . The vertices on the Hamiltonian path are, in order, $a, a^2, \dots a^{n-1}, a^{n-1}c_1, a^{n-1}c_1a, \dots, \dots, a^{n-1}c_1a^{n-1} \cdots a^{n-1}c_ka^{n-1}$.
□

Importantly, this theorem pushes the problem of Hamiltonian Cayley digraphs into more interesting territory, because it immediately suggests that there might be whole classes of groups for which all Cayley digraphs have Hamiltonian paths or cycles.

6. Cayley Graphs of Finite Abelian Groups are Guaranteed a Hamiltonian Cycle.

Notice that it is much easier to find Hamiltonian cycles in Cayley graphs than in Cayley digraphs, because we allow ourselves to travel along edges of our graph Γ in either direction. While several have shown this independently, we present Marusic's proof (1982) that Cayley graphs of finite Abelian groups always have a Hamiltonian cycle. Marusic's proof is similar its approach to the proof of the previous theorem: it uses a similar sort of inductive step, and ends with an explicit construction of a Hamiltonian cycle given a generating set M.

Let G be a group. If M is a subset of the elements of G, then we let $M_0 = M - e$, where e is the identity element of G.

Let G be a finite group with elements $g_1, \dots g_n$, and some minimal generating set M. Let M^{-1} be the set $\{m_1^{-1}, m_2^{-1}, \dots m_n^{-1}\}$. Furthermore, let $M^* = M \cup M^{-1}$. Then a sequence S in G is called a *Hamiltonian M^*-cycle on G* if it has unique partial products $\prod_{i=1}^{j} s_i = g_j$, and a total of n elements such that $\prod_{i=1}^{n} s_i = g_1$, and $s_1, s_2, \dots, s_n$ are elements of M^*. In other words, any Hamiltonian M^*-cycle through G traces out a Hamiltonian cycle in the Cayley graph – *not digraph* – corresponding to group G with generating set M. If we moved through vertices in Γ corresponding to $s_1, s_1s_2, \dots, \prod_{i=1}^{n} s_i \in G$, we would hit every vertex in our graph exactly once before returning to the starting vertex.

Let $S = [s_1, s_2, \dots s_n]$ be a finite sequence. Then define $S^{-1} = [s_n^{-1}, s_{n-1}^{-1}, \dots, s_1^{-1}]$, and $\check{S} = [s_2, s_3, \dots s_{n-1}]$ if $n \geq 3$, and $\check{S}$ is the empty sequence with no terms if $n = 2$. Finally, for $n \geq 2$, let $\overline{S} = [s_1, s_2, \dots s_{n-1}]$, and $l_s = s_n$.

Let S and T be two finite sequences, with $S = [s_1, s_2, \dots, s_n]$, and $T = [t_1, t_2, \dots, t_i]$. Then ST denotes the sequence $[s_1, s_2, \dots s_n, t_1, t_2, \dots t_i]$. If $i \geq 3$ is odd, then $(S, T) = [t_1]\check{S}^{-1}[t_2]\check{S}[t_3]\check{S}^{-1} \dots [t_{i-1}]\check{S}$. If $i \geq 4$ is even, then (S, T) denotes the sequence $[t_1]\check{S}^{-1}[t_2]\check{S} \dots [t_{i-2}]\check{S}$. If $i \leq 2$ then (S, T) is the empty sequence.

Lemma 6.1. *Suppose G is a finite abelian group, with generating set M. If, for some nonempty subset of M_0, denoted M', we have a Hamiltonian $(M')^*$-cycle on $\langle M' \rangle$, denoted S, then there is some sequence Q in G such that $\overline{S}Q$ is a Hamiltonian M^*-cycle on G.*

Proof. We use induction on the cardinality of $M_0 \setminus M'$. If $M_0 \setminus M' = \emptyset$, then $\langle M' \rangle = \langle M_0 \rangle = G$ and take $Q = s_n$.

Now suppose there is some g in $M_0 \setminus M'$. Then let $H = \langle M_0 \setminus g \rangle$, and by inductive hypothesis, there is some Q in H such $\overline{S}Q$ is a Hamiltonian $(M_0 \setminus g)^*$-cycle on H. Let $W = \overline{S}Q$. Let j be the smallest integer such that g^j is in H. Then if j is odd, let

$$T = \overline{Q}(W, [g]^j) l_w [g^{-1}]^{j-1} \tag{1}$$

Then $\overline{S}T$ is a Hamiltonian M^*-cycle on G. We can see this by noting, as in Theorem 4.3, that removal of a nonredundant generator g yields disjoint isomorphic subgraphs. Similar to Theorem 4.4, we move through copies of $H, gH, \ldots g^{j-1}H$ and then backtrack by way of g^{-1}. If j is even, let

$$T = \overline{Q}(W, [g]^j)[g](\overline{W})^{-1}[g^{-1}]^{j-1}. \tag{2}$$

Then $\overline{S}T$ is a Hamiltonian M^*-cycle on G. □

Corollary 6.2. *Every Cayley graph of a finite Abelian group G such that $|G| \geq 3$ has a Hamiltonian cycle.*

Proof. To apply the Lemma, we simply need to find a subset M of any generating set such that some S is a Hamiltonian $(M')^*$-cycle through $\langle M' \rangle$.

Suppose $|G| \geq 3$ has generating set M. Then if M has an element c with $|c| = n \geq 3$, let our subset $M' = \{c\}$. Then $\langle M' \rangle = \langle c \rangle$. Then $\langle M' \rangle = \langle c \rangle = \{1, c, c^2, \ldots c^{n-1}\}$, and $S = [c]^n$. If no such c exists, M has two distinct elements x, y that each have order 2. (Otherwise, $|G| \leq 3$.) So let $M' = \{x, y\}$, and $S = [xyxy]$ is a Hamiltonian M'-sequence on $\langle M' \rangle$. □

7. Conclusion; Further Applications and Research.

Concerning Cayley digraphs, it has been shown (Witte 1985) that every Cayley digraph of a group of prime order is Hamiltonian. More recently, Alspach, Chen, and Dean (2010) have proven that every Cayley graph of a dihedral group D_{2n} with n even has a Hamiltonian cycle.

Of course, perhaps the most famous application and conjecture about Cayley graphs is due to Lovasz, who conjectured in 1969 that every connected, vertex transitive graph (other than four exceptions) has a Hamiltonian cycle. By vertex transitive, we mean that for any two vertices v and v' in a graph, we can find an isomorphism $\phi(v) = v'$. This conjecture is heavily disputed: Babai states that in his view, such conjectures are evidence only of our poor understanding of Hamiltonicity obstacles.

Notably, every one of the four graphs for which the Lovasz conjecture is known to fail is not a Cayley graph; moreover, every Cayley graph is vertex-transitive by Frucht's theorem. Rapaport-Strasser was among the first who asked if every Cayley graph has a Hamiltonian cycle. This is an open question.

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