THE NECKLACE-SPLITTING PROBLEM

KAROLINA SIMCIC

Abstract. The general setting of the necklace-splitting problem is where two thieves steal a necklace with n types of beads and want to use as few cuts as possible to split their loot evenly. This paper will serve to explain the necklace-splitting problem, and show how the Borsuk-Ulam theorem provides a solution; we will also prove this theorem in dimensions up to to $n = 2$ using the notion of fundamental groups.

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1. Necklace-Splitting

Imagine a scenario in which two thieves steal a necklace and want to split it between them evenly, but want to waste the least amount of material in cutting the necklace. The logical solution is to find the smallest amount of cuts needed to split the necklace, but one problem complicates the situation: the necklace has n types of beads, and these beads are worth different amounts of money. Assuming that there is an even amount of gems so that they can be split fairly between the two thieves, the question then becomes: "Is there a way to calculate how many cuts are required for a necklace to be split fairly between the two?"

To transform the necklace puzzle into a mathematical problem, we need a function to somehow keep track of the locations where the necklace is being split and output the amounts of each type of gem secured by each thief. Let us denote the unknown amount of cuts needed as m for now.

It turns out that the unit m-sphere is the perfect way to encode all the information we need to track m places where the necklace is split, leaving $m+1$ fragments: $m + 1$ inputs x_i^2 can track individual necklace fragment lengths, whose sum represents the whole necklace. Moreover, there are positive and negative roots for each x_i^2, allowing the sign of each x_i to represent which of the two thieves receives the corresponding fragment. So the point

$$\left(\tfrac{1}{5}, -\tfrac{2}{5}, \sqrt{\tfrac{4}{5}}\right) \in S^2$$

means that the necklace has been split twice, resulting in three segments with lengths $\frac{1}{25}$, $\frac{4}{25}$, and $\frac{4}{5}$ respectively, where thief 1 receives the first and third pieces and thief 2 receives the second.

Once we split the necklace—choose a coordinate on S^m— we would like to record how much of each gem thief 1 receives from the arrangement; since we have n gem types, this corresponds to a particular vector in $\mathbb{R}^n$. This comes in handy later because simply flipping the signs of the coordinate in S^m gives us how much loot thief 2 receives under the same circumstance. Comparing antipodes thus compares how equally the gems are split between the two thieves. To summarize, thinking of the necklace with n types of beads and m splits as a map from $S^m \to \mathbb{R}^n$ represents our conundrum in a rigorous manner.

Note that here we think of each gem not as a discrete point, but as a region that we can cut how we'd like. This allows the necklace to be split between the two thieves anywhere, allowing us to regard the "necklace" function as a continuous map from the m-sphere, despite the discrete nature of the gems. Assuming the gems are regions rather than discrete objects seems like cheating, but if we prove that a continuous solution exists, we have simultaneously proven that a discrete solution does as well. This is because we think of all gems as regions of the same length, so cutting one gem in the middle cuts the others in the same way—we can always shift to the right or left and result in a solution with all gems intact.

With all that taken care of, we invoke the Borsuk-Ulam theorem:

Theorem 1.1 (Borsuk-Ulam)**.** *Given a continuous map $f : S^n \to \mathbb{R}^n$, there exists a point $x \in S^n$ such that $f(x) = f(-x)$.*

Thus, *if we choose the number of cuts m to be equal to the number of gems n*, and there is an even number of each type of gem, *there must be some way to split the loot evenly between the thieves.* Therefore we have our answer!

But what makes this magical Borsuk-Ulam theorem true in the first place? This paper presents proofs of the Borsuk-Ulam theorem up to the $n = 2$ case, using elementary topology and the notion of covering spaces, because $n > 2$ requires the more advanced machinery of homology theory. For brevity, we assume that the reader has basic knowledge of loops, fundamental groups, and basic covering space theory in algebraic topology [1] from this point on.

2. Borsuk-Ulam Theorem in Dimensions Zero and One

The Borsuk-Ulam theorem is a mere tautology about the map from $S^0 = \{-1, 1\}$ to $\mathbb{R}^0 = \{0\}$. This equates to the trivial situation where the necklace-splitting problem requires no cutting when the thieves didn't even steal a necklace!

Though still somewhat trivial, the 1-dimensional case of the Borsuk-Ulam theorem is more concrete and equates to the following physical situation: Imagine that two people, a and b, are stationed on the equator diametrically opposite from one another. They journey to each other's original position while staying antipodal, taking measurements of the local temperatures as they go.

Without loss of generality, assume that the temperature at a's position is greater than or equal to the temperature at b's position at the beginning of the journey. Since temperature varies continuously, we know that some open region around a's starting point is warmer than some open region of b's; at the end, the positions of a and b are switched, so now a is colder than b and was for some open neighborhood around the ending positions. Hence, at some point in the journey, a and b must've experienced the same temperature while being antipodally opposed to one another.

Using the Intermediate Value Theorem, we can now transform the above explanation into a rigorous proof:

Proof. Let $f : S^1 \to \mathbb{R}$ be a continuous function, and define an auxiliary function that measures the difference between antipodal points, $g : S^1 \to \mathbb{R}$, by the formula

$$g(x) = f(x) - f(-x).$$

Notice that $g(x)$ is odd, and moreover $g(x)$ is continuous because $f(x)$ is. Let y be an arbitrary, fixed point on S^1. If $g(y) = 0$, then $f(y) = f(-y)$ and we are done. If $g(y) \neq 0$, then without loss of generality, assume $g(y) > 0$ and therefore $g(-y) < 0$. Then by the Intermediate Value Theorem, we know that there exists a $z \in S^1$ between y and $-y$ where $g(z) = 0$, as desired. $\square$

If the hypothetical necklace has one type of bead and an even number of them, one cut is sufficient to divide the loot fairly. This is obvious from an intuitive standpoint, but the solution is much more complicated for higher dimensional proofs of the Borsuk-Ulam theorem. Now, onto the 2-dimensional case!

3. Borsuk-Ulam Theorem and the Fundamental Group

While much more complicated than the previous cases, the $n = 2$ case of the Borsuk-Ulam theorem and the subsequent conclusion to the necklace-splitting problem allow us deeper insight into the topological approach one takes to solve the $n > 2$ cases. In the $n = 2$ case, we start by assuming the Borsuk-Ulam Theorem is false. This allows us to calculate the homotopy class of a particular loop $[0, 1] \to S^1$ in two different ways, from which we will derive a contradiction.

Proof. Let $f : S^2 \to \mathbb{R}^2$ be a continuous function. Assume that there exists no point x such that $f(x) = f(-x)$. This allows us to define the function g:

$$g : S^2 \to S^1 \quad g(x) = \frac{f(x) - f(-x)}{|f(x) - f(-x)|}.$$

Obviously g is continuous. Without loss of generality, fix the point $* = (1, 0, 0) \in S^2$ and suppose that $g(*) = 1$ (to simplify the fundamental group argument later).

Consider the loop

$$j : [0, 1] \to S^2 \quad j \mapsto (\cos(2\pi s), \sin(2\pi s), 0)$$

which wraps the unit interval around the $z = 0$ equator of S^2. We can compose $\alpha := g \circ j$ to get a map from $[0, 1] \to S^1$, which is a loop in S^1 based at $1 \in S^1$. This allows us to compute the homotopy class of α in two different ways.

First, we know that, by definition, $[\alpha]$ is the image of the class $[j]$ under the induced homomorphism $g_* : \pi_1(S^2, *) \to \pi_1(S^1, 1)$. But because $\pi_1(S^2, *)$ is trivial, we know that $[j]$ is the trivial class and hence $[\alpha]$ must be 0.

On the other hand, we can compute $[\alpha]$ by the covering map:

$$p : \mathbb{R} \to S^1 \qquad s \mapsto \cos(2\pi s) + i \sin(2\pi s).$$

Note, because $g(x)$ is odd, we see that $\alpha(s + \frac{1}{2}) = -\alpha(s)$ by direct calculation:

$$\alpha(s + \tfrac{1}{2}) = g(j(s + \tfrac{1}{2})) = g((-\cos(2\pi s), -\sin(2\pi s), 0)) = g(-j(s)) = -\alpha(s).$$

After lifting α to a path $\tilde{\alpha} : [0,1] \to \mathbb{R}$ based at 0, we see that this property becomes

$$\tilde{\alpha}(s + \tfrac{1}{2}) = \tilde{\alpha}(s) + \tfrac{q}{2}$$

for some odd integer q, because $\cos(2\pi s) + i\sin(2\pi s) = -1$ only when s is a half-integer. By evaluating this equation at $s = 0$ and $s = \frac{1}{2}$, we conclude that

$$\tilde{\alpha}(1) = \tilde{\alpha}(0) + q.$$

Hence this path is homotopic to the path $s \mapsto qs$ via the elementary homotopy

$$h_t(s) = (1-t) \cdot \tilde{\alpha}(s) + t \cdot (qs).$$

By composing with p, we conclude that $[\alpha] = q$.

This is a contradiction as we have computed that $[\alpha]$ is both zero and odd! $\square$

Since the Earth's surface is a 2-sphere (mostly), we can apply this theorem to two real and continuously-varying measurable quantities on the Earth's surface—for example, there must always be two antipodal points on the Earth's surface for which the temperature and pressure are the same.

As with the property in one dimension, now that the Borsuk-Ulam theorem has been proven in two dimensions, we can apply it to the necklace-splitting problem to conclude that if there are two types of gems on the necklace, the thieves can split it evenly with at most two cuts.

4. Conclusion

As the $n = 2$ proof was much more involved than the $n = 1$ case, the $n > 2$ case requires the reader to know homology theory and therefore is blackboxed. As demonstrated, the general theorem provides an elegant solution to the necklace-splitting problem under some restrictions. Since "antipodal" inherently refers to pairs of points, the Borsuk-Ulam style of proof did not at first extend to the case where there are more than two thieves. However recently, a more general version has extended the solution to odd prime numbers of thieves [2].

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References

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