UNCERTAINTY PRINCIPLES WITH FOURIER ANALYSIS

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Abstract. We describe the properties of the Fourier transform on $\mathbb{R}$ in preparation for some relevant physical interpretations. These include the Heisenberg Uncertainty Principle, Energy Conservation, and Dispersivity of the Schrödinger equation. Then, we will give an introduction to complex analysis with the goal of proving the Hardy Uncertainty Principle and giving some applications to current research in PDE's.

Contents

1. Fourier Transform

Let us first view $\mathbb{R}$ as a group under addition and try to find the group homomorphisms between $\mathbb{R}$ and $S^1 \subseteq \mathbb{C}$, the unit circle, which we may view as a group under multiplication. We are looking for a continuous function $\phi : \mathbb{R} \to S^1$ such that $\phi(x+y) = \phi(x)\phi(y)$ for $x, y \in \mathbb{R}$. We can show that all such functions take the form

$$\phi(x) = \exp\left(\pm 2\pi i x\xi\right), \tag{1.1}$$

for $\xi \in \mathbb{R}$ [3]. We now wish to see how ordinary functions decompose into superpositions of these functions ϕ. To do so, we introduce the concept of the Fourier Transform.

Definition 1.1. Given an $L^1(\mathbb{R})$ function f, we may define two operators as follows:

(1) $^\sharp : f(x) \mapsto f^\sharp(\xi) = \int_{\mathbb{R}} f(x) \exp\left(-2\pi i x\xi\right) dx$,
(2) $^\flat : f(\xi) \mapsto f^\flat(x) = \int_{\mathbb{R}} f(\xi) \exp\left(2\pi i x\xi\right) d\xi$.

We say that $f^\sharp$ is the *Fourier Transform of f*, and for now, all we know is that $^\flat$ is another operator. However, those with a music theory background may conjecture that $\left(f^\flat\right)^\sharp = \left(f^\sharp\right)^\flat = f$, and they would be correct under some circumstances. Giving the formal argument for this is the goal of this section.

Here are some results that help in understanding Fourier Transforms.

Proposition 1.2. *Let $f \in L^1(\mathbb{R})$. Then, $f^\sharp$ is bounded and continuous.*

Proof. To show that $f^\sharp$ is bounded, observe that

$$\left|f^\sharp(\xi)\right| = \left|\int_{\mathbb{R}} f(x)\exp\left(-2\pi i x\xi\right)dx\right| \leq \int_{\mathbb{R}} \left|f(x)\exp\left(-2\pi i x\xi\right)\right|dx = \int_{\mathbb{R}} |f(x)| = \|f\|_{L^2}. \tag{1.2}$$

To show that $f^\sharp$ is continuous, consider $f^\sharp(\xi+\eta) = \int_{\mathbb{R}} f(x)\exp\left(-2\pi i x(\xi+\eta)\right)dx$. The integrand is bounded above by $\|f\|_{L^1}$, so we use the Dominated Convergence Theorem to show

$$\lim_{\eta\to 0} f^\sharp(\xi+\eta) = \lim_{\eta\to 0}\int_{\mathbb{R}} f(x)\exp\left(-2\pi i x(\xi+\eta)\right)dx = \int_{\mathbb{R}} f(x)\exp\left(-2\pi i x\xi\right)dx = f^\sharp(\xi). \tag{1.3}$$

□

Before we continue, let $C_0^k(\mathbb{R})$ be the set of C^k functions with compact support.

Proposition 1.3. *Let $f \in L^1(\mathbb{R})$.*

(1) *If $x^\alpha f(x) \in L^1(\mathbb{R})$ for $0 \leq \alpha \leq k$, then $f^\sharp \in C^k(\mathbb{R})$ and*

$$\frac{d^\alpha}{d\xi^\alpha} f^\sharp(\xi) = \left(\left(-2\pi i x\right)^\alpha f\right)^\sharp. \tag{1.4}$$

(2) *Assume $f \in C_0^k(\mathbb{R})$. Then, for $0 \leq \alpha \leq k$, we have*

$$\left(\frac{d^\alpha}{dx^\alpha} f\right)^\sharp(\xi) = \left(2\pi i\xi\right)^\alpha f^\sharp(\xi). \tag{1.5}$$

This proposition shows two things, namely the duality between integrability and regularity of f and $f^\sharp$ and the fact that, up to constant multiplicative factors, the Fourier transform exchanges multiplication by x, ξ and differentiation with respect to ξ, x respectively. For proofs, see Theorem 8.22 in [3].

We now give a result to show the general behavior of the Fourier transform.

Proposition 1.4. *Let $f \in C_0^2(\mathbb{R})$. Then, $\left|f^\sharp(\xi)\right| \leq \frac{A}{1+\xi^2}$ for some $A \in \mathbb{R}$.*

Proof. Observe that

$$\left(1 - \frac{d^2}{4\pi^2 dx^2}\right)\exp\left(-2\pi i x\xi\right) = 1 + \xi^2\exp\left(-2\pi i x\xi\right). \tag{1.6}$$

Then, we have that

$$f^\sharp(\xi) = \int_{\mathbb{R}} f(x)\exp\left(-2\pi i x\xi\right)dx = \int_{\mathbb{R}}\left(1 - \frac{d^2}{4\pi^2 dx^2}\right) f(x)\exp\left(-2\pi i x\xi\right)dx$$

$$= \frac{1}{1+\xi^2}\int_{\mathbb{R}} f(x)\left(1 - \frac{d^2}{4\pi^2 dx^2}\right)\exp\left(-2\pi i x\xi\right)dx. \tag{1.7}$$

This implies

$$\left|f^\sharp(\xi)\right| \leq \frac{1}{1+\xi^2}\left(\|f\|_{L^1} + \frac{1}{4\pi^2}\|f''\|_{L^2}\right), \tag{1.8}$$

which takes the desired form. □

These propositions motivate the definition of the Schwartz space as a collection of very nice functions where the properties of the Fourier Transform are easier to prove.

Definition 1.5. We define the *Schwartz space* $\mathbb{S}(\mathbb{R})$ as the set

$$\mathbb{S}(\mathbb{R}) = \left\{ f \in C^\infty(\mathbb{R}) \,|\, \forall k, l \geq 0, \sup_{x \in \mathbb{R}} \left|x^k\right| \left|f^{(l)}(x)\right| < \infty \right\}. \tag{1.9}$$

Such functions are said to be of *rapid decrease.*

Examples of Schwartz functions are $\exp\left(-\delta x^2\right)$ with $\delta > 0$ and any compactly supported function.

Proposition 1.6. *The Fourier Transform* $^\sharp$ *maps* $\mathbb{S}$ *into itself.*

Proof. Consider the function $H \in \mathbb{S}$ where

$$H(x) = \frac{1}{(2\pi i)^k} \frac{d^k}{dx^k} \left((-2\pi i x)^l f(x) \right). \tag{1.10}$$

By the results above, we deduce

$$H^\sharp(\xi) = \xi^k \frac{d^l}{d\xi^l} \left(f^\sharp(\xi) \right), \tag{1.11}$$

so $H \in \mathbb{S}$ implies that $H^\sharp$ satisfies the properties of the Schwartz space. □

Definition 1.7. Given two functions $f, g : \mathbb{R} \to \mathbb{R}$, we define the *convolution of* f *and* g as

$$(f \star g)(x) = \int_{\mathbb{R}} f(y) g(x-y) dy. \tag{1.12}$$

The following result from multivariable calculus will help us describe properties of the convolution and the Schwartz space.

Lemma 1.8 (Fubini)**.** *Let* $F(x,y) \in \mathbb{S}(\mathbb{R}^2)$*. Then, letting* $F_1(x) = \int_{\mathbb{R}} F(x,y)dy$ *and* $F_2(y) = \int_{\mathbb{R}} F(x,y)dx$*, we deduce that:*

$$\int_{\mathbb{R}} F_1(x) dx = \int_{\mathbb{R}} F_2(y) dy. \tag{1.13}$$

Remark 1.9. For $n \geq 1$, we may define $\mathbb{S}(\mathbb{R}^n)$ according to Definition 1.5, substituting $\|x\|$ for x and partial derivatives for all single variable derivatives.

The previous lemma allows us to show the following result.

Proposition 1.10. *If* $f, g \in \mathbb{S}(\mathbb{R})$*, then:*

(1) $f \star g \in \mathbb{S}(\mathbb{R})$.
(2) $f \star g = g \star f$.
(3) $\frac{d}{dx^k}(f \star g(x)) = f \star \left(\frac{d}{dx^k} g(x)\right)$
(4) $(f \star g)^\sharp = f^\sharp g^\sharp$.

In this paper, we shall only prove (4).

Proof. Let $f, g \in \mathbb{S}(\mathbb{R})$. Then, consider the function $F(x,y) := f(y)g(x-y)\exp(-2\pi i x \xi)$. Then, we may define $F_1(x) = (f \star g)(x)\exp(-2\pi i x \xi)$ and $F_2(y) = f(y) g^\sharp(\xi) \exp(-2\pi i y \xi)$. Then, by Lemma 1.8,

$$(f \star g)^\sharp = \int_{\mathbb{R}} F_1(x) dx = \int_{\mathbb{R}} F_2(y) dy = f^\sharp g^\sharp, \tag{1.14}$$

as desired. □

Additionally, we may check that $f \star g$ is associative and distributive, so that turns $\mathbb{S}(\mathbb{R})$ into an algebra with addition and scalar multiplication defined the usual way and multiplication as convolution. Now, back to sharps and flats.

Theorem 1.11 (Multiplication Theorem)**.** *For $f, g \in \mathbb{S}$, we have that*

$$\int_{\mathbb{R}} f^{\sharp}(x)g(x)dx = \int_{\mathbb{R}} f(x)g^{\sharp}(x)dx. \tag{1.15}$$

Proof. Consider the function $F(x, y) = f(x)g(y)\exp(-2\pi ixy)$. Using the conventions of Lemma 1.8, it is immediate that $F_1(x) = f(x)g^{\sharp}(x)$ and $F_2(y) = f^{\sharp}(y)g(y)$. The conclusion then follows from the lemma. □

Before we go on, we need to calculate an important Fourier transform.

Lemma 1.12. *For $\delta > 0$, we have*

$$\left(\exp\left(-\pi\delta x^2\right)\right)^{\sharp}(\xi) = \frac{1}{\delta^{1/2}} \exp\left(\frac{-\pi\xi^2}{\delta}\right). \tag{1.16}$$

This computation relies on contour integration from complex analysis. For details of the proof, see [3]. Before we proceed, let us note two important things. The first is that there is a duality in the exponents of the Gaussians. The other is that the integral of the Gaussians on the right hand side is always 1, and these concentrate at 0 as $\delta \to 0$. We will use this fact in the following result.

Theorem 1.13 (Inversion Formula)**.** *For $f \in \mathbb{S}(\mathbb{R})$, we have*

$$f(x) = \int_{\mathbb{R}} f^{\sharp}(\xi)\exp(2\pi ix\xi)\, d\xi. \tag{1.17}$$

This gives $f = \left(f^{\sharp}\right)^{\flat}$. We also have $f = \left(f^{\flat}\right)^{\sharp}$, and the map $^{\sharp} : \mathbb{S} \to \mathbb{S}$ is a bijection from the set $\mathbb{S}$ to itself.

Proof. To begin, let $f \in \mathbb{S}(\mathbb{R})$ and we claim that $f(0) = \int_{\mathbb{R}} f^{\sharp}(\xi)d\xi$. To prove the claim, define (Gaussian) functions $B, B^{\sharp}$ where $B = \exp\left(-\pi\delta x^2\right)$ and $B^{\sharp}$ is the Fourier transform of B. Then, by the multiplication theorem, we may write:

$$\int_{\mathbb{R}} f^{\sharp}(x)B(x)dx = \int_{\mathbb{R}} f(x)B^{\sharp}(x)dx. \tag{1.18}$$

As $\delta \to 0$, $LHS \to \int_{\mathbb{R}} f^{\sharp}(\xi)d\xi$ by the Dominated Convergence Theorem. Also, $RHS \to f(0)$ because the Gaussian function becomes more concentrated at 0. This proves the claim.

Then, we construct the function $F(y) = f(x + y)$, which is a member of $\mathbb{S}(\mathbb{R})$ for fixed x. We thus have

$$f(x) = F(0) = \int_{\mathbb{R}} F^{\sharp}(\xi)d\xi = \int_{\mathbb{R}} f^{\sharp}(\xi)\exp(2\pi ix\xi)\, d\xi = \left(f^{\sharp}\right)^{\flat}. \tag{1.19}$$

These equalities follow from the claim, the definition of the Fourier Transform, and a change of variables.

If we note that $f^{\flat}(\xi) = f^{\sharp}(-\xi)$, it is immediate that $f = \left(f^{\flat}\right)^{\sharp}$ by a change of variables. To show that the Fourier Transform is a bijection, recall that $(\mathbb{S})^{\sharp} \subseteq \mathbb{S}$, and that $^{\sharp}$ has an inverse, namely $^{\flat}$. □

2. Plancherel's Formula

We will now give a simple but extremely important result due to Plancherel that will be useful for problems involving dispersivity as well as anything involving L^2 norms.

The feeling behind this result is that under the Fourier transformation, the resulting function is similar to the original function in the sense that *all of the information* was collected through the homomorphisms.

If we consider a function ρ that gives mass density, Plancherel's Theorem states that the total mass of ρ is the total mass of $\rho^\sharp$. Thus, we say that the Fourier Transform is an *isometry* from $\mathbb{S}(\mathbb{R}) \subseteq L^2(\mathbb{R})$ to itself. In this section, we will also abbreviate $\|f\|_{L^2}$ as $\|f\|_2$.

Theorem 2.1 (Plancherel's Theorem)**.** *If $f \in \mathbb{S}(\mathbb{R})$, then the L^2-norm is preserved under Fourier transform, i.e. $\|f\|_2 = \|f^\sharp\|_2$.*

Proof. Let $f \in \mathbb{S}(\mathbb{R})$. Then, letting $\overline{z}$ denote the complex conjugate of z, define $g(x) = \overline{f(-x)}$, so that $g^\sharp(x) = \overline{f^\sharp(\xi)}$. Defining $h = f \star g$, we readily see that $h^\sharp(\xi) = \left|f^\sharp(\xi)\right|^2$ and moreover that $h(0) = \int_{\mathbb{R}} |f(x)|^2 \, dx$, which is the square of the L^2 norm of f. Then, we use Theorem 1.13 to write

$$\int_{\mathbb{R}} |f(x)|^2 \, dx = h(0) = \int_{\mathbb{R}} h^\sharp(\xi) d\xi = \int_{\mathbb{R}} \left|f^\sharp(\xi)\right|^2 d\xi, \tag{2.1}$$

as we wanted to show. □

We will now carry out a calculation using Plancherel's Formula, showing that the Schrödinger equation is dispersive. The idea of dispersive equations is that the solutions retain a constant norm over time, so as time progresses, the solutions become less and less concentrated in order to maintain the norm. This norm has many physical interpretations, but the one we will use in the next calculation is one of energy, so Plancherel's Theorem will highlight the conservation of energy.

Consider the Schrodinger Equation below:

$$i\frac{\partial u}{\partial t} + \frac{\partial^2 u}{\partial x^2} = 0. \tag{2.2}$$

Now, we take the Fourier transform in x to obtain

$$i\frac{\partial u^\sharp}{\partial t} + (2\pi i \xi)^2 u^\sharp = 0, \tag{2.3}$$

where we have used that $\left(\frac{\partial u}{\partial t}\right)^\sharp = \frac{\partial (u^\sharp)}{\partial t}$. This implies

$$u^\sharp(\xi, t) = C \exp\left(-4\pi^2 \xi^2 it\right), \tag{2.4}$$

for some C. Plugging in $t = 0$, we see that $C = u^\sharp(\xi, 0) =: u_0^\sharp(\xi)$. Then, we use Plancherel's Theorem to state that

$$\|u(x,t)\|_2 = \left\|u^\sharp(\xi,t)\right\|_2 = \left\|u_0^\sharp(\xi) \exp\left(-4\pi^2\xi^2 it\right)\right\|_2 = \left\|u_0^\sharp(\xi)\right\|_2 = \|u_0(x)\|_2, \tag{2.5}$$

meaning that u is dispersive as desired.

3. Heisenberg Principle

In this section, we will prove Heisenberg's Uncertainty Principle using the Fourier Transform in two ways, and apply the second method to produce the usual quantum-mechanical statement of Heisenberg.

In the first method, we will provide two equivalent results, allowing us to consider the displacement about a point (x_0, ξ_0) in phase space instead of the origin, giving us more freedom to fix a physical origin.

Theorem 3.1 (Heisenberg's Uncertainty Principle)**.** *Let $\psi \in \mathbb{S}(\mathbb{R})$ with the normalization condition $\int_{\mathbb{R}} |\psi|^2 \, dx = 1$. We then have:*

$$\left(\int_{\mathbb{R}} x^2 |\psi(x)|^2 \, dx\right)\left(\int_{\mathbb{R}} \xi^2 \left|\psi^\sharp(\xi)\right|^2 d\xi\right) \geq \frac{1}{16\pi^2}. \tag{3.1}$$

We may also equivalently write this result as:

$$\left(\int_{\mathbb{R}} (x-x_0)^2 |\psi(x)|^2 \, dx\right) \left(\int_{\mathbb{R}} (\xi-\xi_0)^2 \left|\psi^\sharp(\xi)\right|^2 d\xi\right) \geq \frac{1}{16\pi^2}, \tag{3.2}$$

for all $x_0, \xi_0 \in \mathbb{R}$*. Equality holds iff* ψ *is of the form* $\psi(x) = A\exp\left(-Bx^2\right)$ *with* $B > 0$ *and* $|A|^2 = \sqrt{2B/\pi}$ *.*

Proof. We begin by proving the equivalence of the results in two steps. Given $\psi \in \mathbb{S}(\mathbb{R})$, define $\Phi(x) := \psi(x+x_0) \in \mathbb{S}(\mathbb{R})$ with the normalization condition $\int |\Phi|^2 = \int |\psi|^2 = 1$. Then, we have that

$$\Phi^\sharp(\xi) = \int \exp\left(-2\pi i x\xi\right) \Phi(x) dx = \exp\left(2\pi i x_0 \xi\right) \psi^\sharp(\xi). \tag{3.3}$$

This directly implies

$$\int (x-x_0)^2 |\psi(x)|^2 \, dx = \int x^2 |\Phi(x)|^2 \, dx \ , \ \int (\xi-\xi_0)^2 \left|\psi^\sharp(\xi)\right|^2 d\xi = \int (\xi-\xi_0)^2 \left|\Phi^\sharp(\xi)\right|^2 d\xi. \tag{3.4}$$

Similarly, letting $\varphi(x) := \exp\left(2\pi i x \xi_0\right) \Phi(x) \in \mathbb{S}$, we observe that $\int |\varphi|^2 = \int |\Phi|^2 = 1$. As well, we have

$$\varphi^\sharp(\xi) = \int \exp\left(-2\pi i x\xi\right) \varphi(x) dx = \Phi^\sharp(\xi-\xi_0). \tag{3.5}$$

This gives

$$\int (x-x_0)^2 |\psi(x)|^2 \, dx = \int x^2 |\Phi(x)|^2 \, dx = \int x^2 |\varphi(x)|^2 \, dx, \tag{3.6}$$

$$\int (\xi-\xi_0)^2 \left|\psi^\sharp(\xi)\right|^2 d\xi = \int (\xi-\xi_0)^2 \left|\Phi^\sharp(\xi)\right|^2 d\xi = \int \xi^2 \left|\varphi^\sharp(\xi)\right|^2 d\xi. \tag{3.7}$$

These two equations and the normalization condition allow us to reduce the general displacement case to that about the origin. This proves the equivalence of the results.

To prove the result, we begin from the normalization condition with

$$1 = \int_{\mathbb{R}} |\psi(x)|^2 \, dx = -\int_{\mathbb{R}} x \frac{d}{dx}\left(|\psi(x)|^2\right) dx \tag{3.8}$$

by integration by parts. Also, since $\psi(x)^2 = \psi(x)\overline{\psi(x)}$, so we may use the product rule to compute

$$\left|\frac{d}{dx}\left(|\psi(x)^2|\right)\right| = \left|\psi'(x)\overline{\psi(x)} + \psi(x)\overline{\psi'(x)}\right| \leq 2\left|\psi'(x)\right| |\psi(x)| \tag{3.9}$$

Then, plugging this into the integral, we find

$$1 \leq \int_{\mathbb{R}} \left| x \frac{d}{dx}\left(|\psi(x)^2|\right)\right| dx \leq 2\int_{\mathbb{R}} |x| \left|\psi'(x)\right| |\psi(x)| \, dx \tag{3.10}$$

$$\leq 2\sqrt{\left(\int_{\mathbb{R}} x^2 |\psi(x)|^2 \, dx\right)\left(\int_{\mathbb{R}} \left|\psi'(x)\right|^2 dx\right)} \tag{3.11}$$

by Cauchy-Schwarz. By Proposition 1.3 and Plancherel's Formula, we claim that

$$\left(\int_{\mathbb{R}} \left|\psi'(x)\right| dx\right) = 4\pi^2 \int_{\mathbb{R}} \xi^2 \left|\psi^\sharp(\xi)\right|^2 d\xi, \tag{3.12}$$

finishing the proof of the inequality.

To force equality, we force the expressions to be equal at where we applied Cauchy-Schwarz. It is a well known fact that equality occurs in Cauchy-Schwarz when one vector is a scalar multiple of another, so we deduce that

$$|\psi'(x)| = K\,|x\psi(x)|\,. \tag{3.13}$$

After removing the absolute signs, which takes a little work, we deduce that

$$\psi(x) = Ae^{-Bx^2}, \tag{3.14}$$

where $B > 0$. If $B < 0$, then there is no way that the function would integrate to 1. Again using the normalization condition, we deduce that indeed, $|A|^2 = \sqrt{2B/\pi}$, so we are done. □

We now present an equivalent proof based on the noncommutativity of position and momentum operators.

Definition 3.2. For two operators P and Q, we define *the commutator of P and Q* as $[P,Q] := PQ - QP$.

Alternate Proof. Consider the operators $Pf = xf$ and $Qf = \frac{\partial f}{\partial x}$. Then, by the product rule, we can readily see that $[P,Q] = -I$.

Let ψ be such that $\int_{\mathbb{R}} |\psi|^2 = 1$. Then, for $t \in \mathbb{R}$, consider the inner product

$$0 \leq \langle (tP+Q)\psi, (tP+Q)\psi \rangle$$

$$= t^2 \|P\psi\|_2^2 + t\langle P\psi, Q\psi\rangle + t\langle Q\psi, P\psi\rangle + \|Q\psi\|_2^2\,. \tag{3.15}$$

For $f, g \in \mathbb{S}$, we may check that $\langle Pf, g\rangle = \langle f, Pg\rangle$. Integrating by parts, we have $\langle Qf, g\rangle = -\langle f, Qg\rangle$. Then, plugging this into the original formula, we obtain

$$t^2 \|P\psi\|_2^2 + t\langle \psi, [P,Q]\psi\rangle + \|Q\psi\|_2^2$$

$$= t^2 \|P\psi\|_2^2 - t\|\psi\|_2^2 + \|Q\psi\|_2^2\,. \tag{3.16}$$

This inner product is positive, so as a polynomial in t, it yields

$$\|\psi\|_2^4 - 4\left(\|P\psi\|_2^2 \|Q\psi\|_2^2\right) \leq 0, \tag{3.17}$$

and we use Proposition 1.3 again to write

$$(Q\psi)^\sharp = 2\pi i\xi\psi^\sharp(\xi) \tag{3.18}$$

and use Plancherel's Theorem to claim that the norms are equal. Then, using the normalization condition on ψ, we may write

$$1 \leq 16\pi^2 \int_{\mathbb{R}} x^2 |\psi|^2\,dx \int_{\mathbb{R}} \xi^2 \left|\psi^\sharp\right|^2 d\xi, \tag{3.19}$$

as desired. □

Additionally, let us redefine $P = \hat{x}$ and $Q = \hat{p}$. These operators act on a function ψ in such a way that $\hat{x}[\psi] = x\psi$ and $\hat{p}[\psi] = -i\hbar\frac{\partial}{\partial x}\psi$. These represent the position and momentum of the particle in said state respectively. A cool fact that should seem reasonable is that $\hat{x} = \hat{p}^\sharp$ and $\hat{x}^\sharp = \hat{p}$ as the Fourier transform interlaces derivatives and multiplication by x.

In any case, it can be shown that $[\hat{x},\hat{p}] = i\hbar$, so we can do a similar calculation as before. Starting with Equation 3.16, we write

$$t^2 \|\hat{x}[\psi]\|_2^2 + t\langle \psi, [\hat{x},\hat{p}][\psi]\rangle + \|\hat{p}[\psi]\|_2^2$$

$$= t^2 \|\hat{x}[\psi]\|_2^2 + i\hbar t \|\psi\|_2^2 + \|\hat{p}[\psi]\|_2^2 . \tag{3.20}$$

Then, we take the discriminant to be negative:

$$\hbar^2 \|\psi\|_2^4 - 4 \|\hat{x}[\psi]\|_2^2 \|\hat{p}[\psi]\|_2^2 \leq 0. \tag{3.21}$$

Thinking of $\|\hat{x}[\psi]\|_2$ as the standard deviation of the displacement, written Δx, and similarly for $\hat{p}$, we may write

$$\hbar^2 - 4 (\Delta x)^2 (\Delta p)^2 \leq 0, \tag{3.22}$$

or

$$\Delta x \Delta p \geq \frac{\hbar}{2}, \tag{3.23}$$

which is the usual quantum-mechanical way of seeing Heisenberg's uncertainty principle.

As we prepare to prove Hardy's Uncertainty Principle, we take a detour into complex analysis to establish some fundamental tools used in the proof.

4. Complex Analysis

To begin our discussion of complex analysis, we must first investigate differentiability of complex-valued functions and the consequences thereof.

Definition 4.1. A function $f : \mathbb{C} \to \mathbb{C}$ is said to be *holomorphic at* $\omega \in \mathbb{C}$ if

$$\lim_{z \to \omega} \frac{f(z) - f(\omega)}{z - \omega} \tag{4.1}$$

exists and is finite. In such cases, we denote this limit as $f'(\omega)$. We then say f is *holomorphic on* $\Omega \subseteq \mathbb{C}$ if f is holomorphic at all $\omega \in \Omega$. Finally, if f is holomorphic on $\mathbb{C}$, then f is said to be *entire*.

Definition 4.2. A function $f : \mathbb{C} \to \mathbb{C}$ is said to be *regular at* ω if for all $n \in \mathbb{N}$, $f^{(n)}$ is holomorphic at ω, where $f^{(n)}$ denotes the nth complex derivative of f. As before, we may define a notion of f being *regular on* $\Omega \subseteq \mathbb{C}$, or just *regular* when appropriate.

Definition 4.3. A function $f : \mathbb{C} \to \mathbb{C}$ is said to be *analytic at* ω if there exists a power series expansion

$$f(z) = \sum_{k=0}^{\infty} a_k (z - \omega)^k \tag{4.2}$$

around ω. We may also generalize this to a notion of what it means to be *analytic on* $\Omega \subseteq \mathbb{C}$.

We will now prove the equivalence of these definitions, however we will wait to prove one case until we have encountered the Cauchy Integral Formula.

Proposition 4.4. *Definitions 4.1, 4.2, and 4.3 are equivalent; i.e. if a function f satisfies one of the conditions, then it satisfies all of them.*

Proof. We will momentarily show that ($4.3 \implies 4.2$). Consider a power series expansion

$$f(z) = \sum_{k=0}^{\infty} a_k (z - \omega)^k. \tag{4.3}$$

To show that f is regular, we show that

$$\frac{f(z)-f(\omega)}{z-\omega} \to L \tag{4.4}$$

as $z \to \omega$ with $|L| < \infty$. We calculate

$$\frac{f(z)-f(\omega)}{z-\omega} = \frac{\left(\sum_{k=0}^{\infty} a_k (z-\omega)^k\right) - a_0}{z-\omega} = \sum_{k=1}^{\infty} a_k (z-\omega)^{k-1}, \tag{4.5}$$

which approaches a_1 as $z \to \omega$. We may repeat this proof starting at any natural number, so we are done. □

And now for the theorem that will propel the rest of this paper, the Cauchy Integral Formula.

Theorem 4.5 (Cauchy Integral Formula)**.** *If f is holomorphic on an open set containing a circle C and its interior, then for all ω inside C, we have*

$$f(\omega) = \frac{1}{2\pi i} \oint_C \frac{f(\xi)}{\xi - \omega} d\xi. \tag{4.6}$$

Proof. See [7], Chapter 2, Theorem 4.1. □

Corollary 4.6. *A holomorphic function is also analytic.*

Proof. By Theorem 4.5, fixing z_0 as the center of the circle C, we may write

$$f(\omega) = \frac{1}{2\pi i} \oint_C \frac{f(\xi)}{\xi - \omega} d\xi = \frac{1}{2\pi i} \oint_C \frac{f(\xi)}{(\xi - z_0) - (\omega - z_0)} d\xi. \tag{4.7}$$

Considering the integrand, we may write:

$$\frac{1}{(\xi - z_0) - (\omega - z_0)} = \frac{1}{(\xi - z_0)} \frac{1}{1 - \left(\frac{\omega - z_0}{\xi - z_0}\right)}, \tag{4.8}$$

and we exploit the fact that the second term represents a geometric series with ratio $|(\omega - z_0)/(\xi - z_0)| < 1$. Additionally, note that the series converges uniformly, so we may commute the integral and the series, arriving at the expression:

$$f(\omega) = \sum_{r=0}^{\infty} \left(\frac{1}{2\pi i} \oint_C \frac{f(\xi)}{(\xi - z_0)^{r+1}} d\xi \right) (\omega - z_0)^r, \tag{4.9}$$

which is the desired result. In fact, this is a Taylor expansion about z_0 with

$$f^{(n)}(z_0) = \frac{n!}{2\pi i} \oint_C \frac{f(\xi)}{(\xi - z_0)^{n+1}} d\xi. \tag{4.10}$$

□

This concludes the proof of Proposition 4.4 as the remaining implication is trivial. We now present another corollary that will be essential in determining the nature of $f^\sharp$ given reasonable decay conditions on f.

Corollary 4.7 (Liouville's Theorem)**.** *If a function f is bounded and entire, then f is constant.*

Proof. We will show that $f' = 0$ everywhere, which implies that f is constant as $\mathbb{C}$ is a connected set. For the proof of the sufficiency of this condition, see Corollary 3.4 in Chapter 1 of [7].

Fix $z \in \mathbb{C}$ and let D be a disk with boundary $\partial D = C$ and radius R centered at z. Plugging in $n = 1$ to Equation 4.10, we may write

$$f'(z) = \frac{1}{2\pi i} \oint_C \frac{f(\xi)}{(\xi - z)^2} d\xi. \tag{4.11}$$

We then observe that

$$|f'(z)| = \frac{1}{2\pi} \left| \oint_C \frac{f(\xi)}{(\xi - z)^2} d\xi \right| \leq \frac{f_C}{R}, \tag{4.12}$$

where f_C denotes $\sup_{z \in C} |f(z)|$. Taking the limit as $R \to \infty$ proves the result. □

Theorem 4.8 (Maximum Modulus Principle)**.** *Let f be a holomorphic, non-constant function on an open set $\Omega \subseteq \mathbb{C}$. Then, $|f|$ does not attain its maximum value on Ω.*

This theorem admits the following corollary.

Corollary 4.9. *Let f be a holomorphic function on a bounded open set $\Omega \subseteq \mathbb{C}$. If f is continuous up to $\overline{\Omega}$, then* $\sup_{\Omega} |f| = \sup_{\partial\Omega} |f|$.

To prove the Maximum Modulus Principle, we will rely on the following fundamental lemma. For the proof of this lemma, see Theorem 4.4 in [7], Chapter 3.

Lemma 4.10 (Open Mapping Theorem)**.** *If f is holomorphic and non-constant on a region Ω, then f maps open sets to open sets.*

Proof of Theorem 4.8. Let $\Omega \subseteq \mathbb{C}$ be open. Suppose $|f|$ does attain a maximum value at $\omega \in \Omega$, and without loss of generality, assume that $0 < f(\omega) \in \mathbb{R}$. Then, there is an open neighborhood $D \subseteq \Omega$ around ω so that $f(D) \subseteq \mathbb{C}$ is an open neighborhood of $f(\omega)$. In particular, there is a $z \in D$ such that $f(z)$ is real and $f(z) > f(\omega)$. However, this contradicts the maximality of $|f(\omega)|$. □

We now transition from observing the behavior of nice functions on a bounded region to using simple geometry to study the behavior of functions of a specified growth on unbounded regions. Patching together use of the following theorem for multiple intersecting sectors can yield results about the entire complex plane; we will use this later in the proof of Hardy's Uncertainty Principle.

Theorem 4.11 (Phragmén-Lindelöf)**.** *Fix $\alpha > 1/2$ and consider a sector of the complex plane S_α with angle π/α. Let f be holomorphic on S_α and continuous on its closure. If there exist $\beta, \tau, C, M > 0$ with $\beta < \alpha$, $|f(z)| \leq C \exp\left(\tau |z|^\beta\right)$ for $z \in S_\alpha$, and $|f(z)| \leq M$ on ∂S_α, then $|f(z)| \leq M$ for all $z \in S_\alpha$.*

Proof. Let f be any such function, and suppose that the sector is given by $S_\alpha = \{z \in \mathbb{C} \mid |\arg z| < \pi/2\alpha\}$. Then, choose $\gamma \in (\beta, \alpha)$ and define $h(z) = f(z) \exp(-z^\gamma)$. Note that since z^γ is continuous, h is holomorphic in the interior and continuous on the closure of the sector. Let us use a polar coordinate representation and write $z = re^{i\theta}$ so that $z^\gamma = r^\gamma e^{i\theta\gamma}$, where $|\theta| < \pi/2\alpha$. Using the growth of f, we deduce

$$|h(z)| \leq |f(z)| |\exp(-z^\gamma)| \leq C \exp\left(\tau r^\beta\right) \exp\left(-r^\gamma \cos(\theta\gamma)\right) \leq C \exp\left(\tau r^\beta - r^\gamma \cos(\pi\gamma/2\alpha)\right). \tag{4.13}$$

Because $\gamma \in (\beta, \alpha)$, $|h(z)|$ approaches 0 as $r \to \infty$. On the boundary, we write

$$|h(z)| \leq M \left|\exp\left(-r^\gamma e^{i\theta\gamma}\right)\right| = M \exp\left(-r^\gamma \cos(\pi\gamma/2\alpha)\right) \leq M, \tag{4.14}$$

using the fact that $\gamma < \alpha$. We apply Corollary 4.9 to increasing bounded sectors, giving that $|h| \leq M$ in the unbounded sector. To conclude the proof, we let $h_\epsilon(z) = f(z)\exp(-\epsilon z^\gamma)$, repeat the argument given, and take the limit as $\epsilon \to 0$. □

We now introduce a criterion that will help us show that $f^\sharp$ admits an entire extension under certain conditions.

Theorem 4.12 (Morera's Theorem)**.** *Suppose f is continuous on an open disk D with*

$$\int_C f(\xi)d\xi = 0 \tag{4.15}$$

for every simple closed curve $C \subseteq D$. Then f is holomorphic.

Proof. Fix $\omega \in D$ and choose $z \in D$. Then, we may write $F(z) = \int_\Gamma f(\xi)d\xi$, where $\Gamma : [0,1] \to D$ is any curve that has $\Gamma(0) = \omega$ and $\Gamma(1) = z$. The hypothesis gives that this is well-defined. We may then compute

$$\frac{F(z+h) - F(z)}{h} = \frac{\int_\gamma f(\xi)d\xi}{h} = \frac{\int_\gamma f(\xi) - f(z)d\xi}{h} + f(z), \tag{4.16}$$

where we choose γ to be the line segment joining z and $z + h$. Since we may bound the integral by $\sup_\xi |f(\xi) - f(z)| \int_\gamma d\xi = \sup_\xi |f(\xi) - f(z)|\, h$, we obtain that the first term vanishes as $h \to 0$. Thus, we conclude that F is holomorphic with $F' = f$, implying that f is also holomorphic as desired. □

5. Hardy's Uncertainty Principle and Applications

Before venturing on to prove Hardy's Uncertainty Principle, we give a lemma that guarantees the analyticity of $f^\sharp$. The proof relies mostly on Morera's Theorem.

Lemma 5.1. *If $f \in L^1$ has $\int_\mathbb{R} |f(t)| \exp(2\pi t y)\, dt < \infty$ for all $y \in \mathbb{R}$, then $f^\sharp$ is entire.*

Some examples of such functions are compactly supported functions. The Theorem of Isolated Zeros for holomorphic functions gives that $f^\sharp$ restricted to $\mathbb{R}$ cannot be of compact support. We understand this as another uncertainty principle.

The following theorem is more quantitative in nature and asserts that f and $f^\sharp$ cannot both decrease very rapidly, justifying its title. The proof mainly follows from that given in [1].

Theorem 5.2 (Hardy's Uncertainty Principle)**.** *Suppose f is an integrable function such that*

$$|f(x)| \leq C\exp\left(-\pi x^2\right) \quad , \quad \left|f^\sharp(\xi)\right| \leq C\exp\left(-\pi\xi^2\right) \tag{5.1}$$

for all $x, \xi \in \mathbb{R}$. Then, $f(x)$ is a multiple of $\exp\left(-\pi x^2\right)$.

Proof. The bounds on f allow us to invoke Lemma 5.1 to say that $f^\sharp$ is analytic. Then, we let $\xi = x + iy$ to obtain

$$\left|f^\sharp(\xi)\right| \leq \int_\mathbb{R} |f(t)| \exp(2\pi t y)\, dt \leq C\int_\mathbb{R} \exp\left(-\pi t^2 + 2\pi t y\right) dt = C\exp\left(\pi y^2\right) \leq C\exp(\pi|\xi|^2). \tag{5.2}$$

Now, we will assume that f is even and prove the claim in this case. Afterwards, we construct a similar argument if f is odd.

Let f be even. Then $f^\sharp$ is also even, and we may write a power series expansion $f^\sharp(\xi) = \sum_{n=0}^\infty c_n \xi^{2n}$. We thus define $h(\gamma) := \sum_{n=0}^\infty c_n\gamma^n = f^\sharp(\gamma^{1/2})$. Note that h is entire. The goal of this argument is to prove that $|h(\xi)\exp(\pi\xi)| =: G(\xi)$ is bounded on the upper half plane. The argument for the lower half plane is analogous. Given this, Liouville's Theorem implies that G is constant. This forces $h(\xi) = C\exp(-\pi\xi)$, implying in turn $f^\sharp(\xi) = C\exp\left(-\pi\xi^2\right)$.

By the bound on $f^\sharp$ in Equation 5.2, we know that $\left|h(\gamma^2)\right| \le C\exp(\pi\left|\gamma\right|^2)$. If $\gamma = \xi^{1/2}$ for some ξ, then $|h(\xi)| \le C\exp(\pi|\xi|)$ for all $\xi \in \mathbb{C}$. This sets $\beta = 1$ in the conditions of Theorem 4.11. For $\xi \in \mathbb{R}_+$, we obtain

$$|h(\xi)| = \left|f^\sharp(\xi^{1/2})\right| \le C\exp(-\pi\xi) \le C. \tag{5.3}$$

Now, fix $0 < \delta < \pi$ and consider ξ in polar form, namely $\xi = Re^{i\theta}$, for $0 < \theta < \delta$. The goal is to find a similar bound for G on the line $\mathbb{R}_+ e^{i\delta}$ so we may apply Phragmén-Lindelöf to find a bound for G on the entire sector. To do so, we construct

$$\left|\exp\left(\frac{i\pi\xi\exp(-i\delta/2)}{\sin(\delta/2)}\right)h(\xi)\right| = \exp\left(\frac{-\pi R\sin(\theta-\delta/2)}{\sin(\delta/2)}\right)h(Re^{i\theta}) \le C \tag{5.4}$$

when $\theta = 0$ or $\theta = \delta$. Then, by Phragmén-Lindelöf, we may write

$$|h(\xi)| \le C\exp\left(\frac{-\pi R\sin(\theta-\delta/2)}{\sin(\delta/2)}\right) \tag{5.5}$$

for all ξ in the desired region. Taking the limit as $\delta \to \pi$, we find that $h(\xi) \le C\exp(-\pi R\cos(\theta))$ for θ in range. We may repeat this for the lower half plane as well.

Now, suppose f is odd. Then, $f^\sharp$ is also odd but also entire. Then, we know that $f^\sharp(0) = 0$ and we may write the function $f^\sharp(\xi)/\xi = \sum_{n=0}^{\infty} c_n\xi^{2n}$. We then consider a function H defined in a similar manner as h. We remark that

$$H(\gamma^2) = \left|\frac{f^\sharp(\gamma)}{\gamma}\right| \le C\frac{\exp\left(\pi|\gamma|^2\right)}{|\gamma|} \le C\exp\left(\pi|\gamma|^2\right) \tag{5.6}$$

when $|\gamma| > 1$. If $\gamma < 1$, consider a new (larger) constant D such that $\left|H(\gamma^2)\right| \le D\exp\left(\pi|\gamma|^2\right)$, so $H(\xi) \le D\exp(\pi|\xi|)$ always. We construct a similar bound when $\xi > 0$ and the argument is similar to before. Then, $f^\sharp(\xi)/\xi$ is a constant multiple of $\exp\left(-\pi\xi^2\right)$, however because of the bound on $f^\sharp$ by hypothesis, this number should be 0, so $f = f^\sharp = 0$ in this case.

Then, since the even parts of $f^\sharp$ correspond to the even parts of f and similarly with the odd parts, we notice that the whole function f and $f^\sharp$ satisfies the same bounds as the even and odd parts, so we use the previous argument and the triangle inequality to finish the proof. □

Corollary 5.3. *Let f be a measurable function and $\alpha, \beta, C > 0$ with*

$$|f(x)| \le C\exp\left(-\alpha x^2\right)\ ;\ \left|f^\sharp(x)\right| \le C\exp\left(-\beta x^2\right), \tag{5.7}$$

for $x \in \mathbb{R}$. If $\alpha\beta = \pi^2$, then f is a multiple of $\exp\left(-\alpha x^2\right)$. If $\alpha\beta > \pi^2$, then $f \equiv 0$.

Proof. Let $\alpha = \pi$. If not, scale it appropriately and we reduce it to this critical case. We then apply Hardy's Uncertainty Principle and complete the proof. □

One interesting application of this result used in more modern research is the study of how the Hardy Uncertainty Principle gives us information about the solutions of the Schrödinger equations. For example, consider the initial value problem

$$\begin{cases} i\partial_t u + \Delta u = 0 & \text{in } \mathbb{R}\times\mathbb{R}, \\ u|_{t=0} = u_0 & \text{in } \mathbb{R}. \end{cases} \tag{5.8}$$

The solution to this takes the form

$$u(x,t) = \frac{1}{\sqrt{t}} \int \exp\left(\frac{i\left|x-y\right|^2}{t}\right) u_0(y)dy, \tag{5.9}$$

for $t > 0$. Namely, plugging in $t = 1$ implies

$$u(x,1) = C \exp\left(i\left|x\right|^2\right) \int \exp\left(-2ixy\right) \exp\left(i\left|y\right|^2\right) u_0(y)dy. \tag{5.10}$$

We then take the norm, $\left\|u(x,1)\right\|_2$, to find

$$\left\|u(x,1)\right\|_2 = \left\|\left(\exp\left(i\left|y\right|^2\right) u_0\right)^{\sharp}_2\right\| = \left\|\exp\left(i\left|y\right|^2\right) u_0(y)\right\|_2 = \left\|u_0(y)\right\|_2, \tag{5.11}$$

so the theorem generalizes to the following:

Theorem 5.4. *For a solution u of the Schrödinger Equation, if $\left|u(x,1)\right|, \left|u(x,0)\right| \leq C \exp\left(-\pi\left|x\right|^2\right)$, then the entire solution has a Gaussian decay.*

The next natural question to ask is what happens to the result above if we add a potential term to the PDE, namely something of the form Vu, for an arbitrary potential function V. Here is a simplified version of one such result, proved by Carlos Kenig et. al. in [2].

Theorem 5.5. *Consider the initial value problem*

$$\begin{cases} i\partial_t u + \Delta u + Vu = 0 & \text{in } \mathbb{R} \times \mathbb{R}, \\ u|_{t=0} = u_0 & \text{in } \mathbb{R}. \end{cases} \tag{5.12}$$

Then, if V is continuous and bounded in the L^∞ norm, Theorem 5.4 holds for the IVP given.

The proof of this uses no Fourier Analysis, but mainly techniques in differential equations and the logarithmic convexity of such solutions. Thus, while it is not immediately related to the topic at hand, it makes sense to talk about as a logical extension of how the material at hand is applied to modern research.

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