HARMONIC ANALYSIS ON LCA GROUPS

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Abstract. Fourier analysis is an extremely useful set of tools for characterizing functions on the real line. Harmonic analysis attempts to generalize many of these constructions to locally compact abelian groups. In this paper, we lay out the basic structure of harmonic analysis on such groups, and prove the Pontryagin Duality and Plancherel theorems.

Contents

1. Introduction

Fourier analysis provides a powerful set of tools by which to analyze the periodic characteristics of functions on the real line. We would like to perform similar analysis on functions defined on other domains. The core tool of Fourier analysis is the Fourier transform:

$$\mathcal{F}(f)(s) = \int_{\mathbb{R}} f(t)e^{2\pi i s t}dt$$

Thus, to extend Fourier analysis, we must define an integral transform on a more general setting which sufficiently replicates its properties. To do this, we need enough structure on the domain for regularity and periodicity to make sense, and an integral that is compatible with that structure. If we allow our domain to be a locally compact topological group, the theory of Haar integration will allow us to define the Fourier transform in a unique way.

It remains to determine what the integration kernel of this transform ought be, and investigate its properties. In the real case, the kernel is $e^{2\pi i s t}$, with $s \in \mathbb{R}$. We see that these are continuous homomorphisms from $\mathbb{R}$ into $\mathbb{T}$ (the circle group in $\mathbb{C}^\times$). These functions are indexed by $\mathbb{R}$. Further, they are such that their pointwise product corresponds to addition of the indices:

$$e^{2\pi i s_1 t}e^{2\pi i s_2 t} = e^{2\pi i (s_1+s_2)t}$$

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Further, the compact-open topology of these functions agrees with the usual topology of $\mathbb{R}$.

These observations provide us with plenty of direction for our investigations. The continuous homomorphisms from a group of interest into $\mathbb{T}$ are a good choice for an integration kernel; it remains to determine how many of the observations we made in the real case carry over. If we stipulate that our domain be an abelian group, then pointwise multiplication of these functions forms a group operation, and applying the compact-open topology furnishes us with a locally compact abelian group. However, unlike the real case, we cannot in general identify this group (which we shall call the *dual group*) with the original group. However, we can guarantee that this character group is *itself* a locally compact abelian group - so we can consider its dual group! It shall be the primary focus of this paper to prove the Pontryagin Duality theorem, which states that each locally compact abelian group can be naturally identified with its double-dual group.

Before we begin, we shall take a moment to clarify our assumptions and notation. We shall assume familiarity with the basics of point-set topology, group theory, and functional analysis. By $C_c(G)$, we will mean continuous functions on G with compact support. By $C_0(G)$, we mean continuous functions on G that vanish at infinity in the following sense: for any $\epsilon > 0$, there is a compact $K \subseteq G$ such that $|f(x)| < \epsilon$ outside of K. For a group G and real (or complex) valued function f, we use the notation $L_z f(x) = f(z^{-1}x)$ and $R_z f(x) = f(xz)$; these are, in a sense, "translations" of f.

2. Topological Groups and the Haar Measure

This section will be devoted to defining the topological and measure theoretic structure on a group necessary to do harmonic analysis. The first ingredient of this is a topology that interacts well with the group structure.

Definition 2.1. A *topological group* is a group G with a topology such that the following maps are continuous:

$$((a,b) \mapsto ab) : G \times G \to G \qquad (a \mapsto a^{-1}) : G \to G$$

We will additionally assume for the remainder of this paper that all topological groups are Hausdorff.

There are some immediate consequences of this definition. The first of these is a regularity property of sorts, which follows immediately from the continuity of the product map:

Proposition 2.2. *A subset X of a topological group G is open (likewise compact) if and only if it is a left translate of some open (compact) set containing the identity.*

Thus, we can study the topology near any point of the group by studying the topology around the identity. This gives us a notion of nearness that does not vary across the space. As an example of this, consider the following adaptation of the definition of uniform continuity to topological groups.

Definition 2.3. Let G be a topological group. We say that $f : G \to \mathbb{C}$ is *uniformly continuous* if for every $\epsilon > 0$, there exists an open set $V \ni \{e\}$ such that if $x^{-1}y \in V$, then $d(f(x), f(y)) < \epsilon$.

We see the regularity of the topology in action in this definition. If $x^{-1}y$ falls in a unit neighborhood V, then xV is a neighborhood of x that contains y. We formalize the intuition of these neighborhoods having the same size with the notion of Haar measure.

Definition 2.4. A *Haar measure* on a locally compact group G is a left-invariant Radon measure; i.e. it is a measure μ such that for any Borel set A, we have:

$$\mu(xA) = \mu(A) \quad \forall x \in G$$

Further, there exists a net of open sets U_ϵ containing A such that $\mu(U_\epsilon - A) < \epsilon$, and if $\mu(A) < \infty$, a net of compact $K_\epsilon \subset A$ such that $\mu(A - K_\epsilon) < \epsilon$.

Theorem 2.5 (Haar's Theorem)**.** *Let G be a locally compact group. Then, there exists a Haar measure μ on G. Further, this measure is unique in the following sense: if μ, ν are both Haar measures, then there exists $c \in (0, \infty)$ such that $\mu = c\nu$*

We shall not give a full proof of this theorem, as it is somewhat lengthy and technical; [1] gives a detailed explanation. As an intuitive guide, consider the measure to be, for some fixed compact K_0 and where U are open unit neighborhoods:

$$\mu(A) = \lim_{U \to \{1\}} \frac{\min\{n \mid \exists x_1, \ldots, x_n \text{ such that } A \subseteq \bigcup_{i=1}^n x_i U\}}{\min\{n \mid \exists x_1, \ldots, x_n \text{ such that } K_0 \subseteq \bigcup_{i=1}^n x_i U\}}$$

The choice of compact set determines the particular scaling of the measure, and its presence in the definition ensures the existence of the limit. Note that this definition, while intuitively simpler, is in fact rather difficult to work with. Every source consulted by the author preferred to define the Haar integral directly as a functional, recovering the measure via the Riesz Representation Theorem.

We give some examples of the Haar measure (up to scaling) on familiar locally compact groups.

Examples 2.6.

- The Haar measure on $\mathbb{R}$ is the standard Lebesgue measure.
- The Haar measure on $\mathbb{R}^\times$ is $\frac{dx}{|x|}$.
- The Haar measure on a group is the counting measure if and only if the group is discrete.
- The Haar measure on $\mathbb{T}$ is $d\theta$.

Some immediate consequences of the presence of a Haar measure on a group include that all open sets have non-zero measure and that all compact sets have finite measure. If the group itself is compact, we will usually prefer to normalize the measure so that $\mu(G) = 1$. If the group is discrete, we will prefer to normalize to the ordinary counting measure. Finite groups we shall likewise equip with the counting measure.

Now that we have a consistent measure on the group, we can construct the measure integral on the group. We will typically use the Daniell construction of the integral; note that this is equivalent to the more familiar construction due to Lebesgue. From this point, all functions will be assumed measurable.

We construct another object essential to the integration theory on locally compact groups by following this train of logic: While a Haar measure μ on G is not necessarily right invariant, we know that $\mu(xAy) = \mu(Ay)$, so $\mu_y(A) = \mu(Ay)$ is a Haar measure on G, so by Haar's Theorem it is a constant multiple of μ. This multiple is denoted $\Delta(y)$, which we call the *modular function*.

It is easily seen that this is a group homomorphism $\Delta : G \to \mathbb{R}^{\times}_{>0}$, we can see (but shall not prove here) that it is continuous. This tool grants the following lemma:

Lemma 2.7. *Let G be a locally compact group, and let Δ be its modular function. Then:*

(1) $\Delta = 1$ *if G compact or abelian.*
(2) $\displaystyle\int_G f(xy)dx = \Delta(y^{-1})\int_G f(x)dx$
(3) $\displaystyle\int_G f(x^{-1})\Delta(x^{-1})dx = \int_G f(x)dx$

Proof. (1) The abelian case is clear; the compact case follows as all nontrivial subgroups of $\mathbb{R}^{\times}_{>0}$ are unbounded.

(2) Consider an indicator function $\mathbf{1}_A(x)$. We see that $xy \in A \iff x \in Ay^{-1}$, so $\int_G \mathbf{1}_A(xy) = \int_G \mathbf{1}_{Ay^{-1}}(x) = \mu(Ay^{-1}) = \Delta(y^{-1})\mu(A)$. Since multiplication by $\Delta(y^{-1})$ respects linearity and continuity, Daniell extension gives us the result for general functions.

(3) Consider $\nu(f) = \int_G f(x^{-1})\Delta(x^{-1})$. It is easily seen that this is a positive linear functional. Further, we see that:

$$\begin{aligned}\nu(L_z f) &= \int_G L_z f(x^{-1})\Delta(x^{-1}) = \int_G f(z^{-1}x^{-1})\Delta(x^{-1})\\ &= \int_G \Delta(z^{-1})f(x^{-1})\Delta((xz^{-1})^{-1}) = \int_G f(x^{-1})\Delta(x^{-1})\end{aligned}$$

Thus, it is a left-invariant integral, so by Haar's Theorem it is a constant multiple c of the standard Haar integral. Let A be a symmetric unit neighborhood in $\Delta^{-1}((\epsilon^{-1}, \epsilon))$ $(\epsilon > 1)$. Then

$$c\mu(A) = \nu(\mathbf{1}_A) = \int_G \mathbf{1}_A(x^{-1})\Delta(x^{-1}) = \int_A \Delta(x^{-1})$$

Thus, $\frac{1}{\epsilon}\mu(A) \leq \nu(\mathbf{1}_A) \leq \epsilon\mu(A)$. Since ϵ is arbitrary, we know $c = 1$.

□

The final assertion provides an operation on $L^1(G)$ that has the properties of an involution; this makes $L^1(G)$ into a *-algebra, as we shall see in the following section. We conclude this section by stating a construction principle for Haar measures.

Theorem 2.8 (Quotient Integral Formula)**.** *Let G be a locally compact group and H be a closed subgroup. Then, there exists a left-invariant Radon measure on the quotient G/H if and only if $\Delta_G|_H = \Delta_H$. Further, there is a unique choice of measure μ that satisfies the following version of Fubini's Theorem:*

$$\int_G f(g)dg = \int_{G/H}\int_H f(xh)dhd\mu(x)$$

3. $L^1(G)$, and other Banach *-Algebras

We construct the Banach space $L^1(G)$ in the usual manner. We additionally define a multiplication on $L^1(G)$ as the convolution of two functions, and an involution $f(x) \mapsto f(x^{-1})\Delta(x^{-1})$. Since $\|f * g\|_1 \leq \|f\|_1\|g\|_1$ by Hölder's inequality, and we have already proven $\|f^*\| = \|f\|$, $L^1(G)$ is a Banach *-algebra. In this section, we shall present some results about $L^1(G)$, as well as several that apply to Banach *-algebras in general.

Theorem 3.1. *$L^1(G)$ is commutative if and only if G is abelian.*

Proof. First, we assume G is abelian. We know $\Delta = 1$ on abelian groups. Then, we see:

$$f * g = \int_G f(y)g(y^{-1}x)dy = \Delta(x^{-1}) \int_G \Delta(y^{-1})g(y)f(y^{-1}x)dy = g * f$$

Now we assume $L^1(G)$ is commutative. Then, for arbitrary f, g we have

$$0 = f * g - g * f = \int_G f(xy)g(y^{-1})dy - \int_G g(y)f(y^{-1}x)dy$$

$$= \int_G g(y) \left(f(xy^{-1})\Delta(y^{-1}) - f(y^{-1}x) \right) dy$$

Now, we choose g to be the indicator of a compact set K, and f to be nonnegative. Thus, we know that on any compact K,

$$f(xy^{-1})\Delta(y^{-1}) = f(y^{-1}x)$$

As our group is locally compact, this relation must hold everywhere. Since this must hold for all x, y, setting $x = 1$ lets us see that $\Delta = 1$ everywhere, so now

$$f(xy^{-1}) = f(y^{-1}x)$$

As $L^1(G)$ separates points, G must be commutative. □

We shall shortly prove that $L^1(G)$ has a multiplicative unit if and only if G is discrete. Since we will frequently work with non-discrete groups, we shall find it useful to define an object that approximates the behavior of a unit. We shall use this object in the following proof, so we define it here:

Definition 3.2. A *Dirac net* is a net of nonnegative functions $\phi_i \in C_c(G)$, with index set I being the open sets containing 1 ordered by reverse inclusion, such that, $\forall i \in I$,

- $\displaystyle\int_G \phi_i d\mu = 1$
- $\phi_i(x) = \phi_i(x^{-1})$
- $\mathrm{supp}(\phi_i) \subseteq i$

This allows us to capture the idea of the Dirac delta distribution, in a convenient and rigorous manner.

Lemma 3.3. *Let G be a locally compact group, and let ϕ_i be a Dirac net. Then:*

(1) Let $f \in L^1(G)$. Then, $\phi_i * f$ and $f * \phi_i$ both converge to f in $L^1(G)$.
(2) If $K \subseteq G$ compact and $f \in C(K)$, then $\phi_i * f$ and $f * \phi_i$ both converge to f uniformly on K.

Proof. We shall only prove each claim on one of $\phi_i * f$ and $f * \phi_i$; the computations for each case are nearly identical, so for each claim we choose to present the computation that does not require use of the modular function.

Note that $f(x) = \int_G f(x)\phi_i(y)dy$. The first assertion then follows from this computation:

$$\begin{aligned}\|\phi_i * f - f\|_1 &= \int_G \left| \int_G \phi_i(y)\left(f(y^{-1}x) - f(x)\right) dy \right| dx \\ &\leq \int_G \int_G \left|\phi_i(y)\left(f(y^{-1}x) - f(x)\right)\right| dxdy \\ &= \int_G \phi_i(y)\|L_y f - f\|_1 dy \leq \sup_{y\in \mathrm{supp}(\phi_i)} \|L_y f - f\|_1\end{aligned}$$

The problem of finding an open set U_i such that this quantity can be made small for a particular $f \in L^1$ can be reduced to the same problem on a nearby $g \in C_c$ by a standard $\frac{\epsilon}{3}$ argument. Since g is uniformly continuous and compactly supported,

$$\|L_y g - g\|_1 \leq \mu(\mathrm{supp}(g))\|L_y g - g\|_\infty$$

Since this can be made arbitrarily small by uniform continuity, we have proven the first assertion. For the second argument, note that we already have uniform continuity of f, so we simply compute:

$$|f * \phi_i - f| = \left|\int_G \phi_i(y^{-1}x)f(y) - f(x)dy\right| \leq \sup_{y\in\mathrm{supp}(\phi_i)} |f(y) - f(x)|$$

□

Proposition 3.4. *$L^1(G)$ has a multiplicative unit if and only if G is discrete.*

Proof. If G is discrete, we can consider the function $1_{\{e\}}$, which we see immediately is a unit.

If L^1 has a unit ϕ, then we know that for a Dirac net ϕ_n, $\phi * \phi_n$ converges to ϕ. As ϕ is a unit, ϕ_n converges to ϕ in L^1, and thus ϕ_n also converges pointwise almost everywhere. Since G Hausdorff, for any $x \neq 1$, $\exists i_o$ such that $i \geq i_o$ implies $\phi_i(x) = 0$. If G is not discrete, then $\{1\}$ has measure zero, so $\phi = 0$ almost everywhere, which is a contradiction. □

At this point, we shall develop some of the tools of the Gelfand theory on commutative Banach algebras. These tools will allow us to more effectively characterize not only L^1, but also many of the other function spaces we will be working with.

Definition 3.5. Let $\mathcal{A}$ be a commutative Banach algebra. The *structure space* $\Delta_{\mathcal{A}}$ is defined as the set of all nonzero algebra homomorphisms $m : \mathcal{A} \to \mathbb{C}$.

This is the primary construction that Gelfand theory is concerned with. We give this the operator norm and the weak-* topology. As a consequence of Banach-Alaoglu, $\Delta_{\mathcal{A}}$ is a locally compact Hausdorff space, and in the case of a unital algebra, it is compact. Frequently, we will not be working with unital algebras, but we shall define a convenient method of adding a unit to an existing algebra.

Definition 3.6. Let $\mathcal{A}$ be a commutative Banach algebra. Then, the *unitization* of $\mathcal{A}$ is defined as $\mathcal{A}^e = \mathcal{A} \oplus \mathbb{C}$, with addition and scalar multiplication defined componentwise, unit element $(0, 1)$, and multiplication:

$$(a, \lambda)(b, \nu) = (ab + a\nu + b\lambda, \lambda\nu)$$

Remark 3.7. The map $(m \mapsto ((a,\lambda) \mapsto m(a)+\lambda)) : \Delta_{\mathcal{A}} \to \Delta_{\mathcal{A}^e}$ is a homeomorphism onto its image, and the only element of $\Delta_{\mathcal{A}^e}$ not of this form is $m_\infty(a,\lambda) = \lambda$. Thus, $\Delta_{\mathcal{A}^e}$ is the Alexandrov compactification of $\Delta_{\mathcal{A}}$.

Thus, from the perspective of the structure space, the unitization only minimally changes $\mathcal{A}$. We shall take advantage of this property in the following proof:

Proposition 3.8. *An algebra homomorphism $\phi : \mathcal{A} \to \mathcal{B}$ induces a continuous map $\phi^* : \Delta_{\mathcal{B}} \to \Delta_{\mathcal{A}}$, which is a homeomorphism if it is bijective.*

Proof. The induced map $(m \mapsto m \circ \phi)$ is a continuous map, as the structure spaces are topologized by pointwise convergence of nets of functions, and its clear that this holds by continuity of ϕ. To verify the second claim, we notice now that if $\mathcal{A}$, $\mathcal{B}$ are both unital, then ϕ^* is a continuous bijection between compact spaces, and is thus a homeomorphism. If this is not the case, we can pass to their respective unitizations, then see that the homeomorphism we obtain restricts to a homeomorphism between the original structure spaces. □

Thus, we see that this is a functorial relation, as one might expect. In the case of commutative unital Banach algebras, the structure space encodes information about its ideals, in the following manner.

Theorem 3.9. *If $\mathcal{A}$ is unital and commutative, then the map $(m \mapsto \ker(m))$ is a bijection between the structure space and the set of ideals of $\mathcal{A}$.*

Proof. First, we note that the kernel of an element of the structure space is in fact an ideal; this is immediately verifiable. It remains to prove that this map is injective and surjective.

To see that the map is injective, assume f, g have identical kernel K. By the First Isomorphism Theorem note that $\mathcal{A}/K \cong \mathbb{C}$. Thus, we see that $g \circ f^{-1}$ is an element of $\mathrm{Aut}(\mathbb{C})$. Since this is the trivial group, $f = g$.

To see surjectivity, take an ideal $\mathcal{I}$. Note that every element of $\mathcal{A}/\mathcal{I}$ is invertible. This means that $\mathcal{A}/\mathcal{I} \cong \mathbb{C}$; one way to verify this is the Gelfand-Mazur theorem (see section 2.2 of [1]). Thus, the projection map $\pi : \mathcal{A} \to \mathcal{A}/\mathcal{I}$ gives an element of the structure space. □

In the case of commutative C*-algebras, the structure space encodes enough information to completely characterize the algebra in question. The following theorem, which is a special case of the more general Gelfand-Naimark theorem, provides this characterization.

Theorem 3.10 (Gelfand-Naimark)**.** *For a commutative C*-algebra $\mathcal{A}$, there is an isomorphism*

$$\mathcal{A} \cong C_0(\Delta_{\mathcal{A}})$$

This isomorphism is given by the Gelfand transform $(a \mapsto (m \mapsto m(a)))$

We shall not prove this theorem, but it is an extremely powerful result, which we will use in the following section. One barrier to our application of this theorem to harmonic analysis is the fact that, for a nontrivial abelian group A, $L^1(A)$ is not a C*-algebra. However, we shall now construct a C*-algebra into which $L^1(A)$ has a dense inclusion.

Construction 3.11. Let $f \in L^1(A)$, $\phi, \psi \in L^2(A)$. The Cauchy-Schwarz inequality gives us that $|\langle L_y\phi, \psi\rangle| \leq \|L_y\phi\|_2\|\psi\|_2 = \|\phi\|_2\|\psi\|_2$. Thus, we have that

$$\left|\int_A f(y)\langle L_y\phi, \psi\rangle dy\right| \leq \|f\|_1\|\langle L_y\phi, \psi\rangle\|_\infty \leq \|f\|_1\|\phi\|_2\|\psi\|_2$$

Fixing f and ϕ, we have that $\lambda := \left(\psi \mapsto \int_A f(y)\langle L_y\phi, \psi\rangle dy\right)$ is a bounded antilinear functional. For all such functionals, there is a unique vector in $L^2(A)$ such that $\lambda\psi = \langle L(f)(\phi), \psi\rangle$, for all $\psi \in L^2(A)$. Taking $\psi = L(f)(\phi)$, we find that

$$\langle L(f)(\phi), L(f)(\phi)\rangle = \|L(f)(\phi)\|_2^2 \leq \|f\|_1\|\phi\|_2\|L(f)(\phi)\|_2$$
$$\|L(f)(\phi)\|_2 \leq \|f\|_1\|\phi\|_2$$

Thus, $L(f)$ is a continuous linear function of ϕ. L, defined as above, is a function from $L^1(A)$ into $\mathcal{B}(L^2(A))$ (the space of bounded linear operators on $L^2(A)$).

Remark 3.12. Fubini's Theorem implies that if $\phi \in L^1(A) \cap L^2(A)$, then:

$$L(f)(\phi) = f * \phi$$

Proposition 3.13. *L is a continuous injective *-homomorphism.*

Proof. We have already seen that L is continuous and linear, and multiplicativity follows from the remark. Injectivity likewise follows the remark; by considering $L(f)(\phi_i)$ for a Dirac net ϕ_i, we show that the kernel of L is trivial. That it is a *-homomorphism follows by taking $\phi, \psi \in C_c(A)$ and performing this computation:

$$\langle L(f^*)(\phi), \psi\rangle = \langle f * \phi, \psi\rangle = \int_A\int_A \overline{\psi(x)} f^*(y)\phi(y^{-1}x)dydx$$
$$= \int_A\int_A \overline{\psi(yx)f(y^{-1})}\phi(x)dxdy = \int_A \phi(x)\int_A \overline{\psi(y^{-1}x)f(y)}\Delta(y^{-1})dydx$$
$$= \langle \phi, f * \psi\rangle = \langle \phi, L(f)(\psi)\rangle$$

Thus, $L(f^*) = L(f)^*$. □

Definition 3.14. We call the closure of $L(L^1(A))$ in this space $C^*(A)$. Since $\mathcal{B}(L^2(A))$ is a C*-algebra, so is $C^*(A)$.

We have a dense inclusion of $L^1(A)$ into $C^*(A)$. This is very convenient in itself, but we would like to further characterize the relationship between these spaces as Banach *-algebras. Specifically, we want to show that they are identical from the standpoint of Gelfand theory.

Lemma 3.15. *The induced map $L^* : \Delta_{C^*(A)} \to \Delta_{L^1(A)}$ is a homeomorphism.*

We know by Proposition 3.8 that if we can show that L^* is surjective, then it is a homeomorphism. By considering the canonical unitization and using Theorem 3.9, we reduce this to the following lemma:

Lemma 3.16. *Let $\mathcal{B}$ be a unital commutative algebra, and let $\mathcal{A}$ be a dense unital subalgebra. Then, for any ideal $\mathcal{I}$ of $\mathcal{A}$, $\overline{\mathcal{I}}$ is an ideal in $\mathcal{B}$, and $\overline{\mathcal{I}} \cap \mathcal{A} = \mathcal{I}$.*

Proof. Let $\mathcal{I}$ be an ideal on $\mathcal{A}$. Let $\alpha \in \overline{\mathcal{I}}$, $\beta \in \mathcal{B}$. Then, as $\mathcal{A}$ is dense in $\mathcal{B}$, there is a sequence $a_n \in \mathcal{I}$ converging to α, and $b_n \in \mathcal{A}$ converging to β. Then, the sequence $c_i = a_ib_i$ is in $\mathcal{I}$ for all i, and so $\alpha\beta = \lim_{i\to\infty} c_i \in \overline{\mathcal{I}}$. Thus, $\overline{\mathcal{I}}$ is an ideal on $\mathcal{B}$. We know that $\overline{\mathcal{I}} \cap \mathcal{A} = \mathcal{I}$, as ideals are topologically closed. □

The induced map $L^* : \Delta_{C^*(A)^e} \to \Delta_{L^1(A)^e}$ is surjective, as for each element of $\Delta_{L^1(A)^e}$, the closure of its kernel is a maximal ideal in $C^*(A)^e$, uniquely specifying an element of $\Delta_{C^*(A)^e}$ that restricts to it. This completes Lemma 3.15.

4. The Character Group $\hat{A}$

From this point forward, we will restrict attention to locally compact abelian groups, or LCA groups for short. For these groups, we can define the group of harmonic characters as follows:

Definition 4.1. Let A be an LCA group. The *dual group* $\hat{A}$ of A is the group of continuous homomorphisms from A into the circle group $\mathbb{T}$. The elements of this group are called *characters*, and the group operation is given by the pointwise product of two characters.

We give $\hat{A}$ the compact-open topology, and observe that it is a Hausdorff topological group.

Examples 4.2.

- The dual group of $\mathbb{R}$ is isomorphic to $\mathbb{R}$ itself, with elements $(x \mapsto e^{2\pi ikx})$, with $k \in \mathbb{R}$.
- The dual group of any finite cyclic group $\mathbb{Z}_n$ is isomorphic to itself. Each character has a subgroup of the nth roots of unity as its image, and is determined by which root of unity it maps the generator 1 to. Thus, the dual group of $\mathbb{Z}_n$ is the subgroup of $\mathbb{T}$ comprising the nth roots of unity, which is itself isomorphic to $\mathbb{Z}_n$. As a consequence of this, all finite abelian groups are self-dual.
- The dual group of $\mathbb{Z}$ is isomorphic to $\mathbb{T}$. Like the previous example, each character is determined by where it maps the generator 1.
- The dual group of $\mathbb{T}$ is isomorphic to $\mathbb{Z}$, with elements $(z \mapsto z^n)$ for $n \in \mathbb{Z}$

Note that when a group is self-dual, the isomorphism is *not* canonical. In the first example, the choice $(x \mapsto e^{2\pi ikx})$ is a standard one, but we could have chosen $(x \mapsto e^{rikx})$ for any $r \in \mathbb{R}^\times$. The isomorphism between the nth roots of unity and $\mathbb{Z}_n$ is likewise noncanonical. In the next section, we shall see that the mapping of a group into its double dual is not only always an isomorphism, but also a canonical one.

Looking at these examples, one might notice a pattern: whenever a group is compact, its dual seems to be discrete, and vice versa. This turns out to be true in general.

Theorem 4.3.

a) If A is a compact group, then $\hat{A}$ is discrete.
b) If A is a discrete group, then $\hat{A}$ is compact.

We could immediately prove both parts of this theorem. However, the second argument would at this point consist of an appeal to the Tychonov theorem, and (in the author's opinion) is not particularly intuitive. Thus, we shall postpone this argument until later, where it is proven in a more intuitive manner as a corollary of Theorem 4.6.

Proof of a). As we assume A is compact, the homomorphisms from A into any open set in $\mathbb{T}$ form an open set in $\hat{A}$. However, the image of such a homomorphism must be a subgroup in $\mathbb{T}$. Choose as an open set $B(1, \frac{1}{2}) \cap \mathbb{T}$ (remembering that $\mathbb{T}$ has the subspace topology), and we see that only the trivial character's image lies entirely within this open set. Thus, $\hat{A}$ is discrete. □

Definition 4.4. Let $f \in L^1(A)$. Then let its *Fourier transform* be the function on $\hat{A}$ defined by:

$$\hat{f}(\chi) = \int_A f(x)\overline{\chi(x)}dx$$

Lemma 4.5. *Let $f, g \in L^1(A)$*

(1) $|\hat{f}| \leq \|f\|_1$
(2) $\widehat{f * g} = \hat{f}\hat{g}$

Proof. (1) follows from Hölder's inequality.
(2) follows from this calculation:

$$\widehat{f * g} = \int_A \int_A f(y)g(y^{-1}x)\overline{\chi(x)}dxdy = \int_A f(y) \left(\int_A g(x)\overline{\chi(y)\chi(x)}dx \right) dy = \hat{f}\hat{g}$$

□

We recall the definition of structure spaces from the previous section. We now have the ability to state a surprising equivalence:

Theorem 4.6. *The map from $\lambda : \hat{A} \to \Delta_{L^1(A)} := (\chi \mapsto (f \mapsto \hat{f}(\chi)))$ is a homeomorphism. Thus, $\hat{A}$ is locally compact, and $\hat{f} \in C_0(\hat{A})$.*

Proof. We shall begin by exhibiting an inverse map. For any $m \in \Delta_{L^1(A)}$ note that $|m(a)| \leq \|a\|_1$. Note further that as m is a continuous algebra functional, we can define a complex-valued measure $\mu_m(E) = m(1_E)$. By the above inequality, we see that μ_m is absolutely continuous with respect to μ, the standard Haar measure on A. Let $\lambda^{-1}(m) = \overline{\frac{d\mu_m}{d\mu}}$ be the Radon-Nikodym derivative.

We note that, due to the above inequality, $|\lambda^{-1}(m)| \leq 1$ almost everywhere. We let ϕ_i be a Dirac net, let $\psi_i = L_z\phi_i$, and note that for any $g \in L^1(A)$, $m(\psi_i * g) = m(\psi_i)m(g)$. The right side converges to $m(L_z g)$, and the right side converges to $(\lambda^{-1}(m))(z)m(g)$. Thus, $\lambda^{-1}(m)$ is a homomorphism into $\mathbb{C}^\times$, and as L_z is uniformly continuous, $\lambda^{-1}(m)$ is continuous as well. As it is bounded, its image must be in the circle group $\mathbb{T}$, and so is a character of A. It is clear that this map is an inverse of λ. Thus, λ is a bijection.

It remains to prove that λ is bicontinuous. One can verify that for a net $\chi_i \in \hat{A}$, $\lim_i \lambda(\chi_i) = \lambda(\lim_i \chi_i)$, and likewise for a net $m_i \in \Delta_{L^1(A)}, \lim_i \lambda^{-1}(m_i) = \lambda^{-1}(\lim_i m_i)$; however, we shall not do so here. □

Corollary 4.7. *$\hat{A}$ is homeomorphic to $\Delta_{C^*(A)}$, and $C^*(A) \cong C_0(\hat{A})$.*

Proof. The first assertion is clear, the second follows from the Gelfand-Naimark theorem. □

We see that the theory of harmonic analysis on an LCA group A is in many ways equivalent to the Gelfand theory on $L^1(A)$. Since $L^1(A)$ is, from the perspective of Gelfand theory, almost a C*-algebra, we have a naturally defined operator from $L^1(A)$ into $C_0(\hat{A})$ in the form of the Gelfand transform. As we shall now see, this operator is exactly the Fourier transform.

Corollary 4.8. *The Fourier transform $\mathcal{F} : L^1(A) \to C_0(\hat{A})$ is injective.*

Proof. This follows from the observation that the Gelfand transform of $f \in L^1(A)$ is $(m \mapsto m(f))$, and that for each $m \in \Delta_{L^1(A)} \cong \Delta_{C^*(A)}$ there is some $\chi \in \hat{A}$ such that $m(f) = \int_A f(x)\overline{\chi(x)}dx$. The inclusion of $L^1(A)$ into $C^*(A)$ composed with the Gelfand transform is equal to $\mathcal{F}$, and thus the corollary follows. □

We see that not only do the definitions of Fourier analysis cleanly generalize to LCA groups, they arise from the underlying structure of $L^1(A)$ as a Banach algebra.

As promised, we shall conclude this section by completing the proof of Theorem 4.3, by constructing a specific object that can only occur on a compact space.

Corollary 4.9. *If A is a discrete group, then $\hat{A}$ is compact.*

Proof. If A is discrete, then there exists a Dirac delta function on A, i.e. a function δ such that, $\forall f \in L^1(A)$, $\int_A f(x)\delta(x)dx = f(1)$. Thus, $\hat{\delta} = 1$. By Theorem 4.6, $\hat{\delta} \in C_0(\hat{A})$. This is only possible if $\hat{A}$ is compact. □

5. The Fourier Transform and Pontryagin Duality

We already know that an LCA group and its dual are not, in general, isomorphic. However, in Corollary 4.7 we lay out a powerful correspondence between the two groups. We would like to extend this as much as possible. The following construction is of vital importance in this endeavor, as it allows us to identify functions defined on A with those defined on $\hat{A}$. As we shall see, these functions are particularly well-behaved with respect to the Fourier transform.

Construction 5.1. $C_0(A) \times C_0(\hat{A})$ is a Banach space, taking the maximum of the norms on the component spaces as the norm on the product. We map $C_0(A) \cap L^1(A)$ into this space by $(f \to (f, \hat{f}))$. We call the closure of this inclusion $C_0^*(A)$.

Remark 5.2. The projections $\pi_0 : C_0^*(A) \to C_0(A)$ and $\pi^* : C_0^*(A) \to C^*(A)$ are injective. Thus, when we refer to a function $f \in C_0^*(A)$, we refer to the first coordinate of an object in the space; equivalently, we refer to a sequence of functions in $C_0(A) \cap L^1(A)$ that converges in $C_0^*(A)$ (and thus, uniformly on A as well).

The following two lemmata contain useful relationships between the numerous function spaces we have defined. The first allows us to determine information about the size of $\hat{f}$ from $f(1)$, whenever $f \in C_0^*(A)$; this will be used to define a measure on $\hat{A}$. The second allows us to better characterize convolutions and Fourier transforms of certain functions.

Lemma 5.3. *Let $f \in C_0^*(A)$.*

(1) If $\hat{f}$ real valued, then so is $f(1)$.
(2) If $\hat{f}$ is positive, then so is $f(1)$.

Proof. (1) $\hat{f} = \overline{\hat{f}} = \hat{f^*}$, as the Fourier transform is injective we have $f = f^*$, and so $f(1) = f^*(1)$ is real.

(2) If $\hat{f}$ is nonnegative, we can choose another nonnegative real-valued function g in $C_0(\hat{A})$ such that $g^2 = \hat{f}$. As $C_0(\hat{A}) \cong C^*(A)$, there is a sequence of functions $g_n \in L^1(A)$ whose Fourier transforms converge to g. Note that as the transforms are real-valued, g_n are all self-adjoint, by the reasoning in part (1). Let ϕ_i be a Dirac net. Since for any $\phi \in L^2(A)$, $\phi * \phi^*(1) = \|\phi\|_2^2$, we have:

$$\begin{aligned} &\lim_n \|L(g_n)\phi_i\|_2^2 = \lim_n \left(L(g_n)\phi_i * (L(g_n)\phi_i)^*\right)(1) \\ &= \lim_n \left(g_n * \phi_i * (g_n * \phi_i)^*\right)(1) = \lim_n \left(g_n * \phi_i * g_n * \phi_i\right)(1) \\ &= \lim_n \left(g_n * g_n * \phi_i * \phi_i\right)(1) = \lim_n L(g_n * g_n)(\phi_i * \phi_i)(1) \end{aligned}$$

Since $\widehat{g_n * g_n} = \hat{g_n}\hat{g_n} \to \hat{f}$, we see that

$$\lim_n L(g_n * g_n)(\phi_i * \phi_i)(1) = L(f)(\phi_i * \phi_i)(1) = f * \phi_i * \phi_i(1)$$

As ϕ_i is a Dirac net, this converges to $f(1)$. However, as the norm in the beginning of our chain of equalities is nonnegative, we see that $f(1)$ must too be nonnegative.

□

Lemma 5.4.

(1) $L^1(A) * C_c(A) \leq C_0(A)$

(2) *Let* $f \in C^*(A)$, $\phi, \psi \in C_c(A)$. *Then,* $L(f)(\phi * \psi) \in C_0^*(A) \cap L^2(A) \leq C_0(A)$. *One has* $\widehat{L(f)(\phi * \psi)} = \hat{f}\hat{\phi}\hat{\psi}$.

Proof. (1) Let $f \in L^1(A), \phi \in C_c(A)$. As $C_c(A)$ is dense in $L^1(A)$, there is a sequence $f_n \in C_c(A)$ that converges to f in L^1. We see that $f_n * \phi \in C_c(A)$. We note that, by the Hölder inequality,

$$(f_n - f) * \phi \leq \|f_n - f\|_1 \|\phi\|_\infty < \epsilon \|\phi\|_\infty$$

Thus, $f_n * \phi$ converges uniformly to $f * \phi$. Thus, $f * \phi \in C_0(A)$.

(2) f is the C^* limit of $f_n \in L^1$. By the first part of the lemma, then, we know that $L(f_n)(\phi * \psi) \in C_0(A) \cap L^1(A)$. We can map this into $C_0^*(A)$. If the projections to both $C_0(A)$ and $C_0(\hat{A})$ converge, we know it converges in $C_0^*(A)$. To see convergence in $C_0(\hat{A})$, note that

$$\lim_n \widehat{f_n * \phi * \psi} = \lim_n \widehat{f_n}\hat{\phi}\hat{\psi} = \hat{f}\hat{\phi}\hat{\psi}$$

To see convergence in $C_0(A)$, we know that

$$(f_n - f_m) * \phi * \psi \leq \|(f_n - f_m) * \phi\|_2 \|\psi\|_2 = \|L(f_n - f_m)(\phi)\|_2 \|\psi\|_2$$

As f_n converges in C*, as m, n become large, the operator norm of $L(f_n - f_m)$ goes to zero, and so we see that $L(f_n)(\phi * \psi)$ is Cauchy in $C_0(A)$. We know this is also in L^2 by the definition of L, completing the lemma.

□

The following lemma is extremely useful. It gives us that the image of the Fourier transform of $C_0^*(A) \cap L^2(A)$ is dense in $C_c(\hat{A})$. Thus, it is also dense in $L^2(\hat{A})$ and $C_0^*(\hat{A})$. Thus, we can approximate large classes of interesting functions on $\hat{A}$ by Fourier transforms of functions on A.

Lemma 5.5. *Let $\eta \in C_c(\hat{A})$ be a real-valued function. For any $\epsilon > 0$, there are $f_1, f_2 \in C_0^*(A) \cap L^2(A) \leq C_0(A)$ such that:*

- *$supp(\hat{f}_i) \subseteq supp(\eta)$*
- *$\hat{f}_1 \leq \eta \leq \hat{f}_2$*
- *$\|\hat{f}_2 - \hat{f}_1\|_\infty < \epsilon$.*

In particular, $\mathcal{F}(C_0^*(A) \cap L^2(A)) \cap C_c(\hat{A})$ is a uniformly dense subset of $C_c(\hat{A})$.

Proof. We notice that $\eta \in C_c(\hat{A}) \subseteq C_0(\hat{A}) \cong C^*(A)$. Since this isomorphism is the Fourier transform, $\exists f \in C^*(A)$ such that $\hat{f} = \eta$. Let ϕ_n be a Dirac net, and let $\psi_n = \phi * \phi^*$. Since $\text{supp}(\eta)$ is compact, ψ_n converges uniformly to 1 on this set. Let ψ_δ be such that $\|\hat{\psi_\delta} - 1\|_{\text{supp}(\eta)} < \delta$. Take $\zeta \in C_c^+(A)$ such that $\xi = \zeta * \zeta^*$ has $\hat{\xi} \geq 1$ on $\text{supp}(\eta)$. Define:

$$f_1 = f * (\psi_\delta - \delta\xi) \qquad f_2 = f * (\psi_\delta + \delta\xi)$$

Taking the Fourier transform, we have:

$$\hat{f}_1 = \eta(\hat{\psi_\delta} - \delta\hat{\xi}) \qquad \hat{f}_2 = \eta(\hat{\psi_\delta} + \delta\hat{\xi})$$

It is now clear that $\text{supp}(\hat{f}_i) \subseteq \text{supp}(\eta)$. Next, we see that:

$$\hat{f}_1 = \eta(\hat{\psi_\delta} - \delta\hat{\xi}) \leq \eta(1 + \delta - \delta) = \eta$$
$$\hat{f}_2 = \eta(\hat{\psi_\delta} + \delta\hat{\xi}) \geq \eta(1 - \delta + \delta) = \eta$$
$$\widehat{f_2 - f_1} = \widehat{f * 2\delta}\xi = 2\delta(\eta\hat{\xi})$$

The remaining assertions are clear. □

We know that $\hat{A}$ must have a Haar measure, since it is an LCA group. However, we can now give a characterization of this measure, using functions defined on A. We shall - somewhat suggestively - call this the *Plancherel measure*.

Construction 5.6 (Plancherel Measure)**.** Let $\eta \in C_c(\hat{A})$ be real-valued. Then,

$$I(\eta) := \sup\{f(1) \mid f \in C_0^*(A); \hat{f} \leq \eta\} = \inf\{f(1) \mid f \in C_0^*(A); \hat{f} \geq \eta\}$$

I is well defined by Lemma 5.5, and is clearly a positive linear functional. We extend I to all of $C_c(\hat{A})$ by linearity. I is a Haar integral, and so we have characterized the measure on $\hat{A}$.

Now, we are almost ready to prove the Pontryagin Duality theorem. First, we shall define a mapping between A and its double dual $\hat{\hat{A}}$; this is identical to the canonical injection that exists into any double dual object. This map will furnish the isomorphism we seek.

Definition 5.7. The *Pontryagin map* is defined as $\delta : A \to \hat{\hat{A}} := (x \mapsto (\chi \mapsto \chi(x)))$

We shall prove a limited version of the Inversion Formula, which we shall use in our proof of Pontryagin Duality.

Theorem 5.8 (Inversion Formula)**.** *Let $f \in C_0^*(A)$ be such that $\hat{f}$ is compactly supported. Then,*

$$f(x) = \hat{\hat{f}}(\delta(x^{-1}))$$

Proof.

$$\hat{\hat{f}}(\delta(x^{-1})) = \int_{\hat{A}} \hat{f}(\chi)\overline{\chi(x^{-1})}d\chi = \int_{\hat{A}}\int_A f(y)\overline{\chi(y)\chi(x^{-1})}dyd\chi$$

$$= \int_{\hat{A}}\int_A f(y)\overline{\chi(x^{-1}y)}dyd\chi = \int_{\hat{A}} \widehat{L_{x^{-1}}f}(\chi)d\chi$$

The Plancherel measure tells us this is equal to $L_{x^{-1}}f(1) = f(x)$. □

We are now ready for the Pontryagin Duality theorem.

Theorem 5.9 (Pontryagin Duality)**.** *The Pontryagin map is an isomorphism.*

Proof. It is clear that δ is a homomorphism. Continuity follows as if $x_i \to 1$, $\chi \in \hat{A}$, $\delta(x_i)(\chi) = \chi(x_i) \to \chi(1) = \delta(1)(\chi)$. Thus, we have pointwise convergence of $\delta(x_i)$ to $\delta(1)$. On a compact subset of $\hat{A}$, we see that there is uniform convergence, so δ is continuous. To show injectivity, assume there is some z such that $\delta(z) = 1$. Then, $\forall \chi \in \hat{A}, \chi(z) = 1$. Thus, we compute

$$\widehat{L_z f}(\chi) = \int_A f(z^{-1}x)\overline{\chi(x)}dx = \int_A f(x)\overline{\chi(x)\chi(z)}dx = \hat{f}(\chi)$$

But this contradicts the injectivity of the Fourier transform, unless $z = 1$. Thus, δ is injective.

Next, we verify that the image of the Pontryagin map is dense in $\hat{\hat{A}}$. We assume for contradiction that there exists some open $U \subset \hat{\hat{A}}$ such that $U \cap \delta(A) = \emptyset$. Let f be a real-valued continuous function with support in some compact subset of U. By Lemma 5.5, there exists $\phi \in C_0^*(\hat{A})$ such that $f - \hat{\phi} < \epsilon$, and $\text{supp}(\hat{\phi}) \subset \text{supp}(f) \subset U$. Thus, $\hat{\phi}(\delta(A)) = 0$. Since $\phi \in C_0^*(\hat{A})$, by a consequence of Lemma 5.5 there exists a sequence $\psi_n \in C_0^*(A) \cap L^2(A)$ such that $\hat{\psi_n} \to \phi$. Since the Fourier transform is injective, we have $\hat{\hat{\psi}}_n \to \hat{\phi}$. By the Inversion Formula, $\psi_n(x) = \hat{\hat{\psi}}_n(\delta(x^{-1}))$. This contradicts $\hat{\phi}(\delta(A)) = 0$, so the image of δ must be dense.

We now wish to show that the preimage of any compact $K \subset \hat{\hat{A}}$ is compact. We choose some $\psi \in C_0^*(\hat{A})$ such that $\hat{\psi} \in C_c(\hat{\hat{A}})$ is nonnegative, and $\hat{\psi} \geq 1$ on K. By a similar argument as before, we choose a sequence $\phi_n \in C_0^*(A) \cap L^2(A)$ such that $\hat{\phi_n} \in C_c(\hat{A})$ and $\hat{\hat{\phi}}_n \to \hat{\psi}$ uniformly. Fix ϵ, n such that $\|\hat{\hat{\phi}}_n - \hat{\psi}\|_{\hat{A}} < \epsilon$. Note that $\phi_n \in C_0(A)$, so there is some compact set X outside of which $|\phi_n| < 1 - \epsilon$. We note that for $k \in K$, $\hat{\hat{\phi}}_n(k) > 1 - \epsilon$, so by the Inversion Formula, if k is in the image of δ, $\delta^{-1}(k) \subset X$. δ is continuous, so $\delta^{-1}(K)$ is closed, and thus compact.

By a lemma of point-set topology, any continuous map between Hausdorff spaces such that the preimages of compact sets are compact is a closed map, and so we see that the Pontryagin map is a homeomorphism. □

There are two immediate consequences of the Pontryagin Duality theorem. The first concerns $C_0^*(A)$. We expect the Fourier transform to provide an isomorphism from it to $C_0^*(\hat{A})$, and indeed it does:

Proposition 5.10. *The Fourier transform is an isometric isomorphism* $\mathcal{F} : C_0^*(A) \to C_0^*(\hat{A})$

We are also able to extend the Inversion Formula to its more traditional form:

Proposition 5.11 (Inversion Formula)**.** *Let* $f \in L^1(A)$ *be such that* $\hat{f} \in L^1(\hat{A})$ *Then,* f *is continuous, and*

$$f(x) = \hat{\hat{f}}(\delta(x^{-1}))$$

These statements are easily verified, see [1] for proofs.

We shall conclude with a proof of the Plancherel Theorem.

Theorem 5.12 (Plancherel Theorem)**.** *For a given Haar measure on* A*, there is a measure on* $\hat{A}$ *such that for* $f \in L^1 \cap L^2$*,*

$$\|f\|_2 = \|\hat{f}\|_2$$

Further, the Fourier transform extends to a canonical unitary equivalence $L^2(A) \cong L^2(\hat{A})$*.*

Proof. Consider $\widehat{f * f^*}$. If this is integrable, then we have the following chain of equalities:

$$\|f\|_2^2 = \widehat{\widehat{f * f^*}}(\delta(1)) = \int_{\hat{A}} \widehat{f * f^*} = \int_{\hat{A}} \hat{f}\hat{f}^* = \int_{\hat{A}} \hat{f}\overline{\hat{f}} = \|\hat{f}\|_2^2$$

The first equality comes from the inversion formula. Note that if $f \in L^1 \cap L^2$, then $f * f^* \in C_0 \cap L^1$, and so is in $C_0^*(A)$, and so the Plancherel measure tells us that $\widehat{f * f^*} \in L^1(\hat{A})$.

Lemma 5.5 gives us that $\mathcal{F}(C_0^*(A) \cap L^2(A))$ is dense in $L^2(\hat{A})$, and so it clearly can be extended into a unitary equivalence. □

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