FINITELY GENERATING THE MAPPING CLASS GROUP WITH DEHN TWISTS

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Abstract. We begin with a discussion of the mapping class group of a closed orientable surface, then show that the mapping class group is finitely generated by Dehn twists. We then apply these results to a discussion of Heegaard decompositions of 3-manifolds, and refer briefly to their role in the proof of the Lickorish-Wallace Theorem.

Contents

1. Introduction to the Mapping Class Group

Definition 1.1. Let S be an oriented surface.Then the *mapping class group* of S, denoted $\mathrm{MCG}(S)$, is the group of homeomorphisms of S considered up to isotopy, where the group operation is composition. We call the group of homeomorphisms which preserve orientation $\mathrm{MCG}^+(S)$.

We can see that $\mathrm{MCG}^+(S)$ is a subgroup of $\mathrm{MCG}(S)$ of index 2.

Lemma 1.2. *Suppose $\phi : D^n \to D^n$ restricts to the identity on the boundary. Then ϕ is isotopic to id_{D^n}.*

Proof. Let

$$\bar{\phi} : \mathbb{R}^n \to \mathbb{R}^n$$

$$v \mapsto \begin{cases} \phi(v) & v \in D^n \\ v & \text{otherwise} \end{cases}$$

Then, $\bar{\phi}$ is continuous, and we have an isotopy ϕ_t, where $\phi_t(v) = t\bar{\phi}(\frac{v}{t})$, for $t \in [0, 1]$. We can see that $\phi_1(v) = \bar{\phi}(v)$. Additionally, as $t \to 0$, $\bar{\phi}(\frac{v}{t}) \to \frac{v}{t}$ for more v, so ϕ_t approaches the identity function. This is an isotopy between $\bar{\phi}$ and $\mathrm{id}_{\mathbb{R}^n}$. This will also hold when we restrict both functions to D^n, and so $\phi \simeq \mathrm{id}_{D^n}$. □

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Example 1.3. Mapping Class Group of S^1:

Let $\phi : S^1 \to S^1$ be an element of $\mathrm{MCG}^+(S^1)$. Visualize S^1 in the complex plane, and let $\alpha = \arg(\phi(1))$. Let $\phi_t(z) = e^{-i\alpha t}\phi(z)$. Since $\phi_1(z) = e^{-i\alpha}\phi(z) = \phi(1)^{-1} \circ \phi(z)$ and $\phi_0(z) = \phi(z)$, this is an isotopy, and $\phi_1(1) = 1$. Define $\psi : [0,1] \to [0,1]$ by cutting S^1 at $\phi(1) \in S^1$, so $s^{2\pi i \psi(t)} = \phi_1(e^{2\pi i t})$. Then, since ϕ is orientation preserving, $\psi(0) = 0$ and $\psi(1) = 1$. Then, by Lemma 1.2, $\psi \simeq \mathrm{id}_{[0,1]}$, so, $\phi_1 \simeq \mathrm{id}_{S^1}$, so $\phi \simeq \mathrm{id}_{S^1}$. Thus, every orientation-preserving homeomorphism of S^1 is isotopic to the identity, which means that $\mathrm{MCG}^+(S^1)$ is trivial.

Example 1.4. Mapping Class Group of D^2:

Let ϕ be an orientation-preserving homeomorphism of D^2, and let $\varphi : S^1 \to S^1$ be the restriction of ϕ to S^1. Then, since ϕ is orientation-preserving, φ is orientation-preserving, and since $\mathrm{MCG}^+(S^1)$ is trivial (as we just showed in Example 1.3), this implies that there exists some isotopy $\varphi_t : S^1 \to S^1$ such that $\varphi_0 = \varphi$ and $\varphi_1 = \mathrm{id}_{S^1}$. Let

$$\psi : D^2 \to D^2$$
$$(r,\theta) \mapsto \begin{cases} (r, \varphi(\theta)) & r \geq \frac{1}{2} \\ \frac{1}{2}\phi(2r,\theta) & r \leq \frac{1}{2} \end{cases}$$

A similar argument to the one used in the proof of Lemma 1.2 shows us that $\phi \simeq \psi$.

Define

$$\psi_t : D^2 \to D^2$$
$$(r,\theta) \mapsto \begin{cases} (r, \varphi_{2t(r-\frac{1}{2})})(\theta) & r \geq \frac{1}{2} \\ \frac{1}{2}\phi(2r,\theta) & r \leq \frac{1}{2} \end{cases}$$

Thus, $\psi_1|_{\partial D^2} = \mathrm{id}_{\partial D^2}$, so we can apply Lemma 1.2 and get that $\psi_1 \simeq \mathrm{id}_{D^2}$, so $\phi \simeq \mathrm{id}_{D^2}$, and consequently, $\mathrm{MCG}^+(D^2)$ is trivial.

2. Simple Closed Curves and Dehn Twists

Definition 2.1. Let X be a topological space. A *curve* γ on X is a continuous function $\gamma : [0,1] \to X$. A curve is called *simple* if it is injective when restricted to $(0,1)$. A curve is called *closed* if $\gamma(0) = \gamma(1)$.

We only require a simple curve to be injective on the interior of $[0,1]$ so that we can consider curves that are both simple and closed. A simple closed curve can also be thought of as an injective continuous mapping of S^1 into some topological space. On a surface, a simple closed curve is a loop which does not intersect itself.

Definition 2.2. Let γ_1, γ_2 be simple closed curves on a closed, oriented surface S, and let $[\gamma_i]$ denote the homotopy class of γ_i. Then $\#(\gamma_1 \cap \gamma_2) = \min\{|\gamma_1' \cap \gamma_2'| : \gamma_1' \in [\gamma_1], \gamma_2' \in [\gamma_2]\}$. We call this the *intersection number* of γ_1 and γ_2.

A point of intersection of two simple closed curves is considered positive if the orientation of the intersection agrees with the orientation of the surface. Otherwise, we say it is a negative point of intersection.

Now, let $A = S^1 \times [0,1]$. Define the twist map

$$\tau : A \to A$$
$$(\theta, t) \mapsto (\theta + 2\pi t, t)$$

For a simple closed curve γ on a closed oriented surface S, a tubular neighborhood N of γ is homeomorphic to A. Let $\phi : A \to N$ be such a homeomorphism.

Definition 2.3. Let

$$\tau_\gamma : S \to S$$
$$x \mapsto \begin{cases} \phi \circ \tau \circ \phi^{-1}(x) & x \in N \\ x & x \notin N \end{cases}$$

We call τ_γ a *Dehn twist about* γ. This is illustrated in Figure 1.

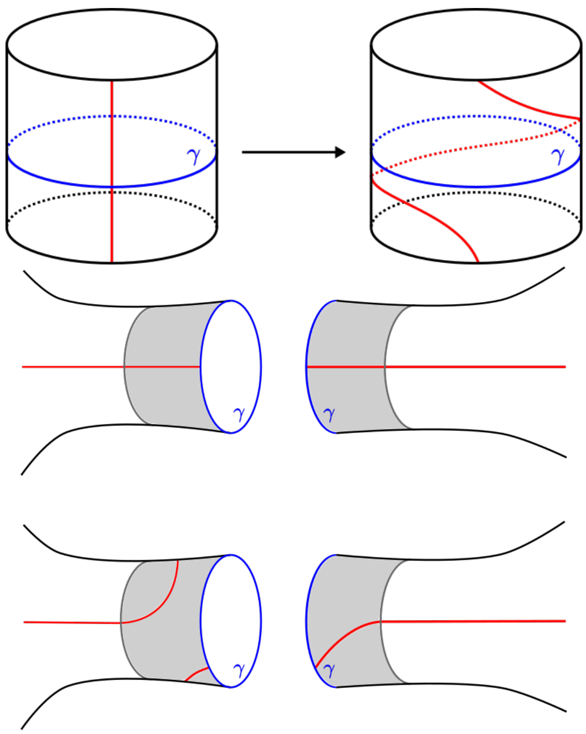


FIGURE 1. Two illustrations of a Dehn twist. In both pictures, γ is the horizontal blue curve. The grey region in the bottom figure is the tubular neighborhood of γ which is twisted. We can visualize a Dehn twist by imagining that the red line "wraps around" γ inside that tubular neighborhood, but is unchanged elsewhere.

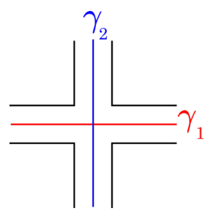
The map τ_γ depends on ϕ and N; however, the isotopy class of τ_γ does not. What's more, the isotopy class of τ_γ does not depend on the chosen representative from the isotopy class of γ. Since τ_γ is a homeomorphism, we can say that $\tau_{[\gamma]} \in \mathrm{MCG}^+(S)$. For the sake of simplicity, we can denote $\tau_{[\gamma]}$ as τ_γ.

Let S be a closed, oriented surface. Let T_S be the subgroup of $\mathrm{MCG}^+(S)$ generated by Dehn twists about simple closed curves in S.

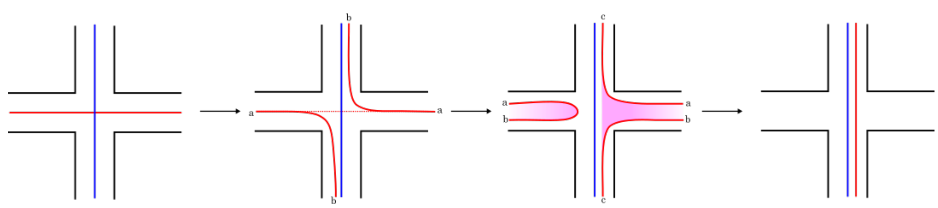
Definition 2.4. Let γ_1, γ_2 be simple closed curves on S. We say that $\gamma_1 \sim \gamma_2$ if there exists some $\rho \in T_S$ such that $\rho(\gamma_1) \simeq \gamma_2$.

Lemma 2.5. *Let γ_1, γ_2 be simple closed curves on a surface S. If $\#(\gamma_1 \cap \gamma_2) = 1$, then $\gamma_1 \sim \gamma_2$.*

Proof. If γ_1 and γ_2 intersect at exactly one point, then we can visualize their intersection as such:



The black lines demarcate tubular neighborhoods of γ_1 and γ_2. Then, we can see that $\tau_{\gamma_1}(\tau_{\gamma_2}(\gamma_1)) \simeq \gamma_2$:



The first arrow indicates a Dehn twist about γ_2, the second indicates a Dehn twist about γ_1, and the third indicates an isotopy along the pink region, which moves down γ_1 until it runs parallel to γ_2.

Since $\tau_{\gamma_1} \circ \tau_{\gamma_2} \in T_S$, $\gamma_1 \sim \gamma_2$. □

Remark 2.6. If γ_1, γ_2 are oriented, then $\tau_{\gamma_1}(\tau_{\gamma_2}(\gamma_1)) = \pm\gamma_2$. We can reverse the directions of the twists to ensure that the orientation of γ_2 is preserved.

Lemma 2.7. *Suppose that γ and δ are simple closed curves on S. Then there exists some $\gamma' \sim \gamma$ such that exactly one of the following statements is true:*

(1) $\gamma' \cap \delta = \emptyset$
(2) $\gamma' \cap \delta = \{p, q\}$, where p and q are points of intersection of opposite sign

Additionally, in both cases, for a tubular neighborhood V of δ, $\gamma' \cap (S \setminus V) \subset \gamma \cap (S \setminus V)$.

Proof. We induct on $\#(\gamma \cap \delta)$. Let $k = \#(\gamma \cap \delta)$. If $k = 0$, then we can take $\gamma' = \gamma$ and (1) holds. If $k = 1$, then $\gamma \sim \delta$ by Lemma 2.5. By "pushing δ out" or "pushing δ off," we see that δ is isotopic of a simple closed curve γ' which is disjoint from δ, and (1) is satisfied. If $k = 2$ and the intersection points have opposite signs, then (2) is satisfied with $\gamma' = \gamma$. Now, assume the result hold for all intersection numbers i such that $i < k$ and that γ and δ have k intersection points. Then, we can reduce to the following two cases:

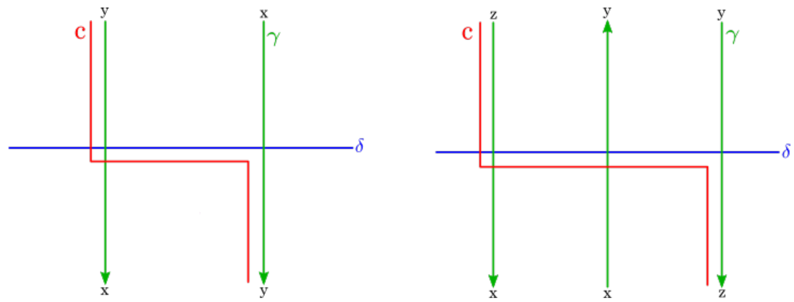
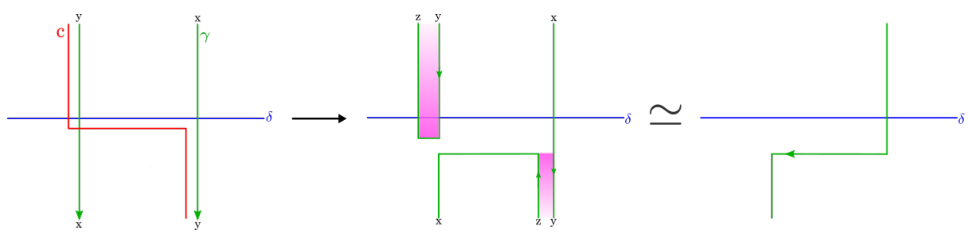


FIGURE 2. The case on the left is a curve which intersects δ twice, where both intersection points have the same sign. The case on the right is a curve which intersects δ three times, and the signs of the intersection points alternate. The simple closed curve c stays within a tubular neighborhood of γ outside of the diagram in both pictures. The letters indicate how γ connects.

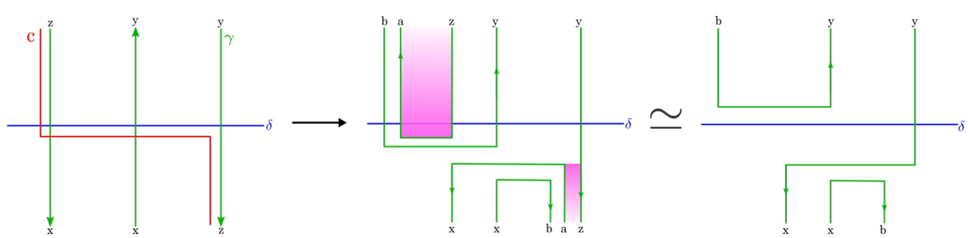
Note that we construct the curve c in both cases. We will first look in detail at the case where there are two adjacent intersection points with the same sign.

If we perform a Dehn twist of γ about c, we get:



Note that we isotope along the pink region to get from the second step to the third. Thus, k intersection points can be reduced to $k-1$ intersection points, and we can apply the inductive hypothesis.

If there are no two adjacent intersection points of the same sign, and (1) or (2) is not already satisfied, then there must be three intersection points of alternating sign, which is the case on the right in Figure 2. Then, we can perform a Dehn twist of γ around c:



Therefore, γ is similar to a curve which intersects δ at $k-2$ points, and we can apply our inductive hypothesis.

Outside of the diagram, in both cases, the final curve follows the same path as our original γ, and so satisfies the second condition of (2).

□

Corollary 2.8. *Suppose $\gamma, \delta_1, \ldots, \delta_k$ are simple closed curves such that all δ_i are pairwise disjoint. Then there exists a $\gamma' \sim \gamma$ such that the previous lemma holds for γ' and all δ_i.*

Proof. We induct on the number k of pairwise-disjoint simple closed curves. Our base case, $k = 1$, is simply the previous lemma. Our inductive hypothesis assumes there exists a $\gamma' \sim \gamma$ such that either (1) or (2) holds for all $i < k$. Lemma 2.7 tells us that there exists a $\gamma'' \sim \gamma'$ such that either (1) or (2) holds for δ_k, and that $\gamma'' \cap (S \setminus v(\delta_k)) \subset \gamma' \cap (S \setminus v(\delta_k))$. Since $\delta_k \cap \delta_i = \emptyset$ for all $i < k$, then $\#(\gamma'' \cap \delta_i) \leq \#(\gamma' \cap \delta_i) \leq 2$. If $\#(\gamma'' \cap \delta_i) = 0$ or 2, then (1) or (2) holds. If $\#(\gamma'' \cap \delta_i) = 1$, then $\gamma'' \sim \delta_i$, which reduces to the same case from the proof of the previous lemma. □

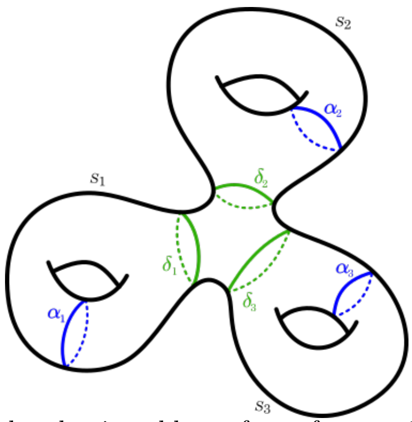


FIGURE 3. A closed orientable surface of genus 3, with disjoint simple closed curves α_i, δ_i drawn in. By the Classification of Surfaces, any closed orientable surface of genus $g > 0$ can be thought of as the connected sum of g tori.

Proposition 2.9. *Let S be a closed orientable surface of genus g with the system of disjoint simple closed curves as shown on Figure 3. Then, for all simple closed curves γ in S, there exists some $\gamma' \sim \gamma$ such that $\gamma' \cap \alpha_i = \emptyset$.*

Proof. Note that we can view each handle S_i in the plane, according to Figure 4.

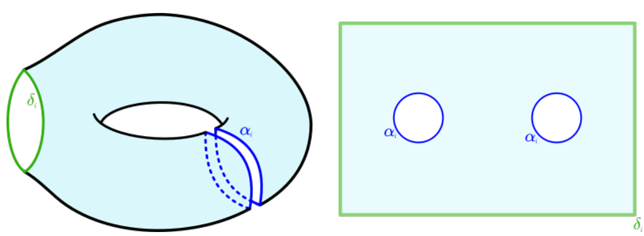
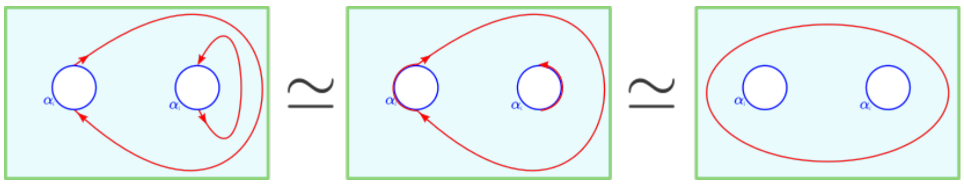


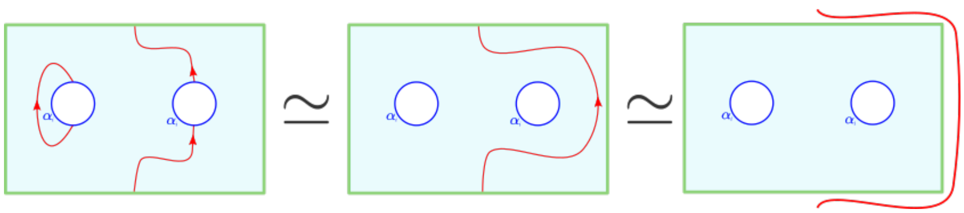
FIGURE 4. An illustration of how S_i can be viewed in the plane, with δ_i as the boundary of a rectangle and α_i as two holes, which are identified to each other.

Apply the previous corollary to show that there exists some $\gamma' \sim \gamma$ such that (1) or (2) holds for all α_i and δ_i. Consider $\gamma' \cap S_i$ for all $i \leq g$. If γ' does not intersect α_i, we are done, so assume it intersects α_i twice. If $\gamma' \cap \delta_i = \emptyset$, we get the following diagram when we view S_i in the plane:



As we can isotope the smaller red loop on the first picture so that it lies on α_i; note that, in the middle picture, the right half of the right circle is identified to the left half of the left circle. Then, γ' rests entirely on S_i for this specific i, and does not intersect any other α_j on S.

If γ' intersects δ_i twice, then we instead get:



As we can again isotope the small loop until it rests on α_i, and then isotope it off. □

3. Generating the Mapping Class Group

Proposition 3.1. *Let S be a closed orientable surface of genus g, and define a system of simple closed curves α_i, δ_i as seen in Figure 3. Then, for any mapping class ϕ of S, there exists some element ψ of T_S such that $\psi \circ \phi|_{\alpha_i} = id_{\alpha_i}$ for all $i \leq g$.*

Proof. Assume inductively that there exists some $\psi_k \in T_S$ such that $\psi_k \circ \phi|_{\alpha_i} = \mathrm{id}_{\alpha_i}$ for all $i \leq k$. Then, let $\gamma = \psi_k \circ \phi(\alpha_{k+1})$. Since $\gamma \cap \alpha_i = \psi_k \circ \phi(\alpha_{k+1}) \cap \psi_k \circ \phi(\alpha_i) = \emptyset$ for all $i \leq k$, the previous proposition gives us that there exists some $\rho \in T_{S \setminus \{\alpha_i, \ldots, \alpha_k\}}$ such that $\rho(\gamma) \cap \alpha_i = \emptyset$ for all i such that $k+1 \leq i \leq g$.

Cut along all $\alpha_1, \ldots, \alpha_g$ to get S^2 with $2g$ holes, as illustrated in Figure 5, and let $\gamma' = \rho(\gamma)$.

If γ' separates the two circles labeled α_{k+1}, then let β_{k+1} be as shown on Figure 5. Since they only intersect once (as simple closed curves on S), then $\gamma' \sim \beta_{k+1} \sim \alpha_{k+1}$. Then, we can find some $\sigma \in T_{S \setminus \{\alpha_i, \ldots, \alpha_k\}}$ such that $\sigma(\gamma') = \alpha_{k+1}$ and $\sigma|_{\gamma'} : \gamma' \to \alpha_{k+1}$ is orientation preserving. Since $\mathrm{MCG}^+(S^1)$ is trivial, we can isotope in such a way that $\sigma \circ \rho \circ \psi_k \circ \psi|_{\alpha_{k+1}} = \mathrm{id}_{\alpha_{k+1}}$.

For the second case, we make an argument using homology classes of simple closed curves. If γ' does not separate the two circles labeled α_{k+1}, we can observe that $[\alpha_{k+1}] \notin \langle [\alpha_1], \ldots, [\alpha_k] \rangle$, where $[\alpha_i]$ is the homology class of α_i. Thus, $[\gamma] \notin \langle [\alpha_1], \ldots, [\alpha_k] \rangle$, and there must exist some $j < k+1$ such that γ' separates the

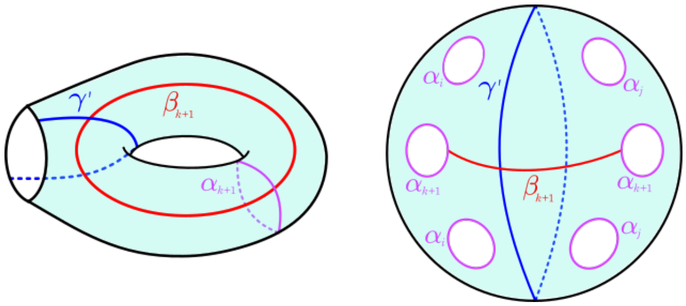


FIGURE 5. The first case in the proof of Proposition 3.1, where γ' separates the two α_{k+1}, as it would appear on both the sphere and the handle.

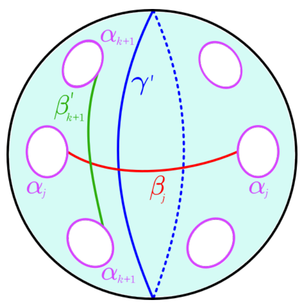


FIGURE 6. An illustration of the second case for the proof of Proposition 3.1.

circles labeled α_j, so $\gamma' \sim \beta_j \sim \beta'_{k+1} \sim \alpha_{k+1}$, where β'_{k+1} is as shown on Figure 6 and we argue as above.

□

This means that we can assume that there exists some mapping class of S which restricts to the identity on α_i. If we cut S along all α_i to get $\bar{S}$ (which is S with $2g$ holes), we can get a mapping class of $\bar{S}$ which restricts to the identity on the boundary of $\bar{S}$, as the boundary is simply all α_i. This is illustrated in Figure 7 for the case of a closed orientable surface of genus 3.

Definition 3.2. Let $\mathrm{MCG}^+(\bar{S}, \partial)$ be the set of homeomorphisms of $\bar{S}$ which restrict to the identity on $\partial\bar{S}$ up to isotopy. Let $T_{\bar{S}}$ be the subgroup of $\mathrm{MCG}^+(\bar{S}, \partial)$ generated by Dehn twists about simple closed curves in $\bar{S}$.

Lemma 3.3. *Given $\phi \in MCG^+(\bar{S}, \partial)$ and γ as drawn in Figure 8, there exists $\psi \in T_{\bar{S}}$ such that $\psi \circ \phi(\gamma) \cap \gamma$ is the set of endpoints of γ.*

Proof. We prove by induction on $\#(\phi(\gamma) \cap \gamma)$. Let p be the first intersection point on γ. Then, the sign of the intersection is either positive or negative. This gives us two cases.

In the first case, we have:

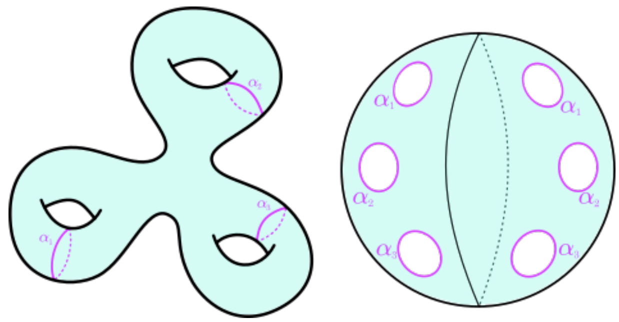


FIGURE 7. For a closed orientable surface of genus 3, we can cut along all α_i to get a sphere with 6 holes in it; the α_i are the boundary of this sphere. We call this sphere $\bar{S}$.

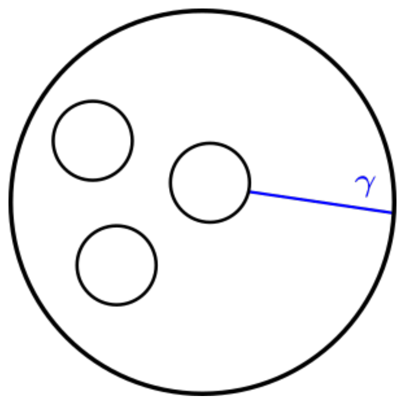
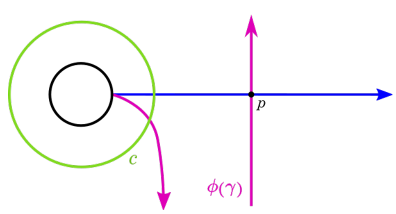
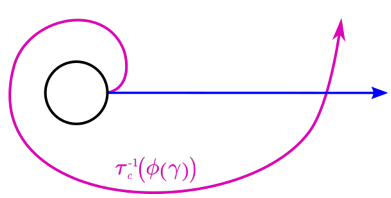


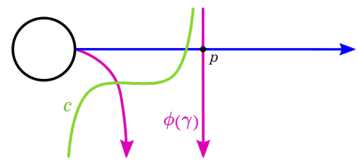
FIGURE 8. The sphere with 4 holes, visualized as a disk with 3 holes.



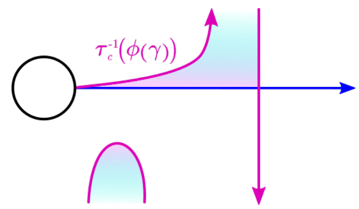
We perform a Dehn twist of $\phi(\gamma)$ about c as is drawn in the figure above.



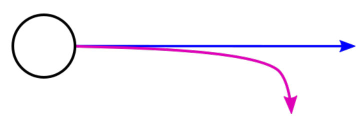
Now, this is equivalent to the second case, which is:



Again, we perform a Dehn twist about c as is drawn in the figure above. This gives us the following picture:



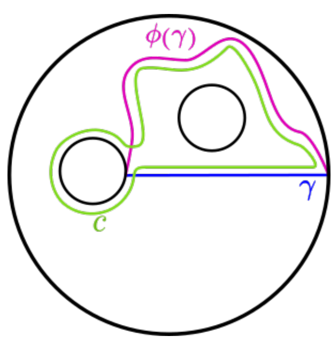
We isotope along the shaded region and see that this is isotopic to



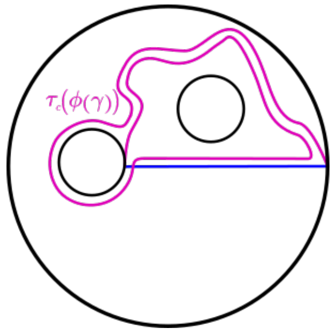
So, case 1 reduces to case 2, and we can see that we can eliminate all non-endpoint intersection points of γ and $\phi(\gamma)$ with Dehn twists. □

Lemma 3.4. *Given $\phi \in MCG^+(\bar{S}, \partial)$, there exists $\psi \in T_{\bar{S}}$ such that $\psi \circ \phi|_\gamma = id_\gamma$.*

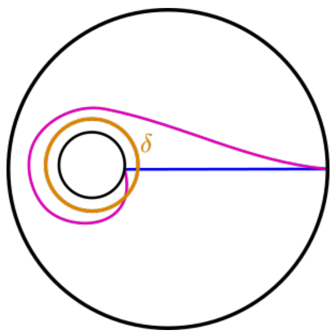
Proof. By Lemma 3.3, we may assume that $\phi(\gamma) \cap \gamma =$ the endpoints of γ. Take c as shown on the diagram, and then twist $\phi(\gamma)$ around c.



Once we take $\tau_c^{-1}(\phi(\gamma))$, we get



This is isotopic to



And we can see that $\tau_\delta^{-1}(\tau_c(\phi(\gamma)))$ is isotopic to γ. □

Proposition 3.5. *Let $\bar{S_n}$ be the disk with n punctures. Then $MCG^+(\bar{S_n}, \partial) = T_{\bar{S_n}}$*

Proof. We induct on the number of punctures. If $n = 0$, then we have $\mathrm{MCG}^+(D^2)$, which is trivial. Assume it holds for a disk with $n-1$ holes. If we have $\psi \in \mathrm{MCG}^+(\bar{S}_n, \partial)$, we can pick some $\phi \in T_{\bar{S}_n}$ such that $\phi' = \psi \circ \phi$ restricts to the identity on γ. Then, we can cut along γ to get $\phi' : \bar{S}_{n-1} \to \bar{S}_{n-1}$, which is isotopic to a composition of Dehn twists by our inductive hypothesis. □

At last, we can prove our main result.

Theorem 3.6. *$MCG^+(S) = T_S$*

Proof. Given a mapping class ϕ of S, there exists some $\psi \in T_S$ such that $\psi \circ \phi$ restricts to the identity on α_i. Then, we can cut along α_i to get $\phi' = \psi \circ \phi : \bar{S}_{2g} \to \bar{S}_{2g}$, which is a composition of Dehn twists by the previous proposition, and since it restricts to the identity on the boundary, this is true when we identify the holes α_i as well. □

4. Applications to 3-manifolds

A *handlebody* is a 3-manifold whose boundary is a closed orientable surface. We say that H_g is a handlebody of genus g, or the handlebody bounded by a g-torus. This also holds if we accept the convention that a sphere is a closed orientable surface of genus 0. For clarity's sake, we will use Σ_g to denote a genus g surface.

Definition 4.1. Let $\phi : \Sigma_g \to \Sigma_g$ be a homeomorphism. Then $Y_\phi := H_g \cup_\phi H_g$ is a 3-manifold, and we say that $H_g \cup_\phi H_g$ is a *Heegaard splitting* for Y_ϕ.

Proposition 4.2. *If ϕ_0 is isotopic to ϕ_1, then $Y_{\phi_0} \simeq Y_{\phi_1}$.*

Proof. We can observe that $H_g \simeq H_g \cup_f \Sigma_g \times I$, where $f : \Sigma_g \times 0 \xrightarrow{\text{id}} \Sigma_g$. So, $Y_{\phi_i} \simeq (H_g \cup_f \Sigma_g \times I) \cup_{\phi_i} H_g$. If Φ is an isotopy between ϕ_0 and ϕ_1, then

$$F : Y_{\phi_0} \to Y_{\phi_1}$$
$$(x,t) \mapsto \begin{cases} \text{id} & (x,t) \in H_g \\ \Phi^{-1}(\phi_0(x),t) & (x,t) \in \Sigma_g \times I \end{cases}$$

is a homeomorphism. □

Thus, the boundary homeomorphism for a Heegaard decomposition need only be specified up to isotopy; in other words, Y_ϕ depends only on the mapping class $\phi \in \text{MCG}(\Sigma_g)$.

Lickorish used the fact that every closed orientable 3-manifold admits a Heegaard splitting, along with the fact that the mapping class group is generated by Dehn Twists, to prove that any closed orientable 3-manifold can be obtained by removing a finite number of disjoint solid tori from S^3 and gluing them back in a different way.[2]

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References

[1] Benson Farb and Dan Margalit, *A Primer on Mapping Class Groups.* Princeton University Press, 2012.

[2] W. B. R. Lickorish, A Representation of Orientable Combinatorial 3-Manifolds. *The Annals of Mathematics*, vol. 76, no. 3, 1962, p. 531., doi:10.2307/1970373.

[3] Allen Hatcher, *Notes on Basic 3-Manifold Topology.* http://www.math.cornell.edu/ hatcher/3M/3Mfds.pdf 2007.

[4] Jake Rasmussen, 3-manifolds. Cambridge University, Cambridge. Notes taken by MurphyKate Montee. 2013.