THE STONE-WEIERSTRASS THEOREM

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Abstract. This paper proves the Stone-Weierstrass Theorem for arbitrary topological spaces. It briefly discusses basic point set topology and then discusses continuous functions and function spaces in more depth before finally proving the Stone-Weierstrass Theorem itself.

Contents

1. Introduction

One useful theorem in analysis is the Stone-Weierstrass Theorem, which states that any continuous complex function over a compact interval can be approximated to an arbitrary degree of accuracy with a sequence of polynomials. Indeed, in his book on analysis for undergraduates, Rudin has a specially marked section dedicated to this theorem and its generalization: that any continuous function over a compact set can be approximated with a sequence from a set of functions that behave like polynomials in certain ways. However, while Rudin proves this only over metric spaces, the Stone-Weierstrass Theorem holds true for any topological space. This paper proves the Stone-Weierstrass Theorem for arbitrary topological spaces. The first part of the paper briefly reviews the basics of point set topology before exploring continuous functions and metric spaces. It then investigates function spaces before moving on to prove various facts about operations on continuous complex functions. Finally, we define an algebra and prove the Stone-Weierstrass Theorem for compact topological spaces.

2. Review of Topology

Generalizing the Stone-Weierstrass Theorem requires a broader idea of topology than simply the metric topology. Indeed, many different types of topologies exist,

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and all share certain useful properties. Before we continue, though, we should remind ourselves what a topology is.

Definition 2.1. Topology

A topology $\mathcal{T}$ over a set X is a collection of subsets of X such that:

1) $\emptyset$ and X are in $\mathcal{T}$.
2) The intersection of any finite number of elements in $\mathcal{T}$ is in $\mathcal{T}$.
3) Arbitrary unions of elements are in $\mathcal{T}$.

We typically refer to sets in $\mathcal{T}$ as the open sets of X and to X as a topological space.

One construct that will prove useful to us while proving the Stone-Weierstrass Theorem is a Basis, defined below:

Definition 2.2. Basis

A basis $\mathcal{B}$ for a topology on a set X is a collection of subsets of X such that:

1) For all x in X, there is an element of $\mathcal{B}$ containing x.
2) If B_1 and B_2, both in $\mathcal{B}$, both contain a point x in X, there exists a third B_3 in $\mathcal{B}$ such that $B_3 \subset B_1 \cap B_2$ and B_3 contains x.

We can build a topology $\mathcal{T}$ from any basis by defining the topology as the collection of sets that are the union of an arbitrary number of basis elements. We call this topology the topology **generated** by the basis.

The proof that this method does create a topology follows clearly from the definition of a basis.

Bases have a number of useful properties and will eventually be quite valuable in proving that a function between two topological spaces is continuous. However, before we talk about continuous functions, we should review a useful concept when discussing topological spaces: **compactness**. Defining this concept requires discussing the notion of an open cover and subcover.

Definition 2.3. Open Cover

Let A be a subset of a topological space X. A collection $\mathcal{G}$ of subsets G of X is an open cover of A if it consists solely of open subsets of X and for each x in A, x is in some G. A subcover is a subset of an open cover that is itself an open cover.

We are now ready to define compactness:

Definition 2.4. Compact

Let A be a subset of a topological space X. A is compact if every open cover of A has a finite subcover.

Compactness is the key to generalizing the Stone-Weierstrass Theorem for arbitrary topological spaces. However, the notion of **closed** sets will also be necessary. A reminder of this definition follows:

Definition 2.5. Closed Set

Let X be a set with a topology $\mathcal{T}$. A subset of X, C, is closed in X if the complement of C is open, that is, $X - C \in \mathcal{T}$.

Remember that as a direct consequence of this definition and DeMorgan's Laws, finite unions and arbitrary intersections of closed sets are closed.

One construct that interacts very nicely with closed sets are limit points. The notion of limit points will be used in the Stone-Weierstrass Theorem.

Definition 2.6. Limit Point

Let X be an arbitrary topological space and let A be a subset of X. A point x in X is a limit point of A if for all open sets U containing X, U and A have at least one point other than x in common, that is, $U \cap A - \{x\} \neq \emptyset$.

If a topology is generated by a basis, we need only prove that for every basis element B that contains x, $B \cap A - \{x\} \neq \emptyset$ to prove that x is a limit point. The proof for this follows trivially from the definition of a basis.

One place where closed sets and limit points interact well is in the closure of a set, which is defined below.

Definition 2.7. Closure

Let A be a set in an arbitrary topological space, X. The closure of A, denoted $\overline{A}$, is A together with its limit points.

Closures have numerous useful properties, including that they are closed. The proof for this follows and is contingent on the fact that a point x is in the closure of a set A if and only if for every open set U containing that point, $U \cap A \neq \emptyset$.

Lemma 2.8. *Closures are Closed: Let $\overline{A}$ be the closure of A, a subset of an arbitrary topological space X. Then $\overline{A}$ is closed in X.*

Proof. Take the union of all open sets U such that $U \cap A = \emptyset$.

$\bigcup U \subset X - \overline{A}$: Let x be in $\bigcup U$. Then there exists at least one open set U containing x such that $U \cap A = \emptyset$. Thus x is not in $\overline{A}$. Therefore x is in $X - \overline{A}$.

$X - \overline{A} \subset \bigcup U$: Let x be in $X - \overline{A}$. Then there exists an open set U such that $U \cap A = \emptyset$. Thus x is in $\bigcup U$.

Therefore $X - \overline{A}$ is equal to the union of some number of open sets. Thus $X - \overline{A}$ is open, so $\overline{A}$ is closed. □

Another useful property of closures is that a set is closed if and only if it is equal to its closure. As a result of this, a set is closed if and only if it contains all of its limit points.

Another property that is quite useful in discussing the Stone-Weierstrass Theorem is Density, defined below:

Definition 2.9. Density

Let X be a subset of a topological space. A subset A of X is dense in X if every point of X is either in A or is a limit point of A.

An alternative way to state this definition is to say that A is dense in X if $\overline{A} = X$. Density can be used to estimate otherwise incalculable things; it also has a number of other uses in fields such as combinatorics. The Weierstrass Theorem, the precursor of the Stone-Weierstrass Theorem, proves the density of polynomials in the space of all bounded continuous functions over a bounded closed interval. Stone's generalization of Weierstrass' Theorem proves the density of certain polynomial-like sets of continuous functions in the set of all continuous functions over a compact set.

The reader may also remember we have a special type of topology that we often apply to the Cartesian Product of sets, the Product Topology. Before defining this topology, though, it makes sense to define the projection function:

Definition 2.10. Projection Function

Let X and Y be sets. The projection function, π, is a function from $X \times Y$ to X or Y such that $\pi(x,y) = x$ if π goes to X or $\pi(x,y) = y$ if π goes to Y for all $(x,y) \in X \times Y$.

If π goes to X, we refer to it as π_X. Similarly, if π goes to Y, we tend to refer to it as π_Y.

Now we are ready to define the product topology:

Definition 2.11. Product Topology

Let X and Y be topological spaces. The product topology on $X \times Y$ is the topology generated by the basis of sets $A \times B \subset X \times Y$ such that A is open in X and B is open in Y.

While the product topology will not be the focus of this paper, we will use it to prove that certain functions are continuous.

3. Continuous Functions

Anyone reading this paper will be familiar with the concept of continuous functions. This concept forms the core of this paper and deserves a more in-depth treatment than we gave basic topology.

Definition 3.1. Continuous

Let X and Y be two arbitrary topological spaces. A function $f : X \to Y$ is continuous if the preimage of every open set of Y is open in X.

This definition, while powerful, is also quite clunky. With this in mind, the rest of this section will be dedicated to finding ways to prove continuity and construct continuous functions without directly using this definition.

First we will show that if the topology on Y has a basis, we need only show that the preimage of every basis element of Y is open in X.

Lemma 3.2. *Let X and Y be topological spaces, and let $\mathcal{B}$ be a basis that generates the topology on Y. Then a function $f : X \to Y$ is continuous if and only if the preimage of every basis element of Y is open in X.*

Proof. If: the preimage of every basis element is open in X, then f is continuous:

Let U be an open set of Y. Then U is equal to the union of some number of basis elements B. For all B, $f^{-1}(B)$ is open in X. The arbitrary union of open sets of X is open in X, so $f^{-1}(U) = f^{-1}(\bigcup B) = \bigcup f^{-1}(B)$ must be open as well.

Only if: The function $f : X \to Y$ is continuous, then the preimage of each basis element of Y is open in X:

Basis elements of Y are open, so since f is continuous, the preimage of a basis element of Y must be open as well. □

This lemma makes proving continuity much easier, though it can still be difficult. The easiest way to prove that a function is continuous is often to prove that it is **continuous at each point** in its domain.

Definition 3.3. Continuous at a Point

Let X and Y be arbitrary topological spaces. A function $f : X \to Y$ is continuous at a point x in X if for all open sets U containing $f(x)$ in Y, there is an open set W containing x in X such that $f(W) \subset U$.

It is also useful to note that if for every basis element B containing $f(x)$, there is an open set W containing x in X such that $f(W) \subset B$, f is continuous at x. This is a trivial corollary following straight from the definition of a topology generated by a basis.

The proof that if a function is continuous at every point of its domain if and only if it is continuous follows:

Lemma 3.4. *Let X and Y be arbitrary topological spaces. Then a function $f : X \to Y$ is continuous if and only if it is continuous at all points x in X.*

Proof. If: The function f is continuous at all points x in X, then f is continuous.

Let U be an open set of Y. If there is no x such that $f(x)$ is in U, then the preimage of U is empty, which is open in X.

However, if there does exist an x such that $f(x)$ is in U, then because f is continuous at x, there exists an open set W containing x in X such that $f(W) \subset U$.

Thus the preimage of U can be written as the union of all such W, so the preimage of U is open in X.

Thus f is continuous.

Only If: f is continuous, then f is continuous at all points x in X.

Fix x in X. Let U be any open set in Y containing $f(x)$.

Then, because f is continuous, the preimage of U is open in X.

Since $f(f^{-1}(U)) \subset U$, f is continuous at x for all x in X. $\square$

The fact that being continuous and being continuous at every point are the same will turn out to be quite helpful when proving that certain functions are continuous. We will now prove some other basic lemmas about continuous functions, including the Composition Lemma. These will be useful in proving the Stone-Weierstrass Theorem.

Lemma 3.5. *Composition Lemma: Let X, Y, and Z be arbitrary topological spaces and let $f : X \to Y$ and $g : Y \to Z$ be continuous functions between these sets. Then $g \circ f : X \to Z$ is continuous.*

Proof. Let U be an open set of Z. Then the preimage of U under g, $g^{-1}(U)$, is open in Y because g is continuous.

Because $g^{-1}(U)$ is open in Y, the preimage of $g^{-1}(U)$ under f,
$f^{-1}(g^{-1}(U)) = (g \circ f)^{-1}(U)$, is open in X because f is continuous.

Therefore, the preimage of U under $g \circ f$ is open in X, so $g \circ f$ is continuous. $\square$

There is one more lemma involving continuous functions to a product topology that will be useful in proving the Stone-Weierstrass Theorem. Before proving it through, we should prove that the projection function, π, is continuous.

Lemma 3.6. *The projection function $\pi_A : A \times B \to A$ is continuous.*

Proof. Let U_A be open in A. Then $\pi_A^{-1}(U_A) = U_A \times B$, which is open in $A \times B$. Thus π_A is continuous.

Note that the logic of this proof also shows that $\pi_B : A \times B \to B$ is continuous. $\square$

Now we can dive into the main proof:

Lemma 3.7. *Let $f : X \to Y \times Z$ be defined by $f(x) = (g(x), h(x))$, where $g : X \to Y$ and $h : X \to Z$. Then f is continuous if and only if g and h are continuous.*

Proof. If: g and h are continuous then f is continuous.

We will prove this by pointwise continuity. Fix x in X and let $f(x) = (u, w)$. Then, all basis elements containing (u, w) are of the form $U \times W$, such that U and W are open sets in Y and Z containing u and w respectively.

Since g and h are continuous, $g^{-1}(U)$ is open in X and $h^{-1}(W)$ is open in X. Note that $g(x) = u$ and $h(x) = w$, so x is in both preimages.

Therefore, $g^{-1}(U) \cap h^{-1}(W)$ is open in X and contains x.

Since $f(g^{-1}(U) \cap h^{-1}(W)) \subset U \times W$, f is pointwise continuous at x.

Since x was chosen arbitrarily, f is continuous.

Only If: f is continuous, then g and h are continuous.

Without loss of generality, we will prove this for g.

For all x in X, $g(x) = \pi_Y \circ f(x)$, so $g = \pi_Y \circ f$. By Lemma 5.8, π_Y is continuous and from our assumption, f is continuous. Therefore, by the composition Lemma, g is continuous. $\square$

4. Metric Topology

Now that we have discussed topological spaces generally, it is time for us to move on to studying the type of topological space most used in analysis: the metric topology. Since the reader is almost certainly familiar with metric topology, this section will be brief.

A metric is a notion of distance formally defined like so:

Definition 4.1. Metric

Let X be a set. A metric is a function $d : X \times X \to \mathbb{R}$ such that:

1) d is symmetric: $d(x, y) = d(y, x)$ for all x, y in X.
2) d is positive definite: $d(x, y) \geq 0$ for all x, y in X and $d(x, y) = 0$ if and only if $x = y$.
3) The triangle inequality holds: $d(x, y) \leq d(x, z) + d(z, y)$ for all x, y, and z in X.

The most common metric on the real numbers is subtraction within absolute value. All readers should be familiar with the fact that for all x, y, and z in $\mathbb{R}$, $|x-y| = |y-x|$, $|x-y| \geq 0$, $|x-y| = 0$ if and only if $x = y$, and $|x-y| \leq |x-z|+|z-y|$.

One useful property of a metric is that it induces a topology on the set we put the metric over. We call this topology the metric topology. Before we define it, though, it will be useful to define the notion of **open balls** in a metric.

Definition 4.2. Open Ball

Let X be a set with a metric d and let x be a point in X. An open ball of radius $\varepsilon > 0$ centered at x, denoted $B(x, \varepsilon)$, is the set $\{y \in X | d(x, y) < \varepsilon\}$.

Open balls have a number of nice properties, chiefly that if y is in an open ball $B(x, \varepsilon)$ centered at x, there exists an open ball $B(y, \delta)$ centered at y such that $B(y, \delta) \subset B(x, \varepsilon)$. The proof for this follows:

Lemma 4.3. *Let $B(x, \varepsilon)$ be a ball of radius ε centered at x. Then, for all y in $B(x, \varepsilon)$, there exists an open ball centered at y, $B(y, \delta)$, such that $B(y, \delta) \subset B(x, \varepsilon)$.*

Proof. Define δ to be $\varepsilon - d(x, y)$. This is positive because $d(x, y) < \varepsilon$.

Now let $z \in B(y, \delta)$. Then $d(y, z) < \varepsilon - d(x, y)$, so $d(y, z) + d(x, y) < \varepsilon$. By the triangle inequality, $d(x, z) < \varepsilon$, so $z \in B(x, \varepsilon)$. $\square$

Now that we have defined open balls, we are ready to define the metric topology:

Definition 4.4. Metric Topology

Let d be a metric for a space X. The metric topology is the topology generated by the basis of all open balls $B(x, \varepsilon)$ in X.

Before continuing, we should make sure this definition makes sense by checking that this open ball basis is actually a basis.

Theorem 4.5. *The open ball basis is a basis.*

Proof. The first condition is trivially met, for all x in X we can always find an open ball centered at x.

For the second condition, consider two open balls, B_1 and B_2 such that y is in B_1 and B_2. By Lemma 6.3, we know that there exist δ_1 and δ_2 such that $B(y, \delta_1) \subset B_1$ and $B(y, \delta_2) \subset B_2$. Let δ be the minimum of δ_1 and δ_2. Then $B(y, \delta)$ will be a subset of both B_1 and B_2, so it will be a subset of their intersection as well.

This fulfills the second condition for a basis, so the open ball basis is a basis. $\square$

Now that we have a concept of distance, we will find it useful to create a notion of **boundedness**.

Definition 4.6. Bounded Subset

Let X be a metric space with a metric d. Let A be a subset of X. A is bounded if there exists some positive real number K such that, for all x and y in A, $d(x, y) \leq K$. In this case, we refer to A as a "bounded subset" of X.

The next concept we will cover in metric spaces is **completeness**. Before discussing completeness, though, we need to discuss **Cauchy Sequences** first.

Definition 4.7. Cauchy Sequence

Let X be a metric space. A Cauchy sequence is a function $a : \mathbb{N} \to X$ such that for all real $\varepsilon > 0$, there exists a N in the natural numbers such that if $m, n \geq N$, $d(a(m), a(n)) < \varepsilon$.

As a convention, we tend to write $a(n) = a_n$ for sequences. Now that we have a definition for Cauchy sequence, we can define what it means for a metric space to be complete:

Definition 4.8. Complete

A metric space X is complete if every Cauchy sequence $a : \mathbb{N} \to X$ converges to a point in X.

Complete metric spaces have a number of properties that prove quite useful in analysis. For example, complete subsets of a metric space are closed. We will see some more of these properties in the next section on function spaces. Examples of complete metric spaces include $\mathbb{R}$ and $\mathbb{C}$ with the usual metric applied to each of them (the usual metric for $\mathbb{C}$ will be defined next section).

There is one lemma we should prove involving complete metric spaces before moving on:

Lemma 4.9. *Let X be a complete metric space with metric d. Then, if K is a closed subset of X and we consider the metric topology induced on K by the metric d restricted to K, then K is a complete metric space.*

Proof. Suppose (k_n) is a Cauchy sequence in K. We know X is complete, so (k_n) converges to some x in X.

Since K is closed, K contains all of its limit points. Thus x is in K. Therefore K is complete. $\square$

5. Complex Function Spaces

Now that we have discussed the metric topology, we are ready to approach function spaces. Just as spaces like $\mathbb{R}^n$ are spaces of tuples, or lists of numbers, function spaces are spaces of functions from one set to another. This can be difficult to visualize at first, so we encourage the reader to slow down and consider the following examples if they are unfamiliar with the topic. One example of this a reader may be familiar with from linear algebra is the set of all polynomials with real coefficients. A very simple, if not particularly useful, function space is the space containing the zero function from the real line to itself. Indeed, any collection of functions from one set to another can be considered a function space.

In this paper, the function spaces we are most interested in are spaces of bounded continuous functions. These include the space of complex polynomials over a closed bounded interval in the real line as well as the sets of all continuous bounded complex and real functions over an arbitrary topological space. However, before diving into these more specific examples, we should briefly review the modulus function.

The modulus function $|\cdot| : \mathbb{C} \to \mathbb{R}$ for a complex number $c = a + ib$ is defined by $|c| = \sqrt{a^2 + b^2}$. The modulus has a number of useful properties which are easy to check: For c and λ in the complex numbers:

Proposition 5.1.

1) $|\lambda \cdot c| = |\lambda| \cdot |c|$
2) $|\lambda + c| = |c + \lambda|$, $|c| \geq 0$, *and* $|c| = 0$ *if and only if* $c = 0$.
3) $|\lambda + c| \leq |\lambda| + |c|$

It is also easy to show that $d(x, y) = |x - y|$ is a metric on $\mathbb{C}$. This modulus metric is the metric most often used on the complex numbers.

Now we are almost ready to discuss complex function spaces. However, although we have informally discussed spaces of bounded functions, we have technically not defined what it means for a function to be bounded:

Definition 5.2. Bounded Function

Suppose $f : X \to \mathbb{C}$ is a function. Then f is bounded if for all x in X, $|f(x)| < K$, where K is some fixed real number.

With that out of the way, we can now begin to discuss function spaces. We will begin our investigation of function spaces through as broad a lens as possible, and gradually telescope into the Stone-Weierstrass Theorem. The first function space we will look at is the set of all bounded complex functions over an arbitrary set.

It is possible for function spaces to be metric spaces, and, fortunately, this one is. Thus, our first substantive order of business will be to establish a sensible metric on our space:

Definition 5.3. Supremum Norm

Let X be a set and let $\mathcal{B}(X)$ be the set of bounded functions from X to the complex numbers. Define the supremum norm as $||f||_\infty = \sup\{|f(x)| : x \in X\}$.

Informally, this norm simply defines the "height" of a function as the furthest it gets from the origin of the complex plane. Now that we have a notion of "height" in our space we can easily establish a notion of distance, and, thus, a metric. We define the distance between two functions in this space as the supremum norm of one function minus the other, that is $d(f,g) = ||f - g||_\infty$. Note that we can add bounded complex functions together and multiply them by complex scalars and still get a bounded complex function. Now we need to show that this notion of distance does actually define a metric. We will refer to this notion of distance as the **supremum metric** .

Lemma 5.4. *The Supremum Norm Defines a Metric and* $|\lambda| \cdot ||f||_\infty = ||\lambda f||_\infty$.

The supremum norm defines a metric:

Proof. To determine whether the supremum norm defines a metric, we need to show that the supremum metric is symmetric, that it is always greater than or equal to zero, and that it obeys the triangle inequality. Throughout this proof, f, g, and h are functions in $B(X)$.

1) The supremum metric is symmetric: This property is clearly inherited from the symmetry of the modulus function.
2) The supremum metric is always greater than or equal to zero and is equal to zero if and only if the functions inputted are identical: This property is clearly inherited from the corresponding property of the modulus function.
3) The supremum metric obeys the triangle inequality: By Proposition 5.1, for all x in X, $|f(x) - g(x)| \leq |f(x) - h(x)| + |h(x) - g(x)|$. Thus,

$$\sup\{|f(x)-g(x)| : x \in X\} \leq \sup\{|f(x)-h(x)| : x \in X\}+\sup\{|h(x)-g(x)| : x \in X\}.$$

Therefore $||f - g||_\infty \leq ||f - h||_\infty + ||h - g||_\infty$.

□

$|\lambda| \cdot ||f||_\infty = ||\lambda f||_\infty$*:*

Proof. For all complex numbers λ and all f in $\mathcal{B}(X)$,

$$||\lambda f||_\infty = \sup\{|\lambda f(x)| : x \in X\} = \sup\{|\lambda|\cdot|f(x)| : x \in X\} = |\lambda| \sup\{|f(x)| : x \in X\} = |\lambda|\cdot||f||_\infty$$

□

Now that we have determined that the supremum norm defines a metric on $\mathcal{B}(X)$, the next logical question to ask is whether $\mathcal{B}(X)$ is a complete metric space. Fortunately, it is, as we will show in this next proof:

Theorem 5.5. $\mathcal{B}(X)$ *is a Complete Metric Space.*

Proof. This proof is rather long, so we will divide it into three steps. First, we will show that for any Cauchy sequence (f_n) in $\mathcal{B}(X)$, there is a function g such that for all x, $g(x) = \lim_{n\to\infty}(f_n(x))$. Then we will show that (f_n) does, in fact, converge to g. Finally, we will show that g is in $\mathcal{B}(X)$.

1) Finding g:
Let (f_n), $n \in \mathbb{N}$, be a Cauchy sequence in $B(X)$.
Then for all $\varepsilon > 0$, there exist a natural number N such that if $k, m \geq N$, then $||f_k - f_m||_\infty < \varepsilon$.
Therefore $\sup\{|f_k(x) - f_m(x)| : x \in X\} < \varepsilon$. Thus, for all x in X, $|f_k(x) - f_m(x)| < \varepsilon$. Therefore $(f_n(x))$ is a Cauchy sequence in the complex numbers for all x in X.
The complex numbers are complete, so $(f_n(x))$ converges to some c_x in $\mathbb{C}$ for all x in X.
Let $g : X \to \mathbb{C}$ be defined by $g(x) = c_x$.
2) Prove that (f_n) converges to g:
It is highly probable that g is the function that (f_n) converges to, but this still must be proved.
Fix $\varepsilon > 0$. We know that (f_n) is a Cauchy sequence, so we can find a natural number M such that for all $n, m \geq M$, $||f_n - f_m||_\infty < \frac{\varepsilon}{4}$.
Since $||f_M - g||_\infty$ is the supremum of all $|f_M(x) - g(x)|$, there exists an x such that

$$|f_M(x) - g(x)| > ||f_M - g||_\infty - \frac{\varepsilon}{4}. \tag{5.6}$$

Because $g(x)$ is the limit of $f_n(x)$ as n approaches infinity, we know that there exists a natural number N' such that for all natural numbers $n \geq N'$, $|g(x) - f_n(x)| < \frac{\varepsilon}{4}$.
Now let $N = \max(M, N')$. For all natural numbers $m \geq N$, we know from the triangle inequality that

$$|g(x) - f_M(x)| \leq |g(x) - f_m(x)| + |f_m(x) - f_M(x)| < |g(x) - f_m(x)| + \frac{\varepsilon}{4}. \tag{5.7}$$

Combining the inequalities (5.6) and (5.7), we obtain

$$|g(x) - f_m(x)| > ||f_M - g||_\infty - \frac{\varepsilon}{2}.$$

Because (f_n) is a Cauchy sequence and by the triangle inequality,

$$|g(x) - f_m(x)| + \frac{\varepsilon}{4} > ||f_m - f_M||_\infty + ||f_M - g||_\infty - \frac{\varepsilon}{2} \geq ||f_m - g||_\infty - \frac{\varepsilon}{2}. \tag{5.8}$$

Adding $\frac{\varepsilon}{2}$ to both sides of the inequality (5.8), we get:

$$|g(x) - f_m(x)| + \frac{3\varepsilon}{4} > ||f_m - g||_\infty$$

Because $|g(x) - f_n(x)| < \frac{\varepsilon}{4}$, we now have

$$\varepsilon > ||f_n - g||_\infty$$

for all $n \geq N$. Therefore (f_n) converges to g.
3) Prove that g is bounded and is thus in $\mathcal{B}(X)$:
Choose the real number 1. By part 2 of this proof, there exists a natural number N such that $||f_N - g||_\infty < 1$.
Because f_N is bounded, by the definition of the supremum norm there exists some real number K such that $||f_N(x)||_\infty < K$.
Thus, by the triangle inequality, $||g||_\infty < K + 1$, so g is bounded.
Thus g is in $\mathcal{B}(X)$.

□

We have now established that $\mathcal{B}(X)$ is a complete metric space. However, the Stone-Weierstrass theorem does not deal with the whole space of bounded functions, but rather the subset of that space made of continuous functions. The rest of the paper will deal with this space. Since we are now dealing with continuous functions, we need our domain to be a topological space. Thus, from this point forward, let X refer to a topological space rather than an arbitrary set.

Definition 5.9. $\mathcal{C}(X)$

$$\mathcal{C}(X) = \{f \in \mathcal{B}(X) | f \text{ is continuous}\}.$$

We will call $\mathcal{C}(X)$ the space of bounded continuous functions.

We will also refer to the set of all bounded continuous real functions over X. We will refer to this set as $\mathcal{C}_{\mathbb{R}}(X)$. All of our proofs for properties of $\mathcal{C}(X)$ are identical to the proof of those properties for $\mathcal{C}_{\mathbb{R}}(X)$.

$\mathcal{C}(X)$ has a number of useful properties, the first of which is that $\mathcal{C}(X)$ is a closed subset of $\mathcal{B}(X)$:

Lemma 5.10. *$\mathcal{C}(X)$ is closed in $\mathcal{B}(X)$.*

Proof. Let g be a limit point of $\mathcal{C}(X)$. Then for all $\varepsilon > 0$, there exists some function f in $\mathcal{C}(X)$ such that $||g - f||_\infty < \frac{\varepsilon}{3}$.

We will now prove that g is continuous using pointwise continuity.

Fix x in X and let $U_0 \subset \mathbb{C}$ be an open set containing $g(x)$.

By the definition of a basis, there exists an open ball centered at $g(x)$ of radius ε such that $B(g(x), \varepsilon) \subset U'$.

From the triangle inequality, we know that for all y in X,

$$|g(y) - g(x)| \leq |g(y) - f(y)| + |f(y) - f(x)| + |f(x) - g(x)|.$$

By the definition of the supremum norm and f, we know that

$$|g(y) - f(y)| + |f(y) - f(x)| + |f(x) - g(x)| < \frac{2\varepsilon}{3} + |f(y) - f(x)|.$$

Because f is continuous, there exists an open set $U \subset X$ such that for all y' in U, $|f(y') - f(x)| < \frac{\varepsilon}{3}$. Using this, we now have

$$|g(y') - g(x)| < \frac{2\varepsilon}{3} + |f(y') - f(x)| < \varepsilon$$

for all y' in U.

Therefore, there exists an open set $U \subset X$ such that $U \subset g^{-1}(B(g(x), \varepsilon))$. Thus for all x, g is pointwise continuous. Therefore g is continuous.

Thus g is in $\mathcal{C}(X)$, so $\mathcal{C}(X)$ contains all of its limit points. Thus $\mathcal{C}(X)$ is closed. □

Now that we know that $\mathcal{C}(X)$ is closed in terms of the supremum norm, it is quite easy to show that $\mathcal{C}(X)$ is complete.

Theorem 5.11. *$\mathcal{C}(X)$ is complete.*

Proof. By Theorem 5.5, $\mathcal{B}(X)$ is complete. By Lemma 5.10, we know that $\mathcal{C}(X)$ is closed in $\mathcal{B}(X)$. Thus, by Lemma 4.9, $\mathcal{C}(X)$ is complete. □

Knowing that $\mathcal{C}(X)$ is complete will be key in proving the Stone-Weierstrass Theorem. However, before we begin proving the Theorem, there are still quite a few lemmas about operations involving $\mathcal{C}(X)$ we need to prove first. Most of these involve proving that certain operations, such as function addition and multiplication, are continuous. However, we also need to be able to add functions in $\mathcal{C}(X)$ together or multiply them by scalars and end up with a function in $\mathcal{C}(X)$.[1] The next proof shows we can do just that:

Lemma 5.12. *$\mathcal{C}(X)$ is closed under function addition and scalar multiplication.*

Proof. Let f and g be functions in $\mathcal{C}(X)$ and let λ be a complex number.

1) $\mathcal{C}(X)$ is closed under function addition:
   To prove this, we will show that addition $+ : \mathbb{C} \times \mathbb{C} \to \mathbb{C}$ defined as $+((x, y)) = x + y$ is continuous and use the Composition Lemma to show that $f + g$ will be continuous as well. We will then show that $f + g$ is bounded.
   a) $+ : \mathbb{C} \times \mathbb{C} \to \mathbb{C}$ is continuous:
      Place the product topology on $\mathbb{C} \times \mathbb{C}$. Now, to prove pointwise continuity, fix c in $\mathbb{C}$ and $\varepsilon > 0$. We now have an open ball $B(c, \varepsilon)$ centered at c with a radius ε.
      Now consider the set $B(c, \frac{\varepsilon}{2}) \times B(0, \frac{\varepsilon}{2}) \subset \mathbb{C} \times \mathbb{C}$.
      Let (x, y) be a point in the basis element $B(c, \frac{\varepsilon}{2}) \times B(0, \frac{\varepsilon}{2})$ of $\mathbb{C} \times \mathbb{C}$.
      Then, by the triangle inequality, $| + (x, y)| < |c| + \varepsilon$.
      Thus $B(c, \frac{\varepsilon}{2}) \times B(0, \frac{\varepsilon}{2}) \subset +^{-1}(B(c, \varepsilon))$.
      Therefore $+$ is pointwise continuous for all (x, y) in $\mathbb{C} \times \mathbb{C}$, so $+$ is continuous.
   b) $f + g$ is continuous:
      We now know that f, g and $+$ are continuous. Since the composition of continuous functions is continuous, $f + g = +(f, g)$ is continuous as well.
   c) $f + g$ is bounded:
      Because f and g are bounded, we know that there exist K and L in the real numbers such that $|f(x)| \leq K$ and $|g(x)| \leq L$ for all x in X. By the triangle inequality, $|f(x) + g(x)| \leq |f(x)| + |g(x)| \leq K + L$. Therefore $f + g$ is bounded.
2) $\mathcal{C}(X)$ is closed under scalar multiplication:
   We will prove this by proving that scalar multiplication, $\lambda : \mathbb{C} \to \mathbb{C}$ defined by $\lambda(c) = \lambda c$ for all c in $\mathbb{C}$ is continuous by pointwise continuity and then applying the Composition Lemma. Fix c in $\mathbb{C}$, λ in $\mathbb{C}$, and $\varepsilon > 0$.
   a) $\lambda : \mathbb{C} \to \mathbb{C}$ is continuous:
      Consider the open ball $B(\lambda c, \varepsilon)$ in $\mathbb{C}$.
      Now consider the open ball $B(c, \frac{\varepsilon}{\lambda})$ in the preimage. For all y in $B(c, \frac{\varepsilon}{\lambda})$, $|\lambda y - \lambda c| < \varepsilon$. Thus $B(c, \frac{\varepsilon}{\lambda}) \subset \lambda^{-1}(B(c, \varepsilon))$, so λ is pointwise continuous for all $c \in \mathbb{C}$.
      Thus λ is continuous, so by the Composition Lemma, λf is continuous for all f in $\mathcal{C}(X)$.

[1] For those readers familiar with linear algebra, we are proving that $\mathcal{C}(X)$ is a linear subspace of $\mathcal{B}(X)$.

b) λf is bounded.
We know f is bounded, so there exists a K in $\mathbb{R}$ such that $|f(x)| \leq K$ for all x in X. Therefore, $|\lambda f(x)| = |\lambda||f(x)| \leq |\lambda|K$ for all x in X, so λf is bounded.

$\square$

6. Continuous Operations on Function Spaces

Now that we have established some ground rules for spaces of continuous bounded complex functions, the last step we need to go through before delving into the Stone-Weierstrass Theorem is to show that a number of operations on bounded functions are continuous. We also need to define an **algebra** and some other properties of collections of functions.

Although we already proved that $\mathcal{C}(X)$ is closed under function addition and scalar multiplication, we did not technically prove that function addition and scalar multiplication are themselves are continuous. This subtle distinction will be important when we eventually prove that the closure of an algebra is itself an algebra.

Lemma 6.1. $+ : \mathcal{B}(X) \times \mathcal{B}(X) \to \mathcal{B}(X)$ *defined by* $+(g, h) = g + h$ *is continuous.*

Proof. We'll prove this through pointwise continuity.

Fix (g, h) in $\mathcal{B}(X)$. Then $+(g, h) = f$ for some f in $\mathcal{B}(X)$. Now fix $\varepsilon > 0$. Consider the open ball $B(f, \varepsilon)$ centered at f in $\mathcal{B}(X)$.

By the definition of product topology, $B(g, \frac{\varepsilon}{2}) \times B(h, \frac{\varepsilon}{2})$ is open in $\mathcal{B}(X) \times \mathcal{B}(X)$.

Let (a, b) be in $B(g, \frac{\varepsilon}{2}) \times B(h, \frac{\varepsilon}{2})$.

Then, by the triangle inequality, $||a + b - f||_\infty < \varepsilon$, so $+(a, b)$ is in $B(f, \varepsilon)$.

Thus $B(g, \frac{\varepsilon}{2}) \times B(h, \frac{\varepsilon}{2}) \subset +^{-1}(B(f, \varepsilon))$. Therefore $+$ is pointwise continuous at all (g, h) in $\mathcal{B}(X) \times \mathcal{B}(X)$, so $+$ is continuous. $\square$

Now we will prove that scalar multiplication, $\cdot : \mathcal{B}(X) \times \mathbb{C} \to \mathcal{B}(X)$ is continuous:

Lemma 6.2. $\cdot : \mathcal{B}(X) \times \mathbb{C} \to \mathcal{B}(X)$ *defined by* $\cdot(f, \lambda) = \lambda f$ *is continuous.*

Proof. As usual, we'll prove this by pointwise continuity.

Let (f, λ) in $\mathcal{B}(X) \times \mathbb{C}$. Then $\cdot(f, \lambda) = \lambda f$.

Now fix $\varepsilon > 0$. We now have an open ball $B(\lambda f, \varepsilon)$.

By our definition of product topology, $B(f, \min(\sqrt{\varepsilon}, \frac{\varepsilon}{3|\lambda|})) \times B(\lambda, \min(\frac{\varepsilon}{3||f||_\infty}, \frac{\sqrt{\varepsilon}}{3}))$ is open in $\mathcal{B}(X) \times \mathbb{C}$.

From the definition of open ball, for all μ in $B(\lambda, \min(\frac{\varepsilon}{3||f||_\infty}, \frac{\sqrt{\varepsilon}}{3}))$ we know that $|\mu| < |\lambda| + \frac{\sqrt{\varepsilon}}{2}$.

Now based on the open balls defined above,

$$||\lambda f - \mu g||_\infty = ||\mu(f - g) + (\lambda - \mu)f||_\infty \leq |\mu| \cdot ||f - g||_\infty + |\lambda - \mu| \cdot ||f||_\infty <$$

$$|\lambda|||f - g||_\infty + \frac{\sqrt{\varepsilon}}{3}||f - g||_\infty + \frac{\varepsilon}{3} < \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon.$$

Therefore, $B(f, \min(\sqrt{\varepsilon}, \frac{\varepsilon}{3|\lambda|})) \times B(\lambda, \min(\frac{\varepsilon}{3||f||_\infty}, \frac{\sqrt{\varepsilon}}{3})) \subset \cdot^{-1}(B(\lambda f, \varepsilon))$. Thus $\cdot$ is continuous for all f in $\mathcal{B}(X)$, so $\cdot$ is continuous. $\square$

Having proven that function addition and scalar multiplication are continuous, we only need to prove prove that function multiplication is continuous and that an operation called the complex conjugation of a function is continuous before delving into algebras. However, doing this requires a definition of complex conjugate, which follows:

Definition 6.3. Complex Conjugate

Let c be a complex number. Then c is equal to $a + ib$, for some real numbers a and b. The complex conjugate of c, $\overline{c}$, is defined as $\overline{c} = a - ib$.

Note that it is clear from the definition of complex conjugate that $|c| = |\overline{c}|$ and that $\overline{x} - \overline{y} = \overline{x - y}$ for all complex numbers c, x, and y. It seems fairly obvious that this operation is continuous. This intuition is accurate, and we prove it formally below:

Lemma 6.4. *$\gamma : \mathcal{B}(X) \to \mathcal{B}(X)$ defined by $\gamma(f(x)) = \overline{f(x)}$ is continuous.*

Proof. As usual, we will prove this via pointwise continuity.

Fix f in $\mathcal{B}(X)$. We know that $\gamma(f) = \overline{f}$. Now fix $\varepsilon > 0$. Consider the open ball $B(\overline{f}, \varepsilon)$ in the codomain of γ, $\mathcal{B}(X)$.

For all g in $\gamma^{-1}B(\overline{f}, \varepsilon)$, $||\overline{g} - \overline{f}||_\infty < \varepsilon$.

For all x in X, $|\overline{g(x)} - \overline{f(x)}| = |\overline{g(x) - f(x)}| = |g(x) - f(x)|$.

Thus $|g(x) - f(x)| < \varepsilon$ for all x in X. Therefore, by the definition of the supremum norm, $||g - f||_\infty \leq \varepsilon$.

If $||g - f||_\infty = \varepsilon$, then by the reverse of the above argument, $||\overline{g} - \overline{f}||_\infty = \varepsilon$. This contradicts that $||\overline{g} - \overline{f}||_\infty < \varepsilon$, so $||g - f||_\infty < \varepsilon$.

Therefore $\gamma^{-1}B(\overline{f}, \varepsilon) \subset B(f, \varepsilon)$. We can also use the reverse of the above argument to show that $B(f, \varepsilon) \subset \gamma^{-1}B(\overline{f}, \varepsilon)$.

Then $\gamma^{-1}B(\overline{f}, \varepsilon) = B(f, \varepsilon)$. Thus $\gamma^{-1}B(\overline{f}, \varepsilon)$ is open in $\mathcal{B}(X)$. Since $\gamma(\gamma^{-1}B(\overline{f}, \varepsilon)) \subset B(\overline{f}, \varepsilon)$, γ is continuous at all f in $\mathcal{B}(X)$. Therefore, γ is continuous. $\square$

Now that we know that the taking the complex conjugate of a function is continuous, we only need to prove that function multiplication is continuous as well before moving on to algebras.

Lemma 6.5. *$* : \mathcal{B}(X) \times \mathcal{B}(X) \to \mathcal{B}(X)$ defined by $*(f, g)(x) = f(x)g(x)$ is continuous.*

The proof for this is close to identical to the proof for lemma 6.2 and is left as an exercise to the reader. We commonly denote $*(f, g)$ as fg.

Now we will define some terms used in the Stone-Weierstrass Theorem:

Definition 6.6. Algebra

A collection of complex functions on a set X, $\mathcal{A}$, is an algebra if for all functions f and g in $\mathcal{A}$ and all complex constants c, $f + g$, fg, and cf are in $\mathcal{A}$. That is to say, $\mathcal{A}$ is an algebra if it is closed under addition, function multiplication, and scalar multiplication. If $\mathcal{A}$ consists only of real functions, then $\mathcal{A}$ must be closed under scalar multiplication only for real scalars to be an algebra.

One example of an algebra that the reader will be familiar with is the set of all real or complex polynomials over a set. We would also like $\mathcal{C}(X)$ to be an algebra. We proved in Lemma 7.12 that $\mathcal{C}(X)$ is closed under function addition and scalar multiplication, so we need only confirm that $\mathcal{C}(X)$ is closed under function multiplication to prove that it is an algebra.

Lemma 6.7. *$\mathcal{C}(X)$ is closed under function multiplication.*

Proof. This is identical to the proof of Lemma 8.2 and is left as an exercise to the reader. □

Corollary 6.8. *$\mathcal{C}(X)$ is an algebra.*

Proof. Lemmas 7.12 and 8.7 prove this. □

Now that we have confirmed that $\mathcal{C}(X)$ is an algebra, we can move on to proving some general theorems about algebras of bounded functions. One nice but not immediately obvious feature of algebras is that an algebra's closure is itself an algebra. This will be useful in proving Stone's generalization of the Stone-Weierstrass Theorem and is the culmination of much of our work on continuous operations on function spaces.

Theorem 6.9. *Let $\overline{\mathcal{A}}$ be the closure of an algebra $\mathcal{A}$ of bounded functions. Then $\overline{\mathcal{A}}$ is a closed algebra.*

Proof. Let f and g be in $\overline{\mathcal{A}}$, and let c be a complex scalar.

The proof that $f+g$, fg, and cf are in $\overline{\mathcal{A}}$ are nearly identical, so we will only present the proof that $f+g$ is in $\overline{\mathcal{A}}$.

The operation $+$ is continuous, so for all open sets U containing $f+g$, there exist open balls $B(f,\varepsilon)$ and $B(g,\delta)$ such that

$$B(f,\varepsilon)\times B(g,\delta)\subset +^{-1}(U).$$

Since f and g are in $\overline{\mathcal{A}}$, there exist functions h and j in $\mathcal{A}$ such that (h,j) is in $B(f,\varepsilon)\times B(g,\delta)$.

Therefore, $h+j$ is in $U\cap\mathcal{A}$, so for all open sets U containing $f+g$, there is an element of $\mathcal{A}$ in U. Therefore $f+g$ is a limit point of $\mathcal{A}$, so $f+g$ is in $\overline{\mathcal{A}}$.

Repeating this proof for fg and cf proves that $\overline{\mathcal{A}}$ is an algebra.

Closures are closed, so $\overline{\mathcal{A}}$ is closed. Thus $\overline{\mathcal{A}}$ is a closed algebra. □

The uses of this theorem may not immediately be obvious, but will be quite useful in proving Stone's generalization of Weierstrass' Theorem. Before delving into the Theorem, though, we need to define two more terms and prove one more lemma:

Definition 6.10. Separating Points

Let $\mathcal{A}$ be a collection of functions from a set X to a set Y. $\mathcal{A}$ separates points on X if for all distinct points x_1 and x_2 in X, there exists a function f in $\mathcal{A}$ such that $f(x_1)\neq f(x_2)$.

This basically says that a collection of functions separates points if no two inputs share the same output for every function in the collection. We define this property, along with the next, to exclude certain cases of algebras that would otherwise invalidate the Stone-Weierstrass Theorem.

Definition 6.11. Vanishing

Let $\mathcal{A}$ be a collection of complex functions on a set X. Then $\mathcal{A}$ vanishes at a point x in X if for all functions f in $\mathcal{A}$, $f(x)=0$. If no such point exists, then we say that $\mathcal{A}$ vanishes at no point of X.

A collection of functions vanishes at no point of X if for every point of X, there is a function f in that collection such that $f(x) \neq 0$. Note that given a complex number and an algebra that vanishes at no point of X, we can find a function in that algebra that maps to that complex number via scalar multiplication. When you also add the condition that the collection of functions separates points, we end up with a fairly sensible and useful result: given any two complex constants and any two distinct points in X, we can find a function in the algebra that equals the first complex constant at the first point of X and the second complex constant in the second point of X. We will now prove this:

Lemma 6.12. *Suppose that $\mathcal{A}$ is an algebra of complex functions on a set X that separates points and vanishes at no point of X. Suppose that there exist distinct points x_1 and x_2 of X and let c_1 and c_2 be complex constants (or real constants if $\mathcal{A}$ consists solely of real functions). Then there exists a function f in $\mathcal{A}$ such that $f(x_1) = c_1$ and $f(x_2) = c_2$.*

Proof. Because $\mathcal{A}$ separates points and vanishes at no point of X, there exist functions g, h, and k in $\mathcal{A}$ such that $g(x_1) \neq g(x_2)$, $h(x_1) \neq 0$, and $k(x_2) \neq 0$.

Now let $u = gk - g(x_1)k$ and $v = gh - g(x_2)h$. Since $\mathcal{A}$ is an algebra, u and v are in $\mathcal{A}$. Also, note that $u(x_1) = v(x_2) = 0$ and that $u(x_2) \neq 0$ and that $v(x_1) \neq 0$.

Then $f = \frac{c_1 v}{v(x_1)} + \frac{c_2 u}{u(x_2)}$ is equal to c_1 at x_1 and c_2 at x_2, so our desired function is in $\mathcal{A}$. $\square$

7. Stone-Weierstrass Theorem

We are now ready to prove the Stone-Weierstrass Theorem. The first iteration of the theorem was discovered by Weierstrass and states that the set of all polynomials over any closed interval $[a, b]$ is dense in $\mathcal{C}([a, b])$. Note that the set of polynomials restricted to a closed interval is bounded, is an algebra, separates points, and vanishes at no point of $[a, b]$.

Theorem 7.1. *The set of all polynomials over a closed interval $[a, b] \subset \mathbb{R}$ is dense in $\mathcal{C}([a, b])$. The same applies to the set of all real polynomials over $[a, b]$ and $C_{\mathbb{R}}([a, b])$.*

Proof. This proof works by showing that for any continuous function f and any ε, we can construct a polynomial, P_n, such that $||f - P_n||_\infty < \varepsilon$.

Recall that continuous functions on closed intervals are uniformly continuous.

Let f be a function in $\mathcal{C}([a, b])$.

Without loss of generality, we can assume that $[a, b] = [0, 1]$. We can also assume that $f(0) = f(1) = 0$ because if we prove it in this case, consider the function $g(x) = f(x) - f(0) - x[f(1) - f(0)]$ for x in $[0, 1]$. Here $g(0) = 0$ and $g(1) = 0$. Because g is a limit point of the set of all polynomials, $f - g$ is a polynomial, $g + f - g = f$, and the closure of an algebra is an algebra, f is a limit point of the set of all polynomials.

This theorem requires a fair amount of set up before we even select our function. This preparation follows:

1) Set up:

   Let $Q_n(x) = c_n(1 - x^2)^n$ for all natural numbers n. We choose c_n such that $\int_{-1}^{1} Q_n(x)dx = 1$.

Now consider the function $h(x) = (1-x^2)^n - (1+nx^2)$. Then $h(0) = 0$ and $h' = -2nx(1-x^2)^{n-1} + 2nx$, which is positive on $[0, 1]$.

Thus $(1-x^2)^n \geq 1 - nx^2$ on $[0, 1]$.

From this, we have,

$$\int_{-1}^{1}(1-x^2)^n dx = 2\int_0^1 (1-x^2)^n dx \geq 2\int_0^{\frac{1}{\sqrt{n}}}(1-x^2)^n dx \geq 2\int_0^{\frac{1}{\sqrt{n}}}(1-nx^2)dx = \frac{4}{3\sqrt{n}} > \frac{1}{\sqrt{n}}.$$

Therefore, $c_n < \sqrt{n}$ because $1 = \int_{-1}^{1} c_n(1-x^2)^n dx > \frac{c_n}{\sqrt{n}}$.

Therefore, for all $\delta > 0$, $Q_n(x) < \sqrt{n}(1-\delta^2)^n$ for all x such that $|x|$ is in $[\delta, 1]$.

2) The polynomial P_n:

Set

$$P_n(x) = \int_{-1}^{1} f(x+t)Q_n(t)dt.$$

By a change of variables, we have

(7.2)

$$P_n(x) = \int_{-x}^{1-x} f(x+t)Q_n(t)dt = \int_0^1 f(t)Q_n(t-x)dt = \int_0^1 f(t)c_n(1-(t-x)^2)^n dt.$$

This last expression looks quite a bit like a polynomial in x. We claim it is.

It is clear that $c_n(1-x^2)^n$ is a polynomial in x.

Thus, for any polynomial $P(x)$, we need to show that

$$F(x) = \int_0^1 f(t)P(t-x)\,dt \tag{7.3}$$

is a polynomial in x as well.

$P(x)$ is a polynomial in x, so $P(t-x) = \sum_{i=0}^{n} a_i(t-x)^i$, where a_i is some constant.

Plugging this and the binomial expansion formula, $(t-x)^j = \sum_{i=0}^{j}\binom{j}{i}t^{j-i}x^i$, into equation 9.3, we have

$$F(x) = \sum_{j=0}^{N}\sum_{i=0}^{j} a_j \binom{j}{i} x^i \int_0^1 f(t)t^{j-i}\,dt,$$

which is a polynomial in x.

Thus by equation 9.2 P_n is a polynomial in x.

3) P_n is in $B(f, \varepsilon)$:

Now, fix $\varepsilon > 0$. We now have an open ball $B(f, \varepsilon)$ in $\mathcal{C}([a, b])$, or $\mathcal{C}_{\mathbb{R}}([a, b])$ if f is real.

Since f is uniformly continuous, there exists a $\delta > 0$ such that $\delta < 1$ and if $|y-x| < \delta$, then $|f(y) - f(x)| < \frac{\varepsilon}{2}$.

Now, let $M = ||f||_\infty$.

Because $|1-\delta^2| < 1$, there exists a natural number N such that

$$4M\sqrt{N}(1-\delta^2)^N < \frac{\varepsilon}{2}.$$

Since $\int_{-1}^{1} Q_n(x)dx = 1$, $Q_n(x) < \sqrt{n}(1-\delta^2)^n$, and $Q_n(x) > 0$,

$$|P_N(x) - f(x)| = |\int_{-1}^{1} [f(x+t) - f(x)]Q_N(t)dt| \leq \int_{-1}^{1} |f(x+t) - f(x)|Q_N(t)dt$$

$$\leq 2M \int_{-1}^{-\delta} Q_N(t)dt + \frac{\varepsilon}{2}\frac{-\delta}{\delta} Q_N(t)dt + 2M \int_{\delta}^{1} Q_N(t)dt \leq 4M\sqrt{N}(1-\delta^2)^n + \frac{\varepsilon}{2} < \varepsilon$$

for all x in $[0,1]$.

Thus P_N is in $B(f, \varepsilon)$, so f is in the closure of the set of all polynomials restricted to $[0,1]$. Therefore the set of all polynomials restricted to $[0,1]$ is dense in $\mathcal{C}([0,1])$. The same is also true for the set of all real polynomials restricted to $[0,1]$ and $\mathcal{C}_{\mathbb{R}}$.

□

This is the form in which Weierstrass initially discovered the Theorem, which Stone later expanded to all self-adjoint algebras that separate points and vanish nowhere. After all the work we did proving Weierstrass' original theorem, we only need a much weaker corollary of it to prove Stone's generalization.

Lemma 7.4. *The absolute value function, $|x| = x$ if $x \geq 0$ and $-x$ if $x < 0$, restricted to a closed interval $[a,b]$ is a limit point of all polynomials P such that $P(0) = 0$ restricted to $[a,b]$.*

Proof. By Weierstrass' Theorem, there exists a polynomial P_N in any open ball $B(|\cdot|, \frac{\varepsilon}{2})$ centered on $|\cdot|$.

Now, let $P'_N = P_N - P_N(0)$. Then, for all x in $[a,b]$, $|P_N(x) - |x|| < \frac{\varepsilon}{2}$.

Then $\varepsilon > |P_N(x) - |x|| + ||0| - P_N(0)| \geq |P_N(x) - P_N(0) - |x||$ for all x.

Thus, $||P_N - P_N(0) - |\cdot|||_\infty < \varepsilon$, so P'_N is in $B(|\cdot|, \varepsilon)$. □

This lemma will allow us to prove Stone's generalization of Weierstrass' Theorem, which follows in two parts. The first only applies to real algebras, the second applies to complex algebras as well.

Theorem 7.5. *Suppose that $\mathcal{A}$ is an algebra of real bounded continuous functions over a compact set X that separates points and vanishes at no point of X. Then $\mathcal{A}$ is dense in $\mathcal{C}_{\mathbb{R}}(X)$.*

Proof. Similarly to the proof of Weierstrass' Theorem, this proof requires a fair amount of set up before we choose our function.

An equivalent way to state the theorem is that the closure of $\mathcal{A}$, $\overline{\mathcal{A}}$, is $\mathcal{C}_{\mathbb{R}}(X)$.

The set up for this theorem has two sections, one of which establishes that for all g in $\overline{\mathcal{A}}$, $|g|$ is in $\overline{\mathcal{A}}$, and the second of which is that if g and h are in $\overline{\mathcal{A}}$, then $\max(g,h)$ and $\min(g,h)$ are in $\overline{\mathcal{A}}$ as well.

1) If g is in $\overline{\mathcal{A}}$, then $|g|$ is in $\overline{\mathcal{A}}$.

   Fix $\varepsilon > 0$ and consider $|\cdot|$. We now have an open ball $B(|\cdot|, \varepsilon) \subset \mathcal{C}_{\mathbb{R}}([a,b])$, where $a = -||g||_\infty$ and $b = ||g||_\infty$. By Lemma 7.4, there is a polynomial P such that $P(0) = 0$ and P is in $B(|\cdot|, \varepsilon)$.

   Since P is a polynomial that is zero at $x = 0$, $P(g) = \sum_{i=1}^{n} c_i g^i$ for some natural number n and real constants c.

   By Theorem 6.9, $\overline{\mathcal{A}}$ is a closed algebra, so, for all i, g^i is in $\overline{\mathcal{A}}$. Then $c_i g^i$ is in $\overline{\mathcal{A}}$ as well. Therefore, $\sum_{i=1}^{n} c_i g^i$ is in $\overline{\mathcal{A}}$. Thus $P(g)$ is in $\overline{\mathcal{A}}$.

Since P is in $B(|\cdot|, \varepsilon)$, $P(g)$ is in $B(|g|, \varepsilon)$.

Therefore $|g|$ is a limit point of $\overline{\mathcal{A}}$.

Closures are closed, so since $|g|$ is a limit point of $\overline{\mathcal{A}}$, $|g|$ is in $\overline{\mathcal{A}}$.

2) For all g and h in $\overline{\mathcal{A}}$, $\max(g, h)$ and $\min(g, h)$ are in $\overline{\mathcal{A}}$.

We know that $\max(g, h) = \frac{g+h}{2} + \frac{|g-h|}{2}$. Thus, since $\overline{\mathcal{A}}$ is an algebra and by part 1 of this theorem, $\max(g, h)$ is in $\overline{\mathcal{A}}$.

We also know that $\min(g, h) = \frac{g+h}{2} - \frac{|g-h|}{2}$, so, by the same logic as above, $\min(g, h)$ is in $\overline{\mathcal{A}}$.

Note that this can be extended through induction to show that the maximum or minimum of any finite number of functions in $\overline{\mathcal{A}}$ is in $\overline{\mathcal{A}}$.

We are now ready to dive into the heart of the theorem. This effectively works by "sandwiching" a function in $\overline{\mathcal{A}}$ inside any given ε ball around a continuous function f.

Now, fix f in $\mathcal{C}_{\mathbb{R}}(X)$ and $\varepsilon > 0$. We now have an open ball $B(f, \varepsilon) \subset \mathcal{C}_{\mathbb{R}}(X)$.

By Lemma 6.12, for all x in X, there exists a function g_x in $\overline{\mathcal{A}}$ such that $g_x(x) = f(x)$.

Let $h_x = \max(g_x, f)$. Since all h_x are continuous, there exists an open set U_x in X such that U_x contains x and $U_x \subset h_x^{-1}(B(f(x), \varepsilon))$ for all x in X. Here $B(f(x), \varepsilon)$ denotes an open ball in $\mathbb{R}$.

The collection of all these U_x is an open cover of X. Because X is compact, a finite number of these U_x, U_i, must cover X.

By choice of U_i, $h_i(U_i) \subset B(f(x_i), \varepsilon)$.

Now, take the minimum of all h_i. Since h_i were defined as $\max(g_i, f)$, for all x, $h_i(x) > f(x) - \varepsilon$.

Thus, for all x, $f(x) - \varepsilon < \min(h_i)(x) < f(x) + \varepsilon$.

Thus $||f - \min(h_i)||_\infty < \varepsilon$, so $\min(h_i)$ is in $B(f, \varepsilon)$.

Since by part 2, $\min(h_i)$ is in $\overline{\mathcal{A}}$, f is a limit point of $\overline{\mathcal{A}}$.

Finally, since closures are closed, f is in $\overline{\mathcal{A}}$. Therefore $\mathcal{A}$ is dense in $\mathcal{C}_{\mathbb{R}}(X)$. □

This theorem, sadly, does not hold for complex continuous algebras. However, it does hold for **self-adjoint** algebras. A self-adjoint algebra is an algebra $\mathcal{A}$ such that for all f in $\mathcal{A}$, $\overline{f}$ is in $\mathcal{A}$.

Theorem 7.6. *Suppose that $\mathcal{A}$ is a self-adjoint algebra of bounded complex functions over a compact set X that separates points and vanishes at no point of X. Then $\mathcal{A}$ is dense in $\mathcal{C}(X)$.*

Proof. Let $\mathcal{A}_R$ be the set of all real functions on X in $\mathcal{A}$.

If f is in $\mathcal{A}$, then $f = g + ih$, where g and h are real functions. Then $2g = f + \overline{f}$ and $2h = i\overline{f} - if$. Since $\mathcal{A}$ is a self-adjoint algebra, g and h are in $\mathcal{A}_R$.

If $x_1 \neq x_2$, then by Lemma 6.12, there exists a function f in $\mathcal{A}$ such that $f(x_1) = 1$ and $f(x_2) = 0$.

Then $g(x_1) = 1 \neq 0 = g(x_2)$, so $\mathcal{A}_R$ separates points.

$\mathcal{A}$ vanishes at no point of X, so for any x in X there exists f in $\mathcal{A}$ such that $f(x) \neq 0$. Then either f has a real component g such that $g(x) \neq 0$, or f has a non zero imaginary component h such that $ih(x) \neq 0$, so $h \neq 0$. Either way, for all x in X, there is a function g or h in $\mathcal{A}_R$ such that g or h is not zero at x, so $\mathcal{A}_R$ vanishes at no point in X.

Thus $\mathcal{A}_R$ fulfills the requirements of Theorem 7.5, so $\mathcal{A}_R$ is dense in the set of all real continuous functions on X.

Thus, if f is in $\mathcal{C}(X)$, since $f = g + ih$, and g and h are in $\overline{\mathcal{A}}_R$, then f is in $\overline{\mathcal{A}}$. Therefore $\mathcal{A}$ is dense in $\mathcal{C}(X)$. $\square$

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