STABLE GRAPHS

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Abstract. In "Regularity Lemmas for Stable Graphs" [1] Malliaris and Shelah apply tools from model theory to obtain stronger forms of Ramsey's theorem and Szemeredi's regularity lemma for "stable graphs," graphs which admit a uniform finite bound on the size of an induced sub-half-graph. This paper provides a background to the first theorem of that , an improved form of Ramsey's theorem for stable graphs without model theory as a prerequisite.

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1. Introduction

Ramsey theory studies the conditions under which order arises in mathematical structures. While Ramsey problems are often stated in the language of graphs, the connection to logic has been evident since F.P Ramsey's 1930 "On a Problem of Formal Logic" which proved the central theorem now known as Ramsey's theorem.

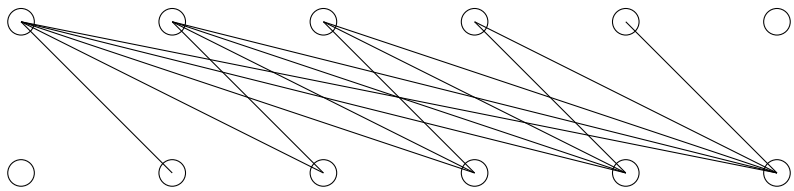
The theorem which will be developed in this paper was proved in a paper ultimately aimed at improving Szemerédi's Regularity Lemma for a certain class of graphs.

Theorem 1.1 (Szemerédi's Regularity Lemma)**.** *For every ϵ, m there exist $N = N(\epsilon, m)$, $m' = m(\epsilon, m)$ such that given any finite graph X of at least N vertices, there is a k with $m \leq k \leq m'$ and a partition $X = X_1 \cup \cdots \cup X_k$ satisfying:*

(1) $||X_i| - |X_j|| \leq 1$ for all $i, j \leq k$ and
(2) all but at most ϵk^2 of the pairs (X_i, X_j) are ϵ-regular.

While it was not known for some time after the original proof whether the ϵk^2 non ϵ-regular pairs were necessary, multiple researchers independently noted that the example of the half graph demonstrates that they are necessary [4]. In essence, the half graph cannot be partitioned nicely using the Szemeredi method. In [1] Malliaris and Shelah show that half graphs are essentially the only difficulty in

creating regular well-behaved partitions. First they show that Ramsey's theorem works better than usual for stable graphs (graphs which admit a bound on the size of an induced half-graph), then use this improved Ramsey theorem to construct a variety of improved regularity lemmas for stable graphs.



A half-graph of height 6

The idea for the work comes from a coincidence with model theory: the half graph witnesses the model theoretic "order property." Its presence aligns with model theoretic instability and in its absence the tools of stability theory can be applied.

The model theoretic notions involved in the construction of the regularity lemmas are beyond the scope of this paper. Rather, we will be concerned only with the improved Ramsey lemma, extending this lemma to hypergraphs, and the exposition uses a minimum of model theory.

After a brief description of Ramsey's theorem, we will examine the orderliness of random or "average" graphs. After a brief reflection on classes of orderly graphs and the Erdős Hájnal conjecture, we will discuss the orderliness of stable graphs and finally of stable hypergraphs.

2. Definitions

We define graphs in the model theoretic way.

Definition 2.1. An *hypergraph* is an ordered pair consisting of a set and a collection of truth valued formulas on that set (G, Δ) where all formulas in Δ are invariant under permutation of the variables and the interpretation Δ_G consists of all $X \subset G$ connected by an edge. For our purposes, hypergraphs are finite.

Note that while this definition may look different from the traditional hypergraph definition, it is equivalent.

Definition 2.2. An hypergraph is *m-uniform* if every forumla in the theory Δ is m-ary. We then write (G, R) for the graph. For our purposes, hypergraphs are uniform.

Definition 2.3. A 2-uniform hypergraph is called a "simple graph" or just a "graph." In this case we sometimes write the relation $R(x, y)$ as xRy.

Definition 2.4. An induced subgraph of a graph (G, R) is a graph (H, R) where $H \subseteq G$. If G has H as an induced subgraph, then G *induces* H. If G has no subgraph isomorphic to H, then G *omits* H.

Definition 2.5. An m regular hypergraph (G, R) is *homogeneous* if there exists a truth value t such that for all $X \subset G$, $|X| = m$, $R(X) = t$.

Homogeneous sets, in the language of simple graphs, are either complete or discrete.

Definition 2.6. A sequence $\langle a_i \colon i \in I \rangle$ in some model is *indiscernible* with respect to a collection of formulas Γ if for every n, and formula $\varphi(x_1, \ldots, x_n) \in \Gamma$, every $i_1 < \cdots < i_n \in I$, $j_1 < \cdots < j_n \in I$, $\varphi(a_{i_1} \ldots a_{i_n}) \iff \varphi(a_{j_1} \ldots a_{j_n})$.

Remark 2.7. In the theory of m-uniform hypergraphs, the m-ary edge relation with equality, an indiscernible sequence $\langle a_i \colon i < n \rangle$ is a either a complete or discrete induced subgraph graph.

Definition 2.8 (Privileged symbols)**.** Let $a, k, m, n < \omega$.

3. Order and Indiscernibles

The two central theorems of Ramsey theory are often phrased in the following way:

Theorem 3.1 ("Coloring Infinite Ramsey")**.** *Every k-edge-coloring of an infinite a-uniform complete hypergraph has an infinite monochromatic complete hypergraph as an induced subgraph.*

Theorem 3.2 ("Coloring Finite Ramsey")**.** *There exists $R = r(a, k, n) < \omega$ such that every k-edge-coloring of a complete a-uniform hypergraph on at least R vertices has a monochromatic complete graph of size n as an induced subgraph.*

In the two color case, we can think of one color as designating the edges in the graph and the other color designating the edges in the graph's complement, giving the following readings.

Theorem 3.3 ("Infinite Ramsey")**.** *Every infinite graph has an infinite homogeneous set.*

Theorem 3.4 ("Finite Ramsey")**.** *For all $s < \omega$ there exists $R(s)$ such that every graph with at least $R(s)$ vertices has a homogeneous set of size s.*

The relation between homogeneity and indiscernible sequences is already clear: for graphs and hypergraphs they are one and the same. In this way we can think of Finite Ramsey as showing the existence of a lower bound on the length of a edge-relation-indiscernible sequence of vertices.

Homogeneous sets are quite orderly in the sense that they are uniform. Using this informal definition, Finite Ramsey's theorem states that large orderly subgraphs become inevitable in sufficiently large graphs, or "complete disorder is impossible." While it is trivial to construct the maximally orderly graphs (i.e. homogeneous graphs) the construction of minimally orderly graphs has long proved challenging, and the best bounds have been obtained not by explicit construction, but using the probabilistic method, initiated by Pál Erdős.

4. Random Graphs

A random graph is a graph in which each edge has probability $1/2$ of being in the graph. The study of these random graphs can give us insight into the orderliness of "normal" graphs. Ramsey's theorem shows that $R(s)$ is well defined, but the calculation of explicit values is difficult: $R(s)$ is only known for $s \leq 4$. Nonetheless, the order of growth is known. Here, we will give some bounds for the size of homogeneous subset of "random" graphs, proving Ramsey's theorem along the way.

Definition 4.1. Let the the Ramsey function $R(s)$ be the smallest positive integer such that every graph on $R(s)$ nodes has a homogeneous set of size s.

Theorem 4.2. $R(s) \leq 2^{2s-3}$

Proof. Let $n = 2^{2s-3}$. Consider a random graph $G(V, E)$ on n vertices where each edge is independently in G or not in G with probability 1/2. We construct a sequence of neighborhoods V_i recursively.

Step 1: Fix a vertex $v_1 \in V$. We have that $\{v \in V : vRv_1\} \cup \{v \in V : \neg(vRv_1)\} = V$ so one set has size at least 2^{2s-3-1}. Let this set be V_1. Give the truth function $\mathbf{t}(V_1) = 1$ if $V_1 = \{v \in V : vRv_1\}$ and 0 otherwise.

Step i+1: Suppose that, after i steps, we have V_i of size 2^{2s-3-i}. Fix a vertex $v_{i+1} \in V_i$. We have that $\{v \in V_i : vRv_{i+1}\} \cup \{v \in V_i : \neg(vRv_{i+1})\} = V_i$ so at least one of the sets has size $2^{2s-3-(i+1)}$. Let V_{i+1} be this set, and give the truth function $\mathbf{t}(V_{i+1}) = 1$ if $V_{i+1} = \{v \in V_i : vRv_{i+1}\}$ and 0 otherwise.

We end this process in $2s-3$ steps. Note that $|V_{2s-3}| \geq 2^{2s-3-(2s-3)} = 1$. Let $v_{2s-2} \in V_{2s-3}$. We have a sequence of $2s-3$ subsets of V, each of which is assigned a truth value of 0 or 1, so by the pigeonhole principle there must be a subsequence of $s-1$ neighborhoods all of which have the same truth value. $V_{j_1}, V_{j_1} \dots V_{j_{s-1}}$ consider the sequence $v_{j_0}, v_{j_1}, \dots, v_{j_{s-1}}, v_{2s-2}$. If the truth value of the sequence is 1, then each vertex in the sequence is connected to every subsequent vertex. If it is 0, each vertex is connected to none of the subsequent vertices. So G has a homogeneous set of size s [6]. □

Remark 4.3. Finite Ramsey theorem follows from the previous result.

Theorem 4.4. *For $s \geq 3$, $R(s) > 2^{s/2}$*

Proof. Let $n = 2^{s/2}$. Consider a graph G on n vertices where each edge is independently in the graph with probability 1/2. The probability that $S \subseteq G, |S| = s$ is homogeneous is $2 \cdot 2^{-\binom{s}{2}}$, the factor of two because S could be complete or discrete. G has $\binom{n}{s}$ distinct subgraphs of size s, and

$$\binom{n}{s} = \frac{n(n-1)\dots(n-s-1)}{s!} < \frac{n^s}{s!}$$

so, using that $n = 2^{s/2}$ and $s \geq 3$, the probability that there is a homogeneous set of size s is

$$\binom{n}{s} \frac{2}{2^{\binom{s}{2}}} < \frac{2n^s}{s! 2^{s(s-1)/2}} = \frac{2^{1+s/2}}{s!} < 1$$

So with nonzero probability, a graph on $2^{s/2}$ vertices has *no* homogeneous set of size s. Thus, $R(s) > 2^{s/2}$ [7]. □

Corollary 4.5. *A random graph on n vertices has a homogeneous set of size $\Theta(log(n))$.*

Proof. Let G be a random graph on more than six vertices. By the previous two theorems we have that G must contain a homogeneous set of size s where

$$(\log_2(n) - 3)/2 \leq s < 2\log_2(n)$$

□

Remark 4.6. The constants have been improved by later work.

5. Orderly graphs

We have given a tight logarithmic lower bound for the growth of homogeneous sets of random graphs. Clearly, we cannot expect to do better than linear growth (no graph on n vertices can have a homogeneous set larger than n), so it makes sense to consider that "orderly graphs" as those which have a homogeneous set which grows polynomially in the size of the graph.

Erdős and Hajnal made an important conjecture regarding orderly graphs, motivating the following definition.

Definition 5.1. A graph H has the Erdős-Hajnal property if there exists a constant $\delta(H) > 0$ such that every graph on n vertices which omits H has a homogeneous set of size at least $n^{\delta(H)}$

Conjecture 5.2 (Erdős-Hajnal conjecture)**.** *Every graph has the Erdős-Hajnal property.*

The conjecture can be interpreted as the statement "orderliness can be expressed in many ways," i.e. through the omission of any graph. For example, two vertex graphs clearly have the EH property, as do homogeneous sets by Ramsey's theorem. [3]

6. Stable Simple Graphs

In this section, we consider stable simple graphs: graphs which admit a finite bound on the size of an induced half-graph. The main result of this section is the proof that the half-graph has the Erdős Hajnal property. The method of the proof, using the model theoretic notion of type trees, can also be applied to prove the Erdős Hajnal property for other graphs. This presentation is based on the one found in [2].

Shelah's [9] is the standard exposition of stability theory. For the purposes of this paper,

Theorem 6.1 (Shelah)**.** *A theory is unstable iff there is some formula $\varphi(\mathbf{x}, \mathbf{y})$ and $\{a_i : i < \omega\}$ and $\{b_k : j < \omega\}$ and $\varphi(a_i, b_i)$ iff $i < j$.*

Definition 6.2. A graph is called *k-edge stable* if there do not exist $a_1, \dots a_k$ $b_1, \dots, b_k$ such that $R(a_i, b_k)$ iff $i < j$.

Note that the statement that G is k-edge stable is equivalent to the statement that G does not have an induced half-graph of height of k or greater.

A key element in the proof is the idea of indexing a graph's nodes with a tree.

Definition 6.3. A *tree* is a partial order $(P, \trianglelefteq)$ such that for each $p \in P$, the set $\{q \in P : p \triangleleft q\}$ is a well order under $\trianglelefteq$.

Given an integer, $n \geq 2$, define

$$2^{<n} = \bigcup_{i=0}^{n-1} \{0,1\}^i,$$

where $\{0,1\}^0 = \langle\ \rangle$ is the *empty string* and for $i > 0$ $\{0,1\}^i$ is the usual cartesian product. This set has a natural tree structure given by $\eta \triangleleft \eta'$ if and only if $\eta = \langle\ \rangle$ or η is an initial segment of η'. Given $\eta \in \{0,1\}^i$, let $|\eta| = i$ denote the *length* of η (the length of the empty string $\langle\ \rangle$ is 0).

An important concept is to take a graph $G = (V, E)$ and arrange G into a tree by indexing its vertex set with the elements of $2^{<n}$. Suppose $G(V, E)$ is a graph and we have an indexing $V = \{a_\eta : \eta \in X\}$ of the vertices of G by some $X \subseteq 2^{<n}$.

Definition 6.4. Given a graph $G = (V, E)$ on n vertices and $A \subseteq 2^{<n}$, we say that an indexing $V = \{a_\eta : \eta \in A\}$ of V by elements of A is a type tree if for each $\eta \in A$ the follwing holds.

- If $\eta \wedge 0 \in A$, then $a_{\eta\wedge 0}$ is not adjacent to a_η. If $\eta_1 \in A$, then $a_{\eta\wedge 1}$ is adjacent to a_η.
- If $\eta \wedge 0$ and $\eta \wedge 1$ are both in A, then for all $\eta' \lhd \eta$, $a_{\eta\wedge 1}$ is adjacent to $a_{\eta'}$ if and only if $a_{\eta\wedge 0}$ is adjacent to $a_{\eta'}$.

Note that we could have replaced the first condition with the following

- If $\eta_1 \lhd \eta_2$ then for all $\eta \lhd \eta_1$, $R(\eta, \eta_1) \iff R(\eta, \eta_2)$

an equivalence which will be useful later.

Lemma 6.5. *Every finite graph $G = (V, E)$ can be arranged into a type tree.*

Proof. Suppose that $|V| = n$. We arrange the vertices of G into a type tree indexed by a subset of $2^{<n}$.

Step 1: Choose any element of G to be $a_{\langle\rangle}$, and set $A_0 = \{a_{\langle\rangle}\}$. Set $X_1 = N(a_{\langle\rangle})$, the neighbors of $a_{\langle\rangle}$, and $X_0 = V \setminus (\{a_{\langle\rangle}\} \cup N(a_{\langle\rangle}))$, the non-neighbors of $a_{\langle\rangle}$. Note that X_1, X_0 partition $V \setminus A_0$.

Step m+1: Suppose we've defined elements of the tree up to height $m \geq 0$ and for each $0 \leq i \leq m$, A_i is the set of vertices of height i. Suppose further that we have a collection of sets of vertices $\{X_{\eta\wedge i} : \eta \in A_m, i \in \{0, 1\}\}$ which partition the unassigned vertices $V \setminus \bigcup_{i=1}^m A_i$ and such that for each $\eta \in A_m$, $X_{\eta\wedge 1} \subseteq N(a_\eta)$ and $X_{\eta\wedge 0} \subseteq V \setminus \{N(a_\eta) \cup \{a_\eta\}$. Then for each $\eta \in A_m$ and $i \in \{0, 1\}$, if $X_{\eta\wedge i} \neq \emptyset$, choose $a_{\eta\wedge i}$ to be any element of $X_{\eta\wedge i}$. Define A_{m+1} to be the set of these $a_{\eta\wedge i}$. Now for each vertex of height $m+1$, $a_v \in A_{m+1}$ and $i \in \{0, 1\}$ let

$$X_{v\wedge 1} = N(a_v) \cap X_v \text{ and}$$
$$X_{v\wedge 0} = (V \setminus (N(a_v) \cup \{a_v\})) \cap X_v.$$

By assumption, $\{X_v : v \in A_{m+1}\}$ is a partition of $V \setminus \bigcup_{i=1}^m A_i$, and by construction, for each $v \in A_{m+1}$, $\{X_{v\wedge 1}, X_{v\wedge 0}\}$ is a partition of $X_v \setminus A_{m+1}$. Therefore, $\{X_{v\wedge i} : v \in A_{m+1}, i \in \{0, 1\}\}$ is a partition of $V \setminus \bigcup_{i=1}^{m+1} A_i$

All elements of V will be chosen after at most n steps. So we obtain an indexing of V by a subset of $2^{<n}$ which is a type tree by construction. □

Definition 6.6. Suppose $G = (V, E)$ is a finite graph.

(1) The tree rank of G, denoted $t(G)$, is the largest integer t such that there is a subset $V' \subseteq V$ and an indexing $V' = \{a_\eta : \eta \in 2^{<t}\}$ which is a type tree. (i.e V' is a full binary type tree of height n.)
(2) The tree height of G, denoted $h(G)$ is the smallest integer h such that every indexing of V which is a type tree has a branch of length h.

Lemma 6.7. *Suppose t, h are integers, and $G = (V, E)$ is a finite graph with tree rank t and a tree height h. Then G contains a complete graph or independent set of size* $\max\{t, h/2\}$.

Proof. By the definition of tree rank, there is a $V' \subseteq V$ and an indexing $V' = \{a_\eta \colon \eta \in 2^{<t}\}$ which is a type tree. By the definition of the standard type tree, $I_1 = \{a_{\langle\,\rangle}, a_0, \ldots, a_{0t-1}\}$ is an independent set of size t. Alternatively, $\{a_{\langle\,\rangle}, a_1, \ldots, a_{1t-1}\}$ is a complete graph of size t.

On the other hand, by the definition of tree height and the lemma which states that every graph can be indexed by a type tree, there is an indexing $V = \{a_\eta \colon \eta \in B \subseteq 2^{<n}\}$ which is a standard type tree which has a branch J of length h. Let a_τ be the last element of J. If $|N(a_\tau) \cap J| \geq |J|/2$ then let $I_2 = N(a_\tau) \cap J$. Otherwise, let $I_2 = (V \setminus N(a_\tau)) \cap J$. In either case, we have that $|I_2| \geq |J|/2 = h/2$.

Consider elements $x, y \in I_2$. Because x and y are on the same branch, we can assume without loss of generality that $x \triangleleft y$. By the definition of I_2, either all $z \in I_2$ are adjacent to a_τ or none of them are, so x is adjacent to a_τ if and only if y is adjacent to a_τ. Because x precedes y, by construction of the type tree we have that a_τ is adjacent to x if and only x is adjacent to y. If $I_2 = N(a_\tau) \cap J$ then a_τ is adjacent to all x and hence all members of I_2 are adjacent to each other, so I_2 is a complete graph. If $I_2 = (V \setminus N(a_\tau)) \cap J$ then a_τ is not adjacent to any x and hence no members of I_2 are adjacent to each other, so I_2 is a discrete graph. So G contains a complete or independent set of size $\max\{|I_1|, |I_2|\} = \max\{t, h/2\}$. □

Definition 6.8. Suppose $G = (V, E)$ is a graph, $A \subseteq 2^{<n}$, and $V = \{a_\eta \colon \eta \in A\}$ is a type tree.

(1) Given an element $a_\eta \in V$, we say there is a full binary tree of height k below a_η if there exists a set $V' \subset \{a_\sigma \colon a_\eta \subseteq a_\sigma\}$ and a bijection $f \colon V' \to 2^{<k}$ with the property that a_σ precedes $a_{\sigma'}$ if and only if $f(a_\sigma) \triangleleft f(a_{\sigma'})$ in $2^{<k}$.
(2) The tree rank of an element $a_\eta \in V$, denoted $t(a_\eta)$, is the largest k such that there is a full binary tree of height k below a_η.

Let $p(a_\eta)$ denote the immediate predecessor to a_η

Theorem 6.9. *Suppose that $n \geq 2$ is an integer and $G = (V, E)$ is a graph of size n. Then*

$$h(G) \geq \frac{(n/t(G))^{\frac{1}{t(G)+1}}}{2}$$

Proof. Note that the minimum tree height is attained when branching is maximal. Thus, a lower bound on tree height which holds for maximally branching trees will hold for all type trees of a graph.

Suppose that $A \subseteq 2^{<n}$ and $V = \{a_\eta \colon \eta \in A\}$ is a type tree with maximal branching. Under these conditions, given a_η with $t(a_\eta) = s$, at most one of the immediate successors to a_η has $t(a_{\eta \wedge i}) = s$ and the other $t(a_{\eta \wedge \neg i}) = s - 1$ or both have $t(a_{\eta \wedge i}) = s - 1$. Let h be the length of the longest branch of this tree, and let $t = \max\{t(a_\eta) \colon \eta \in A\}$. Note that $t \leq t(G)$, the maximum tree height attained by any indexing of G. Given a fixed ℓ and s,

$$\begin{aligned} Z_\ell^s &= \{a_\eta \in V \colon t(a_\eta) = s, ht(a_\eta) = l\} \\ X_\ell^s &= \{a_\eta \in Z_\ell^s \colon t(p(a_\eta)) = s\}, \text{ and} \\ Y_\ell^s &= \{a_\eta \in Z_\ell^2 \colon t(p(a_\eta)) = s + 1\}. \end{aligned}$$

Using words,

- Z_ℓ^s: the nodes in the indexing at height ℓ which admit a complete binary tree of at most height s beneath them.

- X_ℓ^s: the nodes in Z_ℓ^s whose predecessors have a complete binary tree of at most height s beneath them.
- Y_ℓ^s: the nodes in Z_ℓ^s whose predecessors have a complete binary tree of at most height $s+1$ beneath them.

and let $N_\ell^s = |Z_\ell^s|$, $x_\ell^s = |X_\ell^s|$ and $y_\ell^s = |Y_\ell^s|$.

Note that if a_η admits a tree of height s beneath it, then the predecessor to a_η, $p(a_\eta)$ admits a tree of at least height s beneath it, the tree beneath a_η, and at most a tree of height $s+1$. Thus, $Z_\ell^s = X_\ell^s \cup Y_\ell^s$ and $N_\ell^s = x_\ell^s + y_\ell^s$. Note also that every node must admit a tree of height between 0 and t, so we can count the nodes by $n = \sum_{\ell=0}^h \sum_{s=0}^t N_\ell^s$.

We claim that the following facts hold

(i) For all $s \leq t$ and ℓ, $x_{\ell+1}^s \leq N_\ell^s$.
(By maximal branching)

(ii) For all $s < t$ and all ℓ, $y_{\ell+1}^s \leq 2N_\ell^{s+1}$.
(By maximal branching and every node must have at most two direct successors.)

(iii) For all $s < t$ and all ℓ, $N_{\ell+1}^s \leq N_\ell^s + 2N_\ell^{s+1}$.
(Sum of (i) and (ii))

(iv) For all $1 \leq s \leq t$, $N_0^{t-s} = 0$.
(The only element of height 0 is $a_{\langle\rangle}$)

(v) For all ℓ, $N_\ell^t \leq 1$.
(If $N_\ell^2 \geq 2$ then we would have $t(a_{\langle\rangle}) \geq t+1$ which is a contradiction.)

(vi) For all $0 \leq s \leq t$, $N_1^{t-s} \leq 2$.
(The tree is binary, so the second level has at most two elements.)

We now show that for each $0 \leq s \leq t$ and $9 \leq \ell < h$, $N_{\ell+1}^{t-1} \leq (2(\ell+1))^s$. If $s = 0$ this follows immediately from (v).

Case $s = 1$: We wish to show that $0 \leq \ell < h$, $N_{\ell+1}^{t-1} \leq (2(\ell+1))^s$. The case where $\ell = 0$ is done by (vi). Let $\ell > 0$ and suppose by induction that $N_\ell^{t-1} < 2\ell$. By (iii), (v) and the induction hypothesis, we have that

$$N_{\ell+1}^{t-1} \leq N_\ell^{t-1} + 2N_\ell^t \leq 2\ell + 2 = (2(\ell+1))^1$$

Case $s > 1$: Suppose by induction that for all $0 \leq s' < s$, that for all $0 \leq \ell < h$, $N_{\ell+1}^{t-s'} \leq (2(\ell+1))^{s'}$. We wish to show that for all $0 \leq \ell < h$, $N_{\ell+1}^{t-s} \leq (2(\ell+1))^s$. The case of $\ell = 0$ is done by (vi). Let $\ell > 0$ and suppose for the sake of induction that for all $0 \leq \ell' < \ell$, $M_{\ell'+1}^{t-s} \leq (2(\ell'+1))^s$. Then, by (iii) and the induction hypothesis, we have that

$$N_{\ell+1}^{t-s} \leq N_\ell^{t-s} + 2N_\ell^{t-s+1} \leq (2\ell)^s + 2(2\ell)^{s-1} = (2\ell)^s \left(\frac{\ell+1}{\ell}\right) \leq (2(\ell+1))^s$$

Now suppose that for all $0 \leq \ell' < \ell$, $N_{\ell'+1}^{t-1} \leq (2(\ell'+1))^s$.

Having completed this nested induction we conclude that for all $0 \leq \ell < h$,

$$N_{\ell+1} \leq \sum_{0 \leq s \leq t} N_{\ell+1}^s \leq \sum_{0 \leq s \leq t} (2(\ell+1))^s \leq t(2(\ell+1))^t \leq t(2h)^t.$$

Thus,

$$n = N_0 + \sum_{0 \leq \ell < h} N_{\ell+1} \leq 1 + \sum_{0 \leq \ell < h} t(2h)^t \leq t(2h)^{t+1}.$$

Rearranging, we obtain that

$$\frac{(n/t)^{\frac{1}{t+1}}}{2} \leq h$$

and since $t \leq t(G)$, and $h \leq h(G)$,

$$\frac{(n/t(G))^{\frac{1}{t(G)+1}}}{2} \leq h(G)$$

completing the proof. □

Observation 6.10. If G is k-stable, then $t(G) < k$.

Proof. Let G be k-stable and suppose that some indexing of G contained a complete binary tree $\langle a_\eta \colon \eta \in 2^{<k} \rangle$. Consider the branches $\langle a_0, a_{00}, \ldots, a_{0^{k-1}} \rangle$ and $\langle a_{01}, a_{001}, \ldots, a_{0^{k-1}\wedge 1} \rangle$. These form a half graph of height k, which is a contradiction. Thus, $t(G) < k$ □

Observation 6.11. If G contains no homogeneous set of size k, $t(G) < k$.

Proof. Let G be a graph with no homogeneous set of size k. Suppose that some indexing of G contained a complete binary tree $\langle a_\eta \colon \eta \in 2^{<k} \rangle$. Consider the branches $\langle a_0, a_{00}, \ldots, a_{0^{k-1}} \rangle$ and $\langle a_1, a_{11}, \ldots, a_{1^{k-1}} \rangle$. These are homogeneous sets of size k, which is a contradiction. Thus, $t(G) < k$. □

Observation 6.12. Suppose that $G(n)$ is a family of graphs that admit a finite bound on their tree height, $t(G(n))$. Then there exists c such that for all n, $G(n)$ contains a homogeneous set larger than n^c.

Proof. Suppose that for all $n < \omega$, $t(G(n)) < k$. Then we have that $G(n)$ contains an independent set of size at least

$$\frac{(n/k)^{\frac{1}{k+1}}}{4}$$

concluding the proof. □

Conclusion 1. *The half-graph, the complete graph, and the discrete graph have the Erdős-Hajnal property.*

Proof. By the previous observations. □

7. Stable Hypergraphs

In this section we present the proof of the analogous statement for finite hypergraphs.

Definition 7.1. A hypergraph is *k-stable* if for all partitions of the variables, there do not exist sequences of tuples $a_1, \ldots a_k$ $b_1, \ldots, b_k$ such that $R(a_i, b_k)$ iff $i < j$.

Note that the tuples may be of any length.

Fact 7.2. *The stability rank gives a finite bound for the height of a full binary tree which can be embedded in a type tree*

This is a direct consequence case of the Unstable Theory Lemma. See [8] for a presentation.

Fact 7.3. *Let (G, Δ) be a graph. There exists r such that for any $A \subset G$, $|A| \geq 2$, we have that $|S_\Delta(A)| \leq |A|^r$.*

For a proof of the above, see [9] Theorem II.4.10(4) and II.4.11(4) p.74.

Again, we apply the idea of arranging a sequence of nodes into a type tree with the intuition that trees with bounded branching and a bound on the size of an induced complete binary tree must have long branches, which equate to indiscernible sequences.

Theorem 7.4 (for hypergraphs)**.** *If G is an k-regular, stable hypergraph then each $A \subseteq G$ $|A| = n$ has a homogeneous subgraph of size at least $f^k(n)$ where $f(x) = \left|\frac{x}{t}\right|^{\frac{1}{tr+t+1}} - k$ and r, t are constants depending only on the theory of G.*

Proof. We prove by induction on $m \leq k$ that there are $u_m \subseteq n$ such that

I $|u_{m+1}| \geq f(|u_m|)$ where $f(x) = \left|\frac{x}{t}\right|^{\frac{1}{t(r+1)}} - k$ where r and t are defined below

II If $i_0 < \cdots < i_{k-1}$, and $j_0 < \cdots < j_{k-1}$ are from u_m and for all $\ell < k - m$, $i_\ell = j_\ell$, then

$$R(a_{i_0}, \ldots, a_{i_{k-1}}) \iff R(a_{j_0}, \ldots, a_{j_{k-1}})$$

Note that this condition is weaker than indiscernibility for $m < k$.

Case m=0: Let $u_0 = n$. This sequence of length n will serve as the base case for condition (I). Condition (II) is trivially satisfied for $i_0 < \cdots < i_{k-1}$, and $j_0 < \cdots < j_{k-1}$ from u_0 with $i_\ell = j_\ell$ for all $\ell < k$,

$$R(a_{i_0}, \ldots, a_{i_{k-1}}) \iff R(a_{j_0}, \ldots, a_{j_{k-1}})$$

Case m+1: Let u_m be given, and suppose that $|u_m| = \ell_m$. Let $\Delta^m = \{R(x_0, \ldots, x_{k-m}, a_{\ell_m - m}, \ldots, a_{\ell_m - 1})\}$, i.e. the edges where the last m elements are the last m elements of u_m.

Step 0: (Arranging the elements of u_m into a type tree) We wish to arrange u_m into a type tree as we did in the simple graph case, however this time the types are from Δ^m and thus the tree will not necessarily be binary. These type trees are order isomorphic to downward-closed subsets of $\omega^{<\omega}$, the finite strings of naturals, where $v \leq p$ if v is an initial segment of p.

By induction on $\ell < \ell_m$ select elements a_η from u_m with a tree order such that

- If $i \lhd j$, then for all $A \subseteq \{a_\eta : \eta \lhd i\}$, $R(A, \eta_1) \iff R(A, \eta_2)$.
- If $\neg(i \lhd j)$ and $\neg(j \lhd i)$ then there exists $A \subseteq \{a_\eta : (\eta \lhd i) \wedge (\eta \lhd j)\}$ such that $\neg R(A, i) \iff R(A, j)$

Note that these conditions are equivalent to the conditions for a type tree in the case of a binary relationship. In the case of a non binary relationship, greater branching is possible. Consider two distinct successors to the same node. They must satisfy different Δ^m formulas over their common branch. In the case of the binary relationship, this means that the must satisfy different formulas over their immediate common predecessor. However in the finite branching tree, they can disagree about any subset of the elements in their common branch. Thus, in a type tree for a k-regular hypergraph, a node at height h could be as much as $\binom{h}{k-1}$. Nonetheless, as we will show, branching is bounded for stable trees.

Step 1: (Branches of the type tree suffice) Consider the longest branch of the type tree. Let this branch be u_{m+1}, with the order inherited from the type tree. Now consider $i_0 < \cdots < i_{k-1}, j_0 < \cdots < j_{k-1}$ from u_{m+1} such that all but the last m elements are equal, i.e. $\bigwedge_\ell(\ell < k-m \implies i_\ell = j_\ell)$. Without loss of generality, suppose that $j_{k-m} < i_{k-m}$.

Recall that in Δ^m we replace the last m variables in the formula with the last m elements of u_m, thus for $\varphi \in \Delta$ and $\gamma \in \Delta^m$

$$R(a_{i_0}, \ldots, a_{i_{k-1}}) \iff R(a_{i_0}, \ldots, a_{i_{k-m}}, a_{\ell_m - m}, \ldots, a_{\ell_m - 1})$$

by the inductive hypothesis, the first $k-m-1$ indices agree, so

$$R(a_{i_0}, \ldots, a_{i_{k-m}}, a_{\ell_m - m}, \ldots, a_{\ell_m - 1}) \iff R(a_{j_0}, \ldots, a_{j_{k-m-1}}, a_{i_{k-m}}, a_{\ell_m - m}, \ldots, a_{\ell_m - 1})$$

By the construction of u_m, $a_{i_{k-m}}, a_{j_{k-m}}$ are on the same branch so they realize the same Δ^m type over their common initial segment, and by supposition $j_{k-m} < i_{k-m}$, so $a_{j_{k-m}}, a_{i_{k-m}}$ satisfy the same Δ^m formulas over $a_{j_0}, \ldots, a_{j_k - m - 1}$. Thus,

$$R(a_{j_0}, \ldots, a_{j_{k-m-1}}, a_{i_{k-m}}, a_{\ell_m - m}, \ldots, a_{\ell_m - 1}) \iff R(a_{j_0}, \ldots, a_{j_{k-m}}, a_{\ell_m - m}, \ldots, a_{\ell_m - 1})$$

which by the definition of Δ^m gives

$$R(a_{j_0}, \ldots, a_{i_{k-m}}, a_{\ell_m - m}, \ldots, a_{\ell_m - 1}) \iff R(a_{j_0}, \ldots, a_{j_{k-1}})$$

Thus, we have shown that

$$\varphi(a_{i_0}, \ldots, a_{i_{k-1}}) \iff \varphi(a_{j_0}, \ldots, a_{j_{k-1}})$$

in other words, part II of the induction.

Step 2: (Lower bounds on the length of a branch) We now must show that sufficiently long branches exist. The rationale is the same: trees that have a bound on the size of an induced binary tree must have long branches. However we must be more careful in counting, as our type tree is no longer strictly binary.

Step 2B: (Bound on branching) By fact 7.7 we have that at height h, branching is at most $(h+m)^r$ as immediate successors to the same node must satisfy different Δ^m types over their h node long common initial segment and over the m variables specified in Δ^m. We note, as above, that the shortest trees are obtained when branching is maximal. As successors of the same rank must lie along the same branch, and rank successors cannot have a greater rank than their predecessors, this attains when at most one successor to a node i of rank s has rank s and the rest have rank $s-1$.

Step 2C: (Counting nodes) We count the nodes in our type tree by their tree rank. Recall that $t(a_i)$ is the height of the largest binary tree which can be after a_i in the type tree. Because G is stable, its theory has a stability rank t, which is a bound on the tree height embeddable in a Δ type tree of G. Given a fixed ℓ and s, define as before

$$Z_\ell^s = \{a_\eta \in V : t(a_\eta) = s, ht(a_\eta) = l\}$$
$$X_\ell^s = \{a_\eta \in Z_\ell^s : t(p(a_\eta)) = s\}, \text{ and}$$
$$Y_\ell^s = \{a_\eta \in Z_\ell^2 : t(p(a_\eta)) = s+1\}.$$

and let $N_\ell^s = |Z_\ell^s|$, $x_\ell^s = |X_\ell^s|$ and $y_\ell^s = |Y_\ell^s|$, and $Z_\ell^s = X_\ell^s \cup Y_\ell^s$ so $N_\ell^s = x_\ell^s + y_\ell^s$. Note also that every node must admit a tree of height between 0 and t, so we can count the nodes by $n = \sum_{\ell=0}^{h} \sum_{s=0}^{t} N_\ell^s$. We claim that the following facts hold

(i) For all $s \leq t$ and ℓ, $x_{\ell+1}^s \leq N_\ell^s$, by maximal branching
(ii) For all $s < t$ and all ℓ, $y_{\ell+1}^s \leq N_\ell^{s+1} \cdot (\ell+m)^r$, by the bound on branching.
(iii) Thus for all $s < t$ and all ℓ, $N_{\ell+1}^s \leq N_\ell^s + N_\ell^{s+1} \cdot (\ell+m)^r$.
(iv) For all $1 \leq s \leq t$, $N_0^{t-s} = 0$, as the only element of height 0 is the root.
(v) For all ℓ, $N_\ell^t \leq 1$.

Step 2D: (Induction) We observe by induction that $N_{\ell+1}^{t-s} \leq (\ell+m)^{s(r+1)}$. Case $s = 1$: By a nested induction on ℓ using (i) and (iii)

$$N_{\ell+1}^{t-1} \leq \sum_{j \leq \ell} (j+m)^r \leq (\ell+m)^{r+1}$$

Case $s+1$: By induction on ℓ using (iii) and (iv) we have that

$$N_{\ell+1}^{t-(s+1)} \leq N_\ell^{t-(s+1)} + N_\ell^{t-s} \cdot (\ell+m)^r \leq \sum_{j \leq \ell} (\ell+m)^{s(r+1)} \cdot (j+m)^r \leq (\ell+m)^{(s+1)(r+1)}$$

So, the total number of nodes of tree height $\ell+1$, $N_{\ell+1}$ is

$$N_{\ell+1} \leq \sum_{s \leq t} N_{\ell+1}^{t-s} \leq \sum_{s \leq t} (\ell+m)^{s(r+1)} \leq t(\ell+m)^{t(r+1)}$$

Noting that $N_0 = 1$, the total number of nodes in the tree is at most

$$N = 1 + \sum_{\ell < h} N_{\ell+1} \leq 1 + \sum_{\ell_h} t(\ell+m)^{t(r+1)} < t(h+m)^{t(r+1)}$$

Step 2E: (Part (I) of the induction) Suppose for the sake of contradiction that the branch we chose (i.e. u_{m+1} the longest branch in the type tree of u_m) had length h satisfying

$$t(h+m)^{t(r+1)} < |u_m|.$$

We have shown that this is a contradiction, as such a type tree would not be large enough to exhaust all the nodes of u_m. Thus, $h \geq f(|u_m|)$ where

$$f(x) = |x/t|^{\frac{1}{t(r+1)}} - k \leq |x/t|^{\frac{1}{t(r+1)}} - m$$

where here we subtract k instead of m for uniformity.

Thus, after k steps we have extracted a sequence of indices u for an sequence which is edge indiscernible, and this sequence has length at least $f^k(n)$, after k applications of f

□

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