COMBINATORIAL PROOFS OF SOME THEOREMS IN ALGEBRAIC TOPOLOGY

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Abstract. Two important theorems in algebraic topology are the Brouwer Fixed Point theorem and the Borsuk-Ulam theorem. The theorems require the development of homology in their standard proofs. However, each theorem has an equivalent combinatorial result involving triangulating the relevant surface and coloring the vertices of the triangulation. Then by taking the limit of a sequence of finer triangulations, it is possible to prove results about continuous functions. In this paper, I will prove Sperner's lemma and Tucker's lemma and then use them to prove the Brouwer Fixed Point theorem and Borsuk-Ulam theorem, respectively.

Contents

1. Simplices

Definition 1.1. For $n \geq 0$, an n-simplex is the convex hull of $n+1$ points in general position. The points are called the vertices of the simplex.

For instance, a 0-simplex is a point, a 1-simplex is a line segment, a 2-simplex is a triangle, a 3-simplex is a tetrahedron, and so on. For a simplex σ, we denote the set of vertices of σ by $V(\sigma)$.

Definition 1.2. An n-simplex is in standard position if it is represented by the following equations in $\mathbb{R}^{n+1}$.

$$x_1 + x_2 + \ldots + x_n + x_{n+1} = 1.$$

$$\text{For } i \in \{1, 2, \ldots, n+1\}, 0 \leq x_i \leq 1.$$

If we have a simplex not in standard position in some coordinate system, we can turn it into a simplex in standard position by taking as a new basis the vertices of the simplex. In this manner we can consider any simplex to be in standard position.

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Definition 1.3. If σ is a n-simplex, a k-face of σ is a k-simplex τ such that $V(\tau) \subset V(\sigma)$.

Let $F(\sigma)$ denote the set of faces of σ. Let $F_{n-1}(\sigma)$ denote the set of $(n-1)$-faces of σ.

Definition 1.4. The boundary of a simplex is the union of all of the $(n-1)$-faces. The interior of a simplex is the complement of the boundary.

Definition 1.5. A triangulation of an n-simplex σ is a subdivision into nonempty sets $\{\sigma_1, \ldots, \sigma_k\}$ such that

- Each σ_i is itself an n-simplex.
- $\bigcup_{j=1}^{k} \sigma_j = \sigma$.
- if $i \neq j$, $\sigma_i \cap \sigma_j$ is either an $(n-1)$-simplex or the empty set.

We will use the notation $T(\sigma)$ to denote a triangulation of σ. The notation $V(T(\sigma))$ refers to the union of the vertices of the simplices in the triangulation.

2. Sperner's Lemma

Definition 2.1. Let σ be an n-simplex. A Sperner coloring of a triangulation $T(\sigma)$ is a function $c : V(T(\sigma)) \to \{1, 2, \ldots, n+1\}$ satisfying the following conditions:

(1) Each vertex of σ receives a different color.
(2) If v is on a face of σ, then v receives the same color as one of the vertices defining that face.

Lemma 2.2. *Let σ be an n-simplex in standard position, let $T(\sigma)$ be a triangulation, and let $c : (V(T(\sigma)) \to \{1, 2, \ldots, n+1\}$ be a coloring function such that for every color i, each vertex colored i has a nonzero i^{th} coordinate. Then c is a Sperner coloring.*

Proof. The vertices of σ have coordinates $(1, 0, \ldots, 0)$, $(0, 1, \ldots, 0)$, $\ldots$, $(0, 0, \ldots, 1)$. Each of these has only one nonzero coordinate, and this coordinate is different for each vertex. Therefore all the vertices of σ receive different colors, so the first condition is satisfied.

Let $p \in V(T(\sigma))$ be a point on a face f. Let i be some color that is different from all the vertices of f. Then each of the vertices of f has an ith coordinate of 0, and thus every point in their convex hull also has an ith coordinate of 0. Therefore no points on their face can receive the color i. Since this is true for all colors i that are not the same as one of the vertices of f, the point p can only receive a color that matches one of the vertices of f. Therefore the second condition is satisfied. □

Lemma 2.3 (Sperner's Lemma)**.** *In every Sperner coloring of a triangulated n-simplex, there are an odd number of simplices which have vertices colored with all $n+1$ colors. In particular, this means there is at least one simplex whose vertices have all $n+1$ colors.*

Proof. The proof will proceed by induction on n, the dimension of the simplex. The base case for the 0-simplex is trivial because it is only a single point.

Now for the inductive step, let σ be an n-simplex with a triangulation $T(\sigma)$. Let c be a Sperner coloring of $T(\sigma)$. In order to determine the number of n-simplices with the colors $\{1, 2, \ldots, n+1\}$, we will first count the number of faces with the colors $\{1, 2, \ldots, n\}$ over each of the n-simplices in the triangulation. However, we

can switch the order of summation and count this same quantity by counting the number of $(n-1)$-simplices with each of the colors $\{1, 2, \ldots, n\}$ and including it a number of times equal to the number of n-simplices of which it is an $(n-1)$-face.

If τ is a $(n-1)$-simplex, let

$$f(\tau) = \begin{cases} 1 & \tau \text{ has the colors } \{1, 2, \ldots, n\} \\ 0 & \text{otherwise.} \end{cases}$$

Then by switching the order of the sum we have

$$\sum_{\sigma_i \in T(\sigma)} \sum_{\tau \in F_{n-1}(\sigma)} f(\tau) = \sum_{\tau \subset V(T(\sigma))} \sum_{\{\sigma_i \in T(\sigma) | V(\tau) \subset V(\sigma_i)\}} f(\tau), \tag{2.4}$$

where τ is an $(n-1)$-simplex.

First we will examine the right side of the equation. Each $(n-1)$-simplex is either the face of one n simplex, if it is on the boundary, or two n simplices, if it is a face in the interior. Therefore the right hand side is equal to

$$\begin{aligned} &\text{Number of } (n-1)\text{-simplices on the boundary colored with } \{1, 2, \ldots, n\} \\ &+ 2 \cdot \text{Number of } (n-1)\text{-simplices in the interior colored with } \{1, 2, \ldots, n\} \end{aligned}$$

Note that by definition of a Sperner coloring, only one $(n-1)$-face of the boundary can have any $(n-1)$-simplices containing the colors $\{1, 2, \ldots, n\}$, namely the face defined by the vertices of σ that have these n colors. This face is a triangulated $(n-1)$-simplex with a Sperner coloring, so by the inductive hypothesis, there are an odd number of $(n-1)$-simplices on the boundary colored with $\{1, 2, \ldots, n\}$. The first term of the above sum is odd and the second term is even, so the sum in equation 2.4 is odd.

Now we will examine the left hand side of the equation. If a n-simplex has an $(n-1)$-face which contains the colors $\{1, 2, \ldots, n\}$, then the n simplex also contains those colors. For each such n-simplex, there are two possible cases, either it contains all of those colors exactly once and contains the color $n+1$ on the last vertex, or it contains one of the first n colors twice and the rest exactly once. In the former case, it has exactly one face containing $\{1, 2, \ldots, n\}$, the face that is the $(n-1)$-simplex created by the n vertices with colors $\{1, 2, \ldots, n\}$. In the latter case, there are two choices for a face which contains the colors $\{1, 2, \ldots, n\}$. If color i is the repeated color, then there is the face created by the vertices of colors $\{1, 2, \ldots, i-1, i+1, \ldots, n\}$ along with the first vertex of color i, and a face created by the vertices of colors $\{1, 2, \ldots, i-1, i+1, \ldots, n\}$ along with the second vertex of color i. Therefore the left hand side of the equation can be written as

$$\begin{aligned} &\text{Number of } n\text{-simplices with the colors } \{1, 2, \ldots, n+1\} \\ &+ 2 \sum_{i=1}^{n} \text{Number of } n\text{-simplices with the colors } \{1, 2, \ldots, (i-1), i, i, (i+1), \ldots, n\} \end{aligned}$$

We have already established that this sum is odd, and since the second term is even, the first term then must be odd. Therefore there are an odd number of n-simplices with colors $\{1, 2, \ldots, n+1\}$. This completes the induction step and the proof. $\square$

While Sperner's lemma tells us about discrete combinatorial objects, by taking the limit of finer and finer triangulations we can use it to prove results about continuous objects. We will see this in the proof of Brouwer Fixed Point theorem.

3. Brouwer Fixed Point Theorem

Definition 3.1. The n-ball B^n is the set of points in $\mathbb{R}^n$ satisfying $x_1^2 + x_2^2 + \ldots + x_n^2 \leq 1$.

Theorem 3.2 (Brouwer Fixed Point Theorem)**.** *If f is a continuous function $f : B^n \to B^n$, then there is a point fixed by f, i.e. there exists $x \in B^n$ such that $f(x) = x$*

Proof. Since there is a homeomorphism between the n-simplex in standard position and B^n, it suffices to show that every continuous map $f : \sigma \to \sigma$ from a an n-simplex σ to itself has a fixed point.

Suppose for the sake of contradiction that f has no fixed point. We then construct a sequence of triangulations of decreasing size. Let $T_j(\sigma)$ be a triangulation of σ such that for each σ_i in the triangulation, the distance between any two points in σ_i is less than $\frac{1}{j}$. Then let $c_j : V(T_j(\sigma)) \to \{1, 2, \ldots, n+1\}$ be a coloring function defined by

$$c_j(v) = \min\{i \in \{1, 2, \ldots, n+1\} \mid f(v)_i < v_i\}.$$

This means that to each coordinate we associate a color and then for a vertex v, we pick some color i such that the corresponding coordinate decreases when f is applied, so $f(v)_i < v_i$. If there are multiple such colors, we pick the minimum such color. Note that there is always one coordinate that decreases because we assumed f has no fixed points.

We claim that c is a Sperner coloring. If v is a vertex for which the ith coordinate is 0, then f cannot decrease the ith coordinate. Then by definition of c, v does not receive the color i. Therefore, by lemma 2.2, c is a Sperner coloring.

Therefore, by Sperner's lemma, for each j there exists some n-simplex colored with the colors $\{1, 2, \ldots, n+1\}$. Let a_j be some point in this simplex. Then we have a sequence $a_1, a_2, \ldots$ of points in the n-simplex. Since the n-simplex is compact, this sequence has a convergent subsequence $b_1, b_2, \ldots$, which converges to a point $p \in \sigma$.

We claim p is a fixed point. If not, there is some coordinate i for which $f(p)_i > p_i$, since all coordinates add up to 1. By convergence of the subsequence and decreasing size of the triangluations, there is some simplex arbitrarily close to p which has arbitrarily small size, which has a vertex q with color i. Since it is colored i, we have $f(q)_i < q_i$ for points arbitrarily close to p. However, by continuity of f, we cannot then have $f(p)_i > p_i$. This is a contradiction, so therefore p is in fact a fixed point, as required by the statement of the theorem. □

4. Tucker's Lemma

Definition 4.1. The n-sphere S^n is the set of points in $\mathbb{R}^{n+1}$ satisfying $x_1^2 + x_2^2 + \ldots + x_{n+1}^2 = 1$.

Definition 4.2. Two points $x, y \in S^n$ are antipodal if $x = -y$, thought of as vectors in $\mathbb{R}^{n+1}$

Definition 4.3. The boundary of B^n is the set of points where equality holds, i.e. $x_1^2 + x_2^2 + \ldots + x_n^2 = 1$. This is equal to S^{n-1}.

Definition 4.4. A triangulation of B^n is a homeomorphism between B^n and a triangulated n-simplex. The sets in the triangulation of B^n are the preimages under the homeomorphism of the sets in the triangulation of the n-simplex.

Essentially, this means that a triangulation of B^n is a subdivision into n-simplices but where curved edges are allowed.

Definition 4.5. A triangulation $T(\sigma)$ of B^n is antipodally symmetric on the boundary if for every subset S of the boundary, S is a face of some element of $T(\sigma)$ if and only if the set of antipodal points, $\{-s \mid s \in S\}$, is also a face of some element of $T(\sigma)$.

Definition 4.6. The closed positive n-orthant is the set of points in $\mathbb{R}^n$ defined by

$$x_1 \geq 0, \quad x_2 \geq 0, \quad \ldots, \quad x_n \geq 0$$

A closed n-orthant is the set of points of a similar form but where some of the $\geq$ signs in the above equation can be $\leq$ instead. There are 2^n closed n-orthants. It is the generalization of quadrant or octant for two and three dimensions, respectively.

Definition 4.7. Let O denote the set of all 2^n n-orthants. The octahedral triangulation is a triangulation of B^n whose elements are $\{o \cap B^n \mid o \in O\}$, the set of the intersections of the n-orthants with B^n.

Definition 4.8. Let $T(\sigma)$ and $U(\sigma)$ be triangulations of a simplex σ. $U(\sigma)$ is a refinement of $T(\sigma)$ if $U(\sigma)$ is a union of triangulations of the elements of $T(\sigma)$.

In particular, this means that $U(\sigma)$ contains all the edges of $T(\sigma)$ and potentially additional edges.

Lemma 4.9 (Tucker's Lemma)**.** *Let σ be a refinement of the octahedral triangulation of B^n that is antipodally symmetric on the boundary and let $c : V(\sigma) \to \{-n, \ldots, -2, -1, 1, 2, \ldots, n\}$ be a coloring function such that antipodal points receive opposite colors. Then there exists a complementary 1-simplex, i.e. a 1-simplex whose two vertices receive opposite colors.*

It should be noted that the lemma is true without the condition that the triangulation refine the octahedral triangulation, but this version will suffice for proving the Borsuk-Ulam theorem. We impose this condition so that in the proof we can use the fact that each simplex exists in a single orthant.

Proof. If σ_i is a simplex in the subdivision, let $C(\sigma_i) = \{c(v) \mid v \text{ is a vertex of } \sigma_i\}$, the set of colors which vertices of σ_i receive. We call this the coloring of the simplex. In addition to a coloring, we also assign each simplex a label L based on which orthant the simplex is in. If it lies in the positive direction of the ith coordinate we assign $+i$ to the label, if it lies in the negative direction of the ith coordinate we assign $-i$ to the label, so

$$\begin{aligned} L(\sigma_i) = \{&i \mid \sigma_i \text{ has a vertex with a positive } i^{th} \text{ coordinate}\} \\ &\cup \{-i \mid \sigma_i \text{ has a vertex with a negative } i^{th} \text{ coordinate}\} \end{aligned}$$

Note that a simplex can only have one of i or $-i$ in its label because of the condition that the triangulation refine the octahedral triangulation, so a simplex

does not cross the orthant boundaries.

In addition to defining the color and label on the n-simplices in the triangulation, we also define the color and label of all their k-faces for $0 \leq k \leq (n-1)$. Let $F(T(\sigma))$ denote the set of all faces of simplices in $T(\sigma)$. Since these are simplices of lower dimension, the above definitions can be extended to also assign a color to all the elements of $F(T(\sigma))$ in the same way as subsets of $\{-n, \ldots, -2, -1, 1, 2, \ldots, n\}$. Note that in this case, some simplices may lie on the axes and may have a label with less than n elements.

Now we define a graph $G = (N, E)$ on some of the elements of $F(T(\sigma))$ where

$$N = \{f \in F(T(\sigma)) \mid L(f) \subset C(f)\},$$

the simplices for which for every element of the label there exists a vertex of that color. We will use the term node for the elements of the graph, and reserve the terms vertex and vertices for 0-simplices. Edges will refer to edges of the graph, and faces will refer to the simplices. Two nodes n_1 and n_2 of this graph have an edge between them if any of the following are true.

(1) n_1 and n_2 are antipodal simplices on the boundary
(2) n_2 is a face of n_1 and $L(n_1) \subset C(n_2)$
(3) n_1 is a face of n_2 and $L(n_2) \subset C(n_1)$

Since this definition is symmetric, we can think of the graph as being undirected.

Let n_1 be a simplex which is a node of the graph. Then let m be the number of elements in its label. Then n_1 must either be an $(m-1)$-simplex or an m simplex. It cannot be lower-dimensional because it must have as many vertices as elements in its label for it to have been in the graph, and it cannot be higher dimensional because an $m+1$ or larger simplex could not lie in the m dimensional orthant defined by its label.

We will now examine the degrees of these cases of nodes. The claim is that a node has odd degree if and only if it is the origin or if the simplex contains a complementary 1-simplex. Then, since a graph must have an even number of nodes of odd degree and it necessarily contains the origin, there must be at least one complementary 1-simplex.

First consider n_1 to be an $(m-1)$-simplex in the interior. Clearly there are no edges satisfying condition (1). There are also no edges satisfying condition (2) because then n_2 would be a $(m-2)$-simplex with at most $(m-1)$ colors which could not contain n_1's m-element label. Condition (3) is satisfied exactly whenever n_1 is the face of some m-simplex in the same orthant. It could not be adjacent to some simplex in a higher dimensional orthant because it does not contain any additional colors that could be in the new label. Since n_1 is an $(m-1)$-simplex in the interior of a triangulated m-orthant, it is the face of exactly 2 m-simplices. Therefore n_1 has degree 2.

Now consider n_1 to be an $(m-1)$-simplex on the boundary. We claim there is one edge satisfying condition (1). Note that since antipodal simplices receive opposite colors by assumption and opposite labels by definition, two antipodal simplices will either both be in the graph or not in the graph, so any node that is a boundary simplex will have an edge from condition (1). Similarly to the previous case, there are no edges satisfying condition (2). Since the simplex is on the boundary, it will be the face of only one m-simplex in the same orthant. Therefore there is one edge satisfying condition (3), so n_1 also has degree 2 in this case.

Now consider n_1 to be an m-simplex. It has $(m+1)$ vertices, the colors of m of which are determined because it must have the m colors of its label in order to be in the graph. Let the final color be color d. There are 3 possibilities for this final color:

- d is a duplicate of one of the colors in the label, i.e, $d \in L(n_1)$
- d is a complement of one of the colors in the label, i.e. $-d \in L(n_1)$
- d is a new color that is neither a duplicate or a complement of one of the colors in the label, i.e. $d \notin L(n_1)$ and $-d \notin L(n_1)$

Now we will check conditions (1), (2), and (3), keeping these possibilities in mind. Since n_1 is an m-simplex in an m-orthant, it cannot be on the boundary, so there are no edges satisfying condition (1). The adjacent simplices satisfying condition (2) will be exactly the subsets of m-vertices that contain the m required colors. Since there are $(m+1)$ vertices, there will be 2 such subsets if there is a repeat color, and 1 subset if there is no repeat color. Therefore there is one edge of this type (2) if d is not a duplicate color and two edges of this type if d is a duplicate color. There is one exception to this: the empty set of vertices is not a simplex, so a 0-simplex has no edges of this type. The point at the origin is the only 0-simplex that is in the graph, so it is the only such exception. Finally, for an edge to satisfy condition (3), it must be connected to an $(m+1)$-simplex, which lives in an $(m+1)$-orthant. If n_1 has the m colors in its label and an additional color that is not a duplicate or complement of an already existing color, then it will be the face of a simplex n_2 in the orthant defined by the colors of n_1, and n_1 and n_2 will be adjacent. If the final color is a duplicate or complement, $C(n_1)$ cannot contain the label of a simplex in an $(m+1)$-orthant. Therefore there will be one edge satisfying condition (3) if there is no duplicate or complement. Note that the origin was colored with a color that is not a duplicate or complement of its empty label. Therefore n_1 has degree 2 if the final color is a duplicate, 1 if the final color is a complement, 2 if it is not the origin and the final color is not a duplicate or complement, and 1 if it is the origin.

Every node has degree 2 except for simplices which contain complementary colors and the origin, which both have degree 1. Since there are an even number of nodes with odd degree, there must be an odd number of simplices which contain complementary colors, in particular at least one. The 1-simplex connecting the complementary vertices will be a complementary 1-simplex as required in the statement of the lemma. □

5. Borsuk-Ulam Theorem

Theorem 5.1 (Borsuk-Ulam Theorem)**.** *Every continuous map $f : S^n \to R^n$ admits a pair of antipodal points x and $-x$ such that $f(x) = f(-x)$.*

Proof. First define the function $g : B^n \to S^n$ which maps a point x in B^n to the point above it in S^n, i.e. the first n coordinates remain the same, and the $(n+1)^{th}$ coordinate is the nonnegative coordinate such that the point is on S^n.

$$g(x_1, x_2, \ldots, x_n) = (x_1, x_2, \ldots, x_n, 1 - \|x\|)$$

Now we define the function $h : B^n \to R^n$

$$h(x) = f(g(x)) - f(-g(x))$$

We would like to find some point for which $h(x) = 0$, then we would have $f(g(x)) = f(-g(x))$ as desired.

Antipodal points on the boundary receive opposite values of h because if $\|x\| = 1$, then $h(x) = f(x) - f(-x)$ and $h(-x) = f(-x) - f(x)$. Furthermore, we also have that h is continuous by continuity of f and g.

We consider a sequence of refinements of the octahedral triangulation of B^n that are antipodally symmetric on the boundary such that the size of the simplices approaches zero. Let σ_j be such a triangulation of B^n such that for each σ_i in the triangulation, the distance between any two points in σ_i is less than $\frac{1}{j}$. Then let $c_j : V(\sigma_j) \to \{-n, \ldots, -2, -1, 1, 2, \ldots, n\}$ be a coloring function defined by

$$c_j(x) = \min\{i \in \{1, 2, \ldots, n\} \mid (\forall k \in \{1, 2, \ldots, n\})(|h(x)_i| \geq |h(x)_k|)\} \cdot \operatorname{sign}(h(x)_i),$$

where x is a vertex of σ_j.

This function picks the coordinate which achieves the maximum absolute value under h. The color will be positive if the value of this coordinate is positive and negative if it is negative. If multiple values achieve the maximum, pick the smallest one. Note that if all coordinates of a point are zero, then we are already done.

Since h sends antipodal points to opposite values and the coloring is only based on h, this coloring also sends antipodal vertices to opposite colors. Therefore the coloring satisfies the requirements of Tucker's lemma so for each j, there exists points p_j and q_j that form a complementary 1-simplex. Let (k_j) be the sequence midpoints of these 1-simplices. Since this is a sequence in a compact space, it has a convergent subsequence. Let y be the limit point.

We claim that $h(y) = 0$. Suppose $h(y) = z = (z_1, z_2, \ldots, z_n)$ where some $z_i \neq 0$. Let $\epsilon = \max\{|z_1|, |z_2|, \ldots, |z_n|\}$. Then by continuity of h and convergence of the subsequence of complementary 1-simplices, there exists a complementary 1 simplex such for every point p in the simplex, $\|h(p) - h(y)\| < \frac{\epsilon}{3}$. However, for some i, one of the vertices has a positive i^{th} coordinate under h and the other has a negative i^{th} coordinate. Since the ith coordinate of $h(y)$ is within $\frac{\epsilon}{3}$ of both, $|h(y)_i| < \frac{\epsilon}{3}$. Furthermore since the vertices of the simplex p and q were colored with colors $\pm i$, coordinate i achieves the maximum value. We have $|h(p)_i| < \frac{2\epsilon}{3}$, so for all j, $|h(p)_j| < \frac{2\epsilon}{3}$. Therefore for all j, $|h(y)_j| < \frac{3\epsilon}{3} = \epsilon$. This is a contradiction with the definition of ϵ, so we have $h(y) = 0$.

Therefore $f(g(y)) = f(-g(y))$, and thus

$$f(y_1, y_2, \ldots, y_n, 1 - \|y\|) = f(-y_1, -y_2, \ldots, -y_n, \|y\| - 1),$$

so we have two antipodal points which are mapped to the same point as stated by the theorem. □

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References

[1] Jacob Fox. Sperner's lemma and Brouwer's theorem. http://math.mit.edu/ fox/MAT307-lecture03.pdf

[2] Robert M. Freund and Micheal J. Todd. A Constructive Proof of Tucker's Combinatorial Lemma. Journal of Combinatorial Theory, 1981