Characteristic Classes and Cobordism

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Abstract

Characteristic classes associate vector bundles over topological spaces to elements of their cohomology groups. Although their utility is not immediately apparent, they can concisely encode critical information, especially in the case where our topological spaces are manifolds. In this paper we first seek to give a survey of characteristic classes of unoriented real vector bundles with mod 2 coefficients for cohomology. Then without covering the details we look at the cases of complex and oriented vector bundles, and use our knowledge to tackle the problem of computing cobordism rings.

Contents

1 Introduction

In this paper we seek to produce a thorough survey of characteristic classes and to explore their relation to cobordism. Section 2 of the paper presents the main definitions and introduces the universal bundle of the Grassmanian. The discussion of principal bundles is somewhat disjoint from the project of the paper, but it tries to give some motivation for the universal bundle and for the relationship vector bundles have to matrix groups. Section 2 largely draws from [5]. In Section 3 we strive to supplement section 2 with a few basic examples and lemmas in order to make vector bundles more familiar to the reader.

In sections 4 and 5 we explore Stiefel-Whitney classes. Throughout section 4 we use the cohomology of the Grassmanian to show that Stiefel-Whitney classes generate all characteristic classes of real vector bundles with mod 2 coefficients. Computing the cohomology of the Grassmanian is one of the major projects of this paper, and it draws largely from techniques in [1]. In section 5 we use Stiefel-Whitney numbers to relate characteristic classes to cobordism. In section 6 we go on to prove the Thom Isomorphism with mod 2 coefficients, hoping to provide the reader with ample motivation concerning Thom classes and their utility.

Up until the end of section 6 we try to include as many details as possible and for this reason we restrict ourselves to the cases of real vector bundles and mod 2 coefficients. In section 7 we hope to extend the results we have found so far in an intuitive way to the complex and oriented cases. Finally in section 8 we tackle the problem of cobordism, proving the crucial theorem that cobordism rings can be computed as homotopy groups of Thom spaces of universal bundles.

Throughout this paper we primarily follow [1] and [2], and most proofs are borrowed from them in some capacity. However we always try to flesh out the details that are central to our project and give less importance to those which are tangential to it. Furthermore for the core theorems such as the Thom Isomorphism and the Thom-Pontrjagin construction we attempt to make a balance between providing the most canonical proofs and those that provide the reader with the best intuition.

2 Principal Bundles and Classifying Spaces

Definition 2.1. A fiber bundle is an ordered tuple (E, B, F) of topological spaces along with a surjective continuous map $p : E \to B$ such that:

(i) For all $b \in B$, $p^{-1}(b)$ is homeomorphic to F

(ii) Every $b \in B$ has an open neighborhood U such that there exists a homeomorphism $\Phi : U \times F \to p^{-1}(U)$ satifying $(p \circ \Phi)(u, x) = u$.

The map Φ is called a trivialization. Furthermore any map $s : B \to E$ such that $p \circ s$ is the identity is called a section. We will often abuse notation by referring to vector bundles by their projection map. We will study in more detail two types of fiber bundles with added structure.

Definition 2.2. A vector bundle is a fiber bundle where F has a finite-dimensional vector space structure and Φ is a linear isomorphism on each fiber. Specifically a real vector bundle is one where F is a vector space over $\mathbb{R}$ and a complex vector bundle is one where F is a vector space over $\mathbb{C}$. The dimension of a vector bundle is the dimension of F as a vector space.

Definition 2.3. A principal bundle is a fiber bundle where F is homeomorphic to a topological group G. Furthermore E has a right G-action which preserves fibers. Note that this means that B can be identified with the orbit space E/G.

Given two fiber bundles $p : E \to B$ and $p' : E' \to B'$ a bundle map is an ordered pair (f, g) that makes the following diagram commute:

$$\begin{array}{ccc} E & \xrightarrow{f} & E' \\ \Big\downarrow{\scriptstyle p} & & \Big\downarrow{\scriptstyle p'} \\ B & \xrightarrow{g} & B' \end{array}$$

A bundle map over B is one where $B' = B$ and g is the identity. In the case of principal bundles we further require f to be G-equivariant. Additionally in the case of vector bundles we require that both vector bundles have the same dimension and that f is linear on fibers. If f is also a homeomorphism then we call (f, g) an isomorphism. In the case of vector bundles we further require that f is an isomorphism on each fiber. We leave it to the reader to prove the small lemma that every bundle map of G-principal bundles is an isomorphism. This allows us to define the set $\mathcal{E}(B)$ of isomorphism classes of vector bundles over B. Specifically we define $\mathcal{E}_n(B)$ to be the set of isomorphism classes of n-dimensional real vector bundles over B and $\mathcal{E}_G(B)$ to be the set of isomorphism classes of principal G-bundles over B.

Given a fiber bundle $\xi : E \to B$ and a continuous map $f : A \to B$ we define $f^*\xi$ to be the pullback of the maps ξ and f. Concretely this is the set of all ordered pairs $(a, e) \in A \times E$ such that $f(a) = \xi(e)$. This allows $\mathcal{E}(-)$ to be a contravariant functor from the category of topological spaces to the category of sets. In particular $\mathcal{E}(-)$ sends each topological space to the set of isomorphism classes of fiber bundles over it, and sends morphisms between topological spaces to pullbacks of bundles. We note without proof that if f is homotopic to g then $f^*\xi$ will be isomorphic to $g^*\xi$ so that $\mathcal{E}(-)$ factors through the category of homotopy classes of spaces.

In this paper we will primarily be studying vector bundles. However the next theorem explains the utility of principal bundles for our purposes:

Theorem 2.4. *There exists a bijection between the set $\mathcal{E}_n(B)$ and the set $\mathcal{E}_{GL_n\mathbb{R}}(B)$ of $GL_n\mathbb{R}$-principal bundles over B.*

Proof. First we define a map $\psi : \mathcal{E}_n(B) \to \mathcal{E}_{GL_n\mathbb{R}}(B)$. Given some vector bundle $\xi : E \to B$ we let $\psi(\xi)$ be the principal bundle over B where the fiber of b is defined to be the set of n-frames of $\xi^{-1}(b)$. An n-frame is an n-tuple of linearly independant vectors. This is a $GL_n\mathbb{R}$-principal bundle because ordered sets of n linearly independant vectors in $\mathbb{R}^n$ are in correspondance with nonsingular matrices. Now we define a map $\tau : \mathcal{E}_{GL_n\mathbb{R}}(B) \to \mathcal{E}_n(B)$. Given a $GL_n\mathbb{R}$-principlal bundle $\eta : D \to B$ we define $\tau(\nu)$ to be $D \times \mathbb{R}^n$ with the equivalence relation that $(dg, x) \sim (d, gx)$ for all $g \in GL_n\mathbb{R}$. Thus the base space is the orbit space $D/GL_n\mathbb{R}$ and each fiber is $\mathbb{R}^n$. Note that $D/GL_n\mathbb{R}$ is canonically isomorphic to B. Now we need to make sure τ and ψ and inverses. If we start with a vector bundle ξ then $(\tau \circ \psi)(\xi)$ is $\psi(\xi) \times \mathbb{R}^n/\sim$. But there is an isomorphism $e : \psi(\xi) \times \mathbb{R}^n/\sim \to E$ defined by mapping $(\{v_1, ..., v_n\}, (x_1, ..., x_n))$ to $x_1v_1 + ... + x_nv_n$. Since $v_1, ..., v_n$ are linearly independant this must be an isomorphism. Thus $\tau \circ \psi$ is the identity.

To show that $\psi \circ \tau$ is the identity recall that every bundle map of principal bundles is an isomorphism. Now take a principal $GL_n\mathbb{R}$-bundle $\eta : D \to B$ and note that $(\psi \circ \tau)(\eta)$ is the set of n-frames of the vector bundle $D \times \mathbb{R}^n/\sim$.

Thus there is a canonical bundle map $D \to D \times \mathbb{R}^n / \sim$ which sends d to the set $\{(d, e_1), ..., (d, e_n)\}$. $\square$

We note that the bijection from Theorem 2.4 is actually a natural isomorphism of functors, but we will leave the added details to the reader.

Clearly given a fiber bundle $\xi : E \to B$ and a space A we can define a map $\phi_\xi : [A, B] \to \mathcal{E}(A)$ given by $\phi_\xi(f) = f^*\xi$. If we let ξ be a principal G-bundle and $\phi_\xi : [A, B] \to \mathcal{E}_G(A)$ is a bijection for all spaces A then we call B a classifying space for G and E its universal bundle. A classifying space for a toplogical group G is usually denoted BG. The next theorem, which we shall not prove, gives us a way of finding universal bundles.

Theorem 2.5. *If E is weakly contractible then E is a universal bundle. For E to be weakly contractible we require that all its homotopy groups are trivial.*

We define $V_n\mathbb{R}^{n+k}$ to be the set of all n-frames in $\mathbb{R}^{n+k}$ and we let it inherit the subspace toplogy from $\mathbb{R}^{n(n+k)}$. Now if we identify all the n-frames that span the same subspace of $\mathbb{R}^{n+k}$ we get the Grassmanian $G_n\mathbb{R}^{n+k}$, which we endow with the quotient topology. Note that $G_n\mathbb{R}^{n+k}$ is $V_n\mathbb{R}^{n+k}/GL_n\mathbb{R}$ where $GL_n\mathbb{R}$ acts by permuting the n-frames of each n-dimensional subspace. Thus the map $p : V_n\mathbb{R}^{n+k} \to G_n\mathbb{R}^{n+k}$ defines a principal $GL_n\mathbb{R}$-bundle. Similarly if we take the set $V_n^O\mathbb{R}^{n+k}$ consisting only of orthonormal n-frames then $p : V_n^O\mathbb{R}^{n+k} \to G_n\mathbb{R}^{n+k}$ defines a principal $O(n)$-bundle. By taking the direct limit as $k \to \infty$ we get the principal $GL_n\mathbb{R}$-bundle $p : V_n\mathbb{R}^\infty \to G_n\mathbb{R}^\infty$ and the $O(n)$-bundle $p : V_n^O\mathbb{R}^\infty \to G_n\mathbb{R}^\infty$. Following the work of [5] we want to show these are universal bundles.

Theorem 2.6. *$V_n\mathbb{R}^\infty$ and $V_n^O\mathbb{R}^\infty$ are weakly contractible.*

Proof. We note first that by using Gramm-Schmitt orthogonalization we can deformation retract $V_n\mathbb{R}^\infty$ to $V_n^O\mathbb{R}^\infty$ so we only need to prove the theorem for the latter. Furthermore since $V_n^O\mathbb{R}^\infty$ is a direct limit of $V_n^O\mathbb{R}^{k+n}$ we can reduce our problem to showing that $\pi_i V_n^O\mathbb{R}^{n+k}$ is trivial for $i < k$. To do this we define a fibration $p : V_n^O\mathbb{R}^{n+k} \to S^{n+k-1}$ by mapping $(v_1, ..., v_n)$ to v_1. Each fiber of p consists of the subspace generated by $v_2, ..., v_n$ in the orthogonal complement of v_1. Thus the fibers are all $V_{n-1}^O\mathbb{R}^{n+k-1}$, so the long exact sequence associated to this fibration is

$$\pi_{i+1}(S^{n+k-1}) \to \pi_i(V_{n-1}^O\mathbb{R}^{n+k-1}) \to \pi_i(V_n^O\mathbb{R}^{n+k}) \to \pi_i(S^{n+k-1}) \to \pi_{i-1}(V_{n-1}^O\mathbb{R}^{n+k-1})$$
.

We know that for all $i < k$, $\pi_i V_1^O\mathbb{R}^{1+k} = \pi_i S^k = 0$. Now we use induction noting that if $\pi_i V_{n-1}^O\mathbb{R}^{n+k-1} = 0$ and $\pi_i S^{n+k-1} = 0$ then $\pi_i V_n^O\mathbb{R}^{n+k}$ must be trivial as well.

$\square$

This tells us that up to homotopy $BGL_n\mathbb{R} \cong BO(n) \cong G_n\mathbb{R}^\infty$. Using Theorem 2.4 we see that $G_n\mathbb{R}^\infty$ is also a classfying space for $\mathbb{R}^n$-bundles. Looking at the the construction of the bijection in Theorem 2.4 and filling in some details we see that in the univeral $\mathbb{R}^n$-bundle the fiber of each n-plane $X \in G_n\mathbb{R}^\infty$ must be X itself. The total space of this vector bundle will be denoted γ^n. We can define $G_n\mathbb{C}^\infty$ and its universal bundle $\gamma_{\mathbb{C}}^n$ in the same way as in the real case. It turns out that up to homotopy $BU(n) \cong G_n\mathbb{C}^\infty$ and that this bundle classifies complex n-dimensional vector bundles. These results are summarized in the following result:

Corollary 2.7. *Given a space A, every n-dimensional real vector bundle over A is of the form $f^*\gamma^n$ for some map $f : A \to BO(n)$. Similarly every n-dimensional complex vector bundle over A is of the form $f^*\gamma^n_{\mathbb{C}}$ for some map $f : A \to BU(n)$. In both cases f is unique up to homotopy.*

There exist proofs that γ^n is the universal bundle without recourse to principal bundles, for example in [1]. However this paper's approach explains the notation $BO(n)$ and $BU(n)$. Furthermore it should be noted that in categorical language this corollary says that the functors $\mathcal{E}_n(B)$ and $\mathcal{E}^{\mathbb{C}}_n(B)$ are represented by $BO(n)$ and $BU(n)$ respectively.

3 Examples of Vector Bundles

In this section we define and discuss a few important vector bundles.

Example 3.1. If B is a space then $B \times \mathbb{R}^n$ is always a vector bundle. This is called the trivial bundle and it will be denoted ε^n or simply ε.

Example 3.2. The canonical line bundle is a vector bundle over projective space $\mathbb{P}^n$ where the fiber of each $x \in \mathbb{P}^n$ is the line in $\mathbb{R}^{n+1}$ represented by x. Note that this is another way of defining $G_1\mathbb{R}^{n+1}$. It will usually be denoted γ^1_n.

Definition 3.3. Given a space B and two vector bundles $\xi : E \to B$ and $\nu : E' \to B$ over B we can define a new vector bundle $\xi \times \nu$ over $B \times B$ by letting the fiber of each $(b, b') \in B \times B$ be $\xi^{-1}(b) \times \nu^{-1}(b')$. If we take the pullback over the diagonal map $d : B \to B \times B$ then we get a new bundle $d^*(\xi \times \nu) := \xi \oplus \nu$. This is called the Whitney Sum of ξ and ν.

Example 3.4. We will assume the reader is familiar with manifolds. Given a smooth manifold M we can define the tangent bundle τ_M over M by letting the fiber of each point $x \in M$ be the tangent space T_xM. Similarly, given some embedding of M into $\mathbb{R}^n$, we can define the normal bundle ν_M by letting the fiber of every $x \in M$ be the space of all vectors orthogonal to the tangent space.

Lemma 3.5. *If a vector bundle map over some base space B is an isomorphism on each fiber then it is a homeomorphism on the total spaces and thus an isomorphism of vector bundles.*

Theorem 3.6. *An n-dimensional vector bundle is trivial if and only if it has n sections which are linearly independant on each fiber.*

Proof. If an n-bundle ξ over B has n linearly independant sections $s_1, ..., s_n$ then we create the bundle map $f : B \times \mathbb{R}^n \to \xi$ defined by $f(b, (x_1, ..., x_n)) = x_1s_1(b) + ... + x_ns_n(b)$. Since $\{s_1(b), ..., s_n(b)\}$ form a basis for the fiber of ξ at b this is an isomorphism on each fiber. Conversely if ξ is the trivial bundle then it is of the form $B \times \mathbb{R}^n$ so we can let each s_i be the map $b \mapsto (b, e_i)$ for all $b \in B$.

□

Corollary 3.7. *The canonical line bundle γ^1_n is non-trivial for all $n \geq 1$.*

Proof. First we prove the statement for $n = 1$. Assume we have a section $s : \mathbb{P}^1 \to \gamma_1^1$. Then we can precompose this map with the projection $\pi : S^1 \to \mathbb{P}^1$ to get a map $s \circ \pi$ which sends each $x \in S^1$ to a pair $(\overline{x}, v)$ for some v in the line spanned by x. We can thus define a continuous map $f : S^1 \to \mathbb{R}$ where $f(x)$ is the unique real number such that $(s \circ \pi)(x) = (x, cx)$. Since $(s \circ \pi)(x) = (s \circ \pi)(-x)$, we have that $f(x) = -f(-x)$. Thus by the intermediate value theorem f must have a zero in the arc from x to $-x$. For $n > 1$ we note that the restriction of γ_n^1 to S^1 is γ_1^1 and thus must be non-trivial.

□

On the other hand the tangent bundle of S^1 is trivial as we can obtain a continuous non-zero section by taking unit tangent vectors with a consistent orientation.

4 Characteristic Classes

Characteristic classes are the central notion of this paper.

Definition 4.1. A characteristic class of degree q is a natural transformation from the functor $\mathcal{E}_n(-)$ to the cohomology functor $H^q(-;\pi)$ for some abelian group π. Concretely this means that a characteristic class c assigns to every vector bundle $p : E \to B$ an element $c(p) \in H^q(B;\pi)$. Furthermore given a map $f : \xi \to \nu$ of vector bundles, c must satisfy $f^*c(\nu) = c(\xi)$.

From now until the end of section 6 we will assume that $\pi = \mathbb{Z}/2\mathbb{Z}$. We are now going to define the Stiefel-Whitney classes axiomatically. Later we will show how to construct these from the Steenrod squares, but the construction yields less intuition towards their importance.

Definition 4.2. There exist characteristic classes ω_i of degree i for all $i \geq 0$ which are characterized by the following axioms.

(i) $\omega_0(\xi) = 1$ for any vector bundle ξ.
(ii) If ξ is an n-dimensional vector bundle then $\omega_i(\xi) = 0$ for $i > n$.
(iii) $\omega_k(\xi \oplus \nu) = \sum_{i=0}^{k} \omega_i(\xi) \cup \omega_{k-i}(\nu)$.
(iv) For the canonical line bundle γ_1^1 of $\mathbb{P}^1$, $\omega_1(\gamma_1^1) \neq 0$.

The last axiom exists to force Stiefel-Whitney classes to be non-trivial. In some texts naturality is also considered an axiom, but we treat it as part of the definition of a characteristic class. We now compute the simplest Stiefel-Whitney class.

Lemma 4.3. *$\omega_i(\varepsilon) = 0$ for $i \geq 1$.*

Proof. Given $\varepsilon = B \times \mathbb{R}^n$ we can define a bundle map (f, g) from ε to the trivial bundle ε_* over a point $*$. We let $g(b) = *$ for all $b \in B$ and let $f(x, v) = (*, v)$. This map is linear on each fiber and commutes with the projection maps, so it is a bundle map. Furthermore we know that $H^i(*) = 0$ and thus $\omega_i(\varepsilon_*) = 0$ for $i \geq 1$. Thus for $i \geq 1$ we have $\omega_i(\varepsilon) = g^*(\omega_i(\varepsilon_*)) = 0$.

□

Lemma 4.4. *$\omega_1(\gamma_n^1) = x$ where x is the generator of $H^1(\gamma_n^1)$ and $\omega_i(\gamma_n^1) = 0$ for $i > 1$.*

Proof. The last statement follows directly from axiom (ii) since all canonical line bundles are 1-dimensional. Furthermore since $H^1(\mathbb{P}^1) \cong \mathbb{Z}/2\mathbb{Z}$ axiom (iv) tells us that we must have $\omega_1(\gamma_1^1) = x$. To complete the proof we note that the inclusion $i : \omega_1(\gamma_1^1) \to \omega_1(\gamma_n^1)$ is a bundle map. Thus $i^*(\omega_1(\gamma_n^1)) = \omega_1(\gamma_1^1) = x$. So indeed $\omega_1(\gamma_n^1) \neq 0$ and thus $\omega_1(\gamma_n^1)$ is the generator of $H^1(\mathbb{P}^n) \cong \mathbb{Z}/2\mathbb{Z}$.

□

Now we define the total Stiefel-Whitney class $\omega(\xi) \in H^*(B)$ of a vector bundle $\xi : E \to B$ to be the sum $\omega_0(\xi) + \omega_1(\xi) + \omega_2(\xi) + ...$ We note that because of axiom (ii) this is always a finite sum. We also see that because of axiom (iii) we obtain the concise equation $\omega(\xi \oplus \zeta) = \omega(\xi) \cup \omega(\zeta)$. This means that given $\omega(\xi \oplus \zeta)$ and $\omega(\xi)$ we could find $\omega(\zeta)$ by looking for inverses in the cohomology ring. The following theorem provides one case where this strategy is particularly useful.

Theorem 4.5 (Whitney Duality Formula)**.** *Given a smooth manifold M the total Stiefel-Whitney classes of the normal and tangent bundles are inverses, so $\omega(\tau_M) \cup \omega(\nu_M) = 1$.*

Our next task is to compute the cohomology of $BO(n)$, and we do so largely by following the work in [1]. First we want to describe the cell structure of $G_n\mathbb{R}^k$. Let $S \in G_n\mathbb{R}^k$ and look at the sequence of numbers:

$$0 = \dim(S \cap \mathbb{R}^0) \leq \dim(S \cap \mathbb{R}^1) \leq ... \dim(S \cap \mathbb{R}^{k-1}) \leq \dim(S \cap \mathbb{R}^k) = n.$$

Clearly two consective numbers must either be the same or differ by one. Let $\sigma = (\sigma_1, ..., \sigma_n)$ be an n-tuple such that $1 \leq \sigma_1 < \sigma_2 < ...\sigma_{n-1} < \sigma_n \leq k$. We can define e_σ to be the subset of $G_n\mathbb{R}^k$ containing all n-planes such that

$$\dim(S \cap \mathbb{R}^{\sigma_i - 1}) = i - 1 \text{ and } \dim(S \cap \mathbb{R}^{\sigma_i}) = i.$$

Theorem 4.6. *The sets e_σ for every possible σ provide a cell structure for $G_n\mathbb{R}^k$, where each e_σ has dimension $\sum_{i=1}^n (\sigma_i - i)$.*

Proof. For a proof refer to [1].

□

Corollary 4.7. *The number of r-cells in $G_n\mathbb{R}^k$ is the number of partitions of r into at most n non-negative integers which are not greater than $k - n$. Thus the number of r-cells in $BO(n)$ is the number of ways to split r into at most n non-negative integers.*

Proof. Every r-cell of $G_n\mathbb{R}^k$ is uniquely defined by an n-tuple $(\sigma_1, ..., \sigma_n)$ where $\sum_{i=1}^n (\sigma_i - i)$ and each $\sigma_i - i$ is between zero and $k - n$.

□

This construction allows us to prove the following critical theorem:

Theorem 4.8. *The cohomology ring of $BO(n)$ is freely generated by the Stiefel-Whitney classes of γ^n so that as an algebra*

$$H^*(BO(n)) \cong \mathbb{Z}/2\mathbb{Z}[\omega_1(\gamma^n), ..., \omega_n(\gamma^n)].$$

Proof. Let $C^*(BO(n))$ be the cellular cochain complex associated with the above CW-structure of $BO(n)$. Then we know that $\dim(H^r(BO(n))) \leq \dim(C^r(BO(n)))$ which by Theorem 4.7 is the number of partitions of n into at most r non-negative integers.

Now note that the degree r part of the algebra $\mathbb{Z}/2\mathbb{Z}[\omega_1(\gamma^n), ..., \omega_n(\gamma^n)]$ has a basis consisting of terms of the form $\omega_1(\gamma^n)^{r_1}\omega_2(\gamma^n)^{r_2}...\omega_n(\gamma^n)^{r_n}$, where $\sum_{i=1}^n ir_i = r$. We can put these monomials in bijection with partitions of r into at most n integers by sending each monomial to the following partition:

$$(r_1 + ... + r_n) + (r_2 + ... + r_n) + ... + r_n = \textstyle\sum_{i=1}^n ir_i = r.$$

Thus if we prove that the algebra $\mathbb{Z}/2\mathbb{Z}[\omega_1(\gamma^n), ..., \omega_n(\gamma^n)]$ is contained in $H^*(BO(n))$ they would have to be equal.

Since we know each Stiefel-Whitney class is in $H^*(BO(n))$ we just need to show that they do not have any polynomial relations. Assume for sake of contradiction that they do. Given any other bundle ξ, we know by Corollary 2.7 that there is a bundle map $(f, g) : \xi \to \gamma^n$ and thus that $g^*(\omega_i(\gamma^n)) = \omega_i(\xi)$. Thus if there is a polynomial relation between $\omega_1(\gamma^n), ..., \omega_n(\gamma^n)$, there must also be one between $\omega_1(\xi), ..., \omega_n(\xi)$.

Now we let the base space of ξ be $BO(1)^n$ and we let ξ be $(\gamma^1)^n$. If we define π_i to be the projection of $BO(1)^n$ onto its ith factor then ξ is $\pi_1^*(\gamma^1) \oplus \pi_2^*(\gamma^1) \oplus ... \oplus \pi_n^*(\gamma^1)$. By Kunneth's Formula we know that as an algebra $H^*(BO(1)^n)$ has n generators all of dimension 1. Furthermore recall that $\omega(\gamma^1) = 1 + x$. Thus since each π_i pulls x back to a different degree 1 generator of $H^*(BO(1)^n)$, we have

$$\omega(\xi) = \omega(\pi_1^*(\gamma^1))\omega(\pi_2^*(\gamma^1))...\omega(\pi_n^*(\gamma^1)) = (1 + x_1)(1 + x_2)...(1 + x_n)$$

Thus each $\omega_i(\xi)$ is the ith elementary symmetric polynomial in n variables. These are known to have no polynomial relations so we are done. □

Before putting this statement to use we must recall a key fact from category theory.

Theorem 4.9 (Yoneda's Lemma)**.** *Let F and G be two contravariant functors from a category C to the category of sets. Then if F is represented by A with a natural transformation $\Phi : Hom(-, A) \to F$, there is a bijection*

$$\Psi\colon \mathit{Nat}(F, G) \xrightarrow{\sim} G(A)$$

where $Nat(F, G)$ is the set of natural transformations from F to G. Furthermore Ψ is defined by $f \mapsto f(\Phi(Id_A))$.

Corollary 4.10. *There is a bijection Θ between the set of characteristic classes of n-dimensional real vector bundles and $H^*(BO(n))$. This bijection is defined by $c \mapsto c(\gamma^n)$.*

Proof. Corollary 2.7 tells us that $BO(n)$ represents the functor $\mathscr{E}_n(-)$ and characteristic classes are defined as natural tranformations from $\mathscr{E}_n(-)$ to $H^*(-)$. Thus Yoneda's Lemma tells us that Θ exists. Furthermore the correspondance in Corollary 2.7 carries the map $\text{Id}_{BO(n)}$ to $\text{Id}^*_{BO(n)}(\gamma^n) = \gamma^n$ so Θ is defined by $c \mapsto c(\gamma^n)$ for all characteristic classes c.

$\square$

Now since Theorem 4.8 tells us the structure of the cohomology ring $H^*(BO(n))$, we can compute the set of characteristic classes of real n-dimensional vector bundles.

Corollary 4.11. *Every characteristic class of real n-dimension vector bundles is a linear combination of characteristic classes of the form* $\omega_1^{i_1}\omega_2^{i_2}...\omega_n^{i_n}$ *for some non-negative integers* $i_1, ..., i_n$.

5 Stiefel-Whitney Numbers

Definition 5.1. Given a cohomology class $a \in H^n(X)$ and a homology class $b \in H_n(X)$ we define the Kronecker index $\langle a, b\rangle$ to be the evaluation of b at a. We leave it to the reader to prove this is well defined.

Lemma 5.2. *Given a map* $f : X \to Y$ *we have* $\langle f^*a, b\rangle = \langle a, f_*b\rangle$.

Recall that every n-manifold M is $\mathbb{Z}/2\mathbb{Z}$-orientable and thus has a fundamental class μ_M. Furthermore if μ is the fundamental class of the pair $(M, \partial M)$ then $\partial\mu = \mu_{\partial M}$.

Definition 5.3. Let M be a smooth n-dimensional manifold. Given an n-tuple $\mathbf{r} = (r_1, ..., r_n)$ such that $r_1 + 2r_2 + ... + nr_n = n$ we define the Stiefel-Whitney number $\omega_{\mathbf{r}}[M] \in \mathbb{Z}/2\mathbb{Z}$ to be

$$\omega_{\mathbf{r}}[M] := \langle \omega_1^{r_1}(\tau_M)\omega_2^{r_2}(\tau_M)...\omega_n^{r_n}(\tau_M), \mu_M\rangle.$$

Note that in light of Corollary 4.11 the Stiefel-Whitney numbers of a manifold M fully determine every number of the form $\langle c(\tau_M), \mu_M\rangle$ for some characteristic class c. Now we are ready to state the main theorem of this section. Our proof largely emulates the proof in [2].

Theorem 5.4. *If N is a compact (n+1)-manifold and $M = \partial N$ is a smooth n-manifold then all the Stiefel-Whitney numbers of M are zero.*

Proof. Let μ_N be the fundamental class of the pair (N, M) so that we have $\partial\mu_N = \mu_M$. The restriction of the tangent bundle of N to M is spanned by the tangent bundle of M and by a unit inward normal vector. It is a small lemma that if a bundle is the cartesian product of two other bundles in each fiber then it is their Whitney sum. Furthermore the unit inward normal vector spans the trivial bundle and thus we have

$$\tau_N|_M \cong \tau_M \oplus \varepsilon.$$

If we let i be the inclusion map $M \hookrightarrow N$ then it follows that i^* is the restriction of $H^*(N)$ to $H^*(M)$ and thus that we have

$$i^*(\omega_k(\tau_N)) = \omega_k(\tau_N|_M) = \omega_k(\tau_M \oplus \varepsilon) = \omega_k(\tau_M).$$

Now if we let $\omega_{\mathbf{r}}$ be the charactertsic class $\omega_1^{r_1}\omega_2^{r_2}...\omega_n^{r_n}$ then we have

$$\omega_{\mathbf{r}}[M] = \langle \omega_{\mathbf{r}}(\tau_M), \mu_M \rangle = \langle i^*(\omega_{\mathbf{r}}(\tau_N)), \partial\mu_N \rangle = \langle \omega_{\mathbf{r}}(\tau_N), i_*(\partial\mu_N) \rangle.$$

Finally we look at the long exact sequence of the pair (N, M) to see that $i_*(\partial\mu_N) = 0$ and we have that $\omega_{\mathbf{r}}[M] = 0$.

□

Although it is beyond the scope of this paper, the converse of Theorem 5.4 is also true, and was first proved by Thom in [8]. This idea will be further generalized later with the concept of cobordism.

Theorem 5.5 (Thom)**.** *If M is a smooth n-manifold then M is the boundary of some compact $(n + 1)$-manifold if and only if all its Stiefel-Whitney numbers are zero.*

6 Thom Isomorphism

Definition 6.1. Given any n-dimensional vector bundle $\xi : E \to B$ we can define a new fiber bundle $Sph(\xi) : Sph(E) \to B$ by taking the one-point compactification of each fiber so that each fiber looks like S^n. Then we can define a section $s : B \to Sph(E)$ which sends each $b \in B$ to the point at infinity in its fiber. By taking the quotient space $\mathrm{Sph}(E)/s(B)$ we associate all the points at infinity to get the Thom space of the the vector bundle ξ. We shall denote the Thom space as $T\xi$. Additionally given a fiber F let F_0 be $F - \{0\}$ and let E_0 be the fiber bundle over B where each fiber F is replaced with F_0. We can think of E_0 as E with the zero section removed.

Lemma 6.2. *$H^*(T\xi, \infty) \cong H^*(E, E_0)$.*

Proof. Let $T\xi_0$ be $T\xi$ with the zero section removed. Every fiber of $T\xi_0$ looks like $\mathbb{R}^n$ and thus we can deformation retract $T\xi_0$ to ∞. Thus we have that $H^*(T\xi, \infty) \cong H^*(T\xi, T\xi_0)$. Now looking at the excisive triad $(T\xi, T\xi_0, \infty)$ we get that $H^*(T\xi, T\xi_0) \cong H^*(T\xi - \infty, T\xi_0 - \infty) \cong H^*(E, E_0)$.

□

The notion of Thom spaces is very important for the computation of the cobordism ring which will be defined later. The above lemma relates Thom spaces to more traditional notions of orientation.

Definition 6.3. Given a vector bundle $\xi : E \to B$, a Thom class of ξ is some element $z \in H^n(E, E_0)$ such that the restriction of z to (F, F_0) for any fiber F generates $H^n(F, F_0) \cong H^n(\mathbb{R}, \mathbb{R} - 0) \cong \mathbb{Z}/2\mathbb{Z}$.

A Thom class is sometimes instead called an orientation. Note that in light of Lemma 6.2 we also could have defined a Thom class to be some element $z \in H^n(T\xi, \infty)$ such that the restriction of z to any fiber $F \cup \infty$ generates $H^n(F \cup \infty) \cong H^n(S^n) \cong \mathbb{Z}/2\mathbb{Z}$. Now we are ready to prove the Thom Isomorphism, a statement with vast applications in algebraic topology.

Theorem 6.4 (Thom Isomorphism)**.** *Given an n-dimensional vector bundle $\xi : E \to B$ there is a unique Thom class $z \in H^n(E, E_0)$. Furthermore the map $\Phi : H^*(E) \to H^{*+n}(E, E_0)$ defined by $\Phi(x) = x \cup z$ is an isomorphism.*

We are going to need to recall the relative version of Kunneth's Theorem for cohomology with field coefficients.

Theorem 6.5 (Kunneth's Theorem)**.** *Given pairs* (X, A) *and* (Y, B) *with* $H^*(Y, B)$ *finitely generated, we have an isomorphism*

$$\Psi : H^*(X, A) \otimes H^*(Y, B) \to H^*(X \times Y, X \times B \cup A \times Y)$$

where $\Psi(s \otimes t) = p_1^*(s) \cup p_2^*(t)$ *and the* p_i *are projections from* $X \times Y$ *to* X *and to* Y *respectively.*

Proof of 6.4. We begin with the trivial case of $E = B \times \mathbb{R}^n$. Taking the pairs $(B, \emptyset)$ and $(\mathbb{R}^n, \mathbb{R}^n - \{0\})$ and applying Theorem 6.5 we get an isomorphism

$$H^*(B) \otimes H^*(\mathbb{R}^n, \mathbb{R}^n - \{0\}) \to H^*(B \times \mathbb{R}^n, B \times \mathbb{R}^n\{0\}) = H^*(E, E_0)$$

defined by $b \otimes x \mapsto p_1^*(b) \cup p_2^*(x)$. Now note that by uniformly contracting the fibers of a vector bundle $\xi : E \to B$ we can deformation retract E to its zero section, which is identical to B. This tells us that the projection $p_1 : E \to B$ is in fact a homotopy equivalence, so that p_1^* is an isomorphism. Combining this with the above isomorphism we get a new isomorphism

$$\Gamma : H^*(E) \otimes H^*(\mathbb{R}^n, \mathbb{R}^n - \{0\}) \to H^*(E, E_0)$$

defined by $e \otimes x \mapsto e \cup p_2^*(x)$. If we want to find a Thom class $z \in H^n(E, E_0)$ then we must must find some $a \in H^*(E) \otimes H^{n-*}(\mathbb{R}^n, \mathbb{R}^n - \{0\})$ such that $\Gamma(a) = z$. However $H^*(\mathbb{R}^n, \mathbb{R}^n - \{0\})$ is trivial except in the nth degree so we must have $a \in H^0(E) \otimes H^n(\mathbb{R}^n, \mathbb{R}^n - \{0\})$. In order for the restriction of a to any fiber F to generate $H^n(F, F_0)$ we need the first component of a to be the identity element and the second component to be the unique non-trivial element s of $H^n(\mathbb{R}^n, \mathbb{R}^n - \{0\})$. So indeed we have a unique Thom class $z = \Gamma(1 \otimes s) = 1 \cup p_2^*(s) = p_2^*(s)$.

Finally since $H^*(\mathbb{R}^n, \mathbb{R}^n - \{0\}) = \langle s \rangle \cong \mathbb{Z}/2\mathbb{Z}$ we have the canonical isomorphism $\Theta : H^*(E) \to H^*(E) \otimes H^*(\mathbb{R}^n, \mathbb{R}^n - \{0\})$ defined by $e \mapsto e \otimes s$. Thus we get an isomorphism $\Gamma \circ \Theta : H^*(E) \to H^{*+n}(E, E_0)$ defined by $\Gamma(\Theta(e)) = \Gamma(e \otimes s) = e \cup p_2^*(s) = e \cup z$.

Next we use the Mayer-Vietoris sequence. Assume we have a vector bundle $\xi : E \to B$ such that $B = C \cup D$, $\xi^{-1}(C) = U$, $\xi^{-1}(D) = V$, and the Thom isomorphism applies to the restrictions of ξ to U, V, and $W := U \cap V$. Thus by definition there are unique Thom classes u and v corresponding to U and V. Now we can look at the following relative Mayer-Vietoris sequence:

$$\to H^{n-1}(W, W_0) \xrightarrow{i} H^n(E, E_0) \xrightarrow{j} H^n(U, U_0) \oplus H^n(V, V_0) \xrightarrow{\partial} H^n(W, W_0) \to .$$

By definition u and v must restrict to the same thing in W and thus $(u, v) \in ker(\partial)$. Thus by the exactness of the sequence there is some z such that $j(z) = (u, v)$. Furthermore by the Thom isomorphism applied to W we know that $H^{n-1}(W, W_0) = 0$ so j must be injective. Thus there is a unique z such that $j(z) = (u, v)$. This is a Thom class for ξ.

Now to prove the Thom isomorphism for E we look at the following diagram, where the rows are each formed via the Mayer-Vietoris sequence and the downward maps are all cupping with the Thom classes.

$$\begin{array}{ccccccccc}
H^{m-1}(U) \oplus H^{m-1}(U) & \longrightarrow & H^{m-1}(W) & \longrightarrow & H^m(E) & \longrightarrow & H^m(U) \oplus H^m(V) & \longrightarrow & H^m(W) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
H^{m+n-1}(U,U_0) \oplus H^{m+n-1}(V,V_0) & \longrightarrow & H^{m+n-1}(W,W_0) & \longrightarrow & H^{m+n}(E,E_0) & \longrightarrow & H^{m+n}(U,U_0) \oplus H^{m+n}(V,V_0) & \longrightarrow & H^{m+n}(W,W_0)
\end{array}$$

All of the downward maps except for the middle one are isomorphisms since we can apply the Thom isomorphism to U, V, and W. Thus by the five lemma the middle map is an isomorphism as well, and we have proven the Thom isomorphism for E. Since vector bundles are locally trivial we can clearly use this argument inductively to show that the theorem holds for all compact sets.

Leaving most of details to the reader, we can complete the proof by taking inverse limits. Specifically we note that if we let $B^\alpha, \alpha \in I$ be all the compact subsets of B and $E^\alpha = \xi^{-1}(B^\alpha)$ then $H^i(E) \cong \varprojlim H^i(E^\alpha)$ and $H^i(E, E_0) \cong \varprojlim H^i(E^\alpha, E_0^\alpha)$. There is only one $z \in H^n(E, E_0)$ which retricts to the unique Thom class z_α of each $H^n(E^\alpha, E_0^\alpha)$ so z is the unique Thom class. Furthermore in the following commutative diagram we know all the maps except the top one must be isomorphisms, so the top map must be the Thom isomorphism for E.

$$\begin{array}{ccc}
H^m(E) & \xrightarrow{\cup z} & H^{m+n}(E, E_0) \\
\downarrow & & \downarrow \\
\varprojlim H^i(E^\alpha) & \xrightarrow{\cup z_\alpha} & \varprojlim H^i(E^\alpha, E_0^\alpha)
\end{array}$$

□

Combining this with other results from this section we have the following sequence of isomorphisms:

$$H^*(B) \cong H^*(E) \cong H^{*+n}(E, E_0) \cong H^{*+n}(T\xi, \infty)$$

One application of the Thom isomorphism is that it allows us to construct the Stiefel-Whitney classes. If we let Sq^i be the Steenrod squares then we have

$$\omega_i(\xi) = \Phi^{-1} Sq^i \Phi(1) = \Phi^{-1} Sq^i(z)$$

7 Generalizations and Orientation

So far we have purposefully dealt with the easiest cases, looking at real unoriented vector bundles and allowing cohomology to have coefficients in $\mathbb{Z}/2\mathbb{Z}$. In this section we provide a survey of the analogous results for more complicated cases. Nothing major will be proved in this section, but we hope that the results of the previous sections will provide ample intuition for these new cases. Although the results of this section are at times messier to prove, they are just as important, and many are necessary to study cobordism.

Definition 7.1. Given a ring R, an R-orientation of an n-dimensional vector bundle $\xi : E \to B$ is a choice of a generator $a_b \in H^n(\xi^{-1}(b), \xi^{-1}(b)_0; R)$ for each $b \in B$. Furthermore we require that these orientations are locally compatible, so that given any $b' \in B$, there exists an open neighborhood U of b' and an element $u \in H^n(\xi^{-1}(U), \xi^{-1}(U)_0; R)$ which restricts to a_b for every $b \in U$.

Definition 7.2. Given a vector bundle $\xi : E \to B$ with an R-orientation, an R-Thom class is an element $z \in H^n(E, E_0; R)$ which restricts to a_b on each fiber.

Note that this definition is consistent with the definition from the previous section, since $\mathbb{Z}/2\mathbb{Z}$ has a unique generator. With these extra details we can now extend the Thom Isomorphism to cohomology with any coefficients.

Theorem 7.3 (Thom Isomorphism)**.** *Given a vector bundle $\xi : E \to B$ with an R-orientation, it has a unique R-Thom class z. Furthermore the map $\Phi : H^*(E; R) \to H^{*+n}(E, E_0)$ defined by $x \mapsto x \cup z$ is an isomorphism.*

Note that in our proof of the Thom Isomorphism from last section we implicitly proved that every vector bundle has both a $\mathbb{Z}/2\mathbb{Z}$-orientation and a $\mathbb{Z}/2\mathbb{Z}$-Thom class. Further note that Thom classes and orientations turn out to be equivalent concepts. We could define them to be equivalent but we made the distinction to emphasize the point that Thom classes take a local notion of orientation and make it global. For the rest of the section when we refer to orientability it will be with coefficients in $\mathbb{Z}$.

Now we look to complex vector bundles and oriented real vector bundles. We let the functors $\widetilde{\mathcal{E}}_n(-)$ and $\mathcal{E}_n^{\mathbb{C}}(-)$ be the analogues of $\mathcal{E}_n(-)$ in the oriented and complex cases. These functors turn out the be represnted by $G_n(\mathbb{C}^\infty)$ and $\widetilde{G_n}(R^\infty)$, the complex and oriented Grassmanians. Specifically $G_n(\mathbb{C}^\infty)$ is the set of n-planes in $\mathbb{C}^\infty$, constructed just as in the real case, while $\widetilde{G_n}(R^\infty)$ is the set of n-planes in R^∞ with a chosen orientation. Thus in the construction of $\widetilde{G_n}(R^\infty)$ two bases of the same n-plane are identitified only if they have the same orientation. We will refer to these Grassmanians as $BU(n)$ and $BSO(n)$ as they are the classifying spaces for the groups $U(n)$ and $SO(n)$. The universal bundles of these Grassmanians are the obvious ones, which we can denote by $\gamma_{\mathbb{C}}^n$ and $\widetilde{\gamma}^n$.

Clearly the logic of Corollary 4.10 applies to cohomology with coefficients in any ring. The reason why $\mathbb{Z}/2\mathbb{Z}$ coefficients were useful was because we knew the structure of $H^*(BO(n))$. In the case of $BU(n)$ and $BSO(n)$ it is useful to use integral coefficients.

Corollary 7.4. *There is a bijection Θ between the set of characteristic classes of n-dimensional complex vector bundles and $H^*(BU(n); \mathbb{Z})$. This bijection is defined by $c \mapsto c(\gamma_{\mathbb{C}}^n)$. Additionally there is a bijection $\widetilde{\Theta}$ between the set of characteristic classes of n-dimensional oriented real vector bundles and $H^*(BSO(n); \mathbb{Z})$ defined by $c \mapsto c(\widetilde{\gamma}^n)$.*

When looking at complex characteristic classes with coefficients in $\mathbb{Z}$ there exist analogues of Stiefel-Whitney classes called Chern classes. Once again we will skip the construction, but they can be defined axiomatically as follows.

Definition 7.5. There exist characteristic classes c_i of degree $2i$ for all $i \geq 0$ which are characterized by the following axioms.

(i) $c_0(\xi) = 1$ for any vector bundle ξ.
(ii) If ξ is an n-dimensional vector bundle then $c_i(\xi) = 0$ for $i > n$.
(iii) $c_k(\xi \oplus \nu) = \sum_{i=0}^{k} c_i(\xi) \cup c_{k-i}(\nu)$.

(iv) For the canonical line bundle $\gamma_{\mathbb{C}}^1$ over $BU(1)$, $c_1(\gamma_{\mathbb{C}}^1)$ is a fixed generator of $H^2(BU(1))$. We could also require that $c_1(\gamma_{\mathbb{C}}^1)$ be the Euler class of the underlying 2-dimensional real vector bundle. We will define the Euler class later in this section.

This definition is very similar to the axiomatic definition of Stiefel-Whitney classes, but one notable distinction is that c_i has degree $2i$. The analogues of Theorem 4.8 and Corollary 4.11 also apply to Chern classes.

Theorem 7.6. *The cohomology ring of $BU(n)$ is freely generated by the Chern classes of $\gamma_{\mathbb{C}}^n$ so that as an algebra*

$$H^*(BU(n)) \cong \mathbb{Z}[c_1(\gamma_{\mathbb{C}}^n), ..., c_n(\gamma_{\mathbb{C}}^n)].$$

Corollary 7.7. *Every characteristic class of complex n-dimension vector bundles is a linear combination of characteristic classes of the form $c_1^{i_1}c_2^{i_2}...c_n^{i_n}$ for some non-negative integers $i_1, ..., i_n$.*

The cohomology ring of $BSO(n)$ is unfortunately not as nice, but it shall be essential for our study of cobordism.

Definition 7.8. Let $p : E \to B$ be an oriented real vector bundle with Thom class z and let i be the inclusion of pairs $(E, \emptyset) \to (E, E_0)$. Then let the Euler class e be a characteristic class of degree n defined by $e(p) = (p^* \circ i^*)(z)$.

Definition 7.9. Let $\xi : E \to B$ be an oriented real vector bundle. We define the Pontrjagin classes to be characteristic classes of degree $4i$ defined by $p_i(\xi) = (-1)^i c_{2i}(\xi \otimes \mathbb{C})$.

The free part of $H^*(BSO(n); \mathbb{Z})$ can be computed in terms of these classes, and this fact is what will be useful for the next section. Furthermore $H^*(BSO(n); \mathbb{Z})$ consists only of a free part and 2-torsion. To remedy this issue we let our coefficients have 2 as a unit, providing the following theorem.

Theorem 7.10. *Let R be a ring which contains $\mathbb{Z}$ and has 2 as a unit. Then the cohomology ring $H^*(BSO(n))$ is the algebra $R[p_1(\widetilde{\gamma}^n), ..., p_{(n-1)/2}(\widetilde{\gamma}^n)]$ when n is odd and $R[p_1(\widetilde{\gamma}^n), ..., p_{(n-2)/2}(\widetilde{\gamma}^n), e(\widetilde{\gamma}^n)]$ when n is even.*

8 Cobordism

Cobordism provides a way of classifying manifolds that is weaker than homeomorphism and diffeomorphism. However it has been proven that it is not possible to classify all manifolds by diffeomorphism or homeomorphism. The problem of cobordism on the other hand has been solved for decades, dating back to the work of Thom. We will study two types of cobordism in this section: oriented and unoriented.

Definition 8.1. First we define unoriented cobordism. In this case we say two manifolds M and N are cobordant if there exists a manifold W whose boundary is the disjoint union $M \sqcup N$. The nth unoriented cobordism group Ω_n is the abelian group of cobordism classes of n-dimensional manifolds where $[M] + [N] = [M \sqcup N]$. Furthermore the unoriented cobordism ring Ω_* is the graded ring $\bigoplus_{n=0}^{\infty} \Omega_n$ with multiplication defined by $[M][N] = [M \times N]$. We leave it to the reader to prove this product is well defined.

The proof that cobordism is an equivalence relation will be left to the reader as well. We shall state two results about unoriented cobordism.

Lemma 8.2. *Two manifolds are cobordant if and only if they have all the same Stiefel-Whitney numbers.*

Proof. For any n-tuple $\mathbf{r} = (r_1, ..., r_n)$ we have $\omega_{\mathbf{r}}[M \sqcup N] = \omega_{\mathbf{r}}[M] + \omega_{\mathbf{r}}[N]$. Thus this statement is equivalent to Theorem 5.5. □

Lemma 8.3. *Ω_* has characateristic 2 and is thus an algebra over $\mathbb{Z}/2\mathbb{Z}$.*

Proof. The cobordism class $[M] + [M] = [M \sqcup M]$ is clearly the boundary of the manifold $[0, 1] \times M$. Thus indeed for any cobordism class $[M]$, $2[M] = 0$. □

Definition 8.4. To define the oriented cobordism ring we take as our underlying set oriented manifolds. Then we define M and N to be cobordant if there exists an oriented manifold W whose boundary is $M \sqcup -N$. From there we proceed as in the unoriented case to get the oriented cobordism ring $\widetilde{\Omega}_*$.

Note that in the oriented cobordism ring elements do not necessarily have finite order. The following theorem is at the heart of cobordism theory. By applying concepts from differential topology it translates the problem of computing cobordism rings into a problem of algebraic topology. The following proof will be in broad strokes, leaving out many of the details related to differential topology. The main goal is for the reader to understand the mappings and the interplay between differential topology and algebraic topology.

Theorem 8.5 (Thom-Pontrjagin)**.** *For $m > n + 1$ we have $\pi_{m+n}(T\gamma^m, \infty) \cong \Omega_n$. Furthermore in the oriented case we have $\pi_{m+n}(T\widetilde{\gamma}^m, \infty) \cong \widetilde{\Omega_n}$.*

Before continuing we will have to disucss the notion of transversality.

Definition 8.6. Let X and Y be manifolds and let $f : X \to Y$ be a smooth map. Given a submanifold A of Y we say that f is transverse to A if for every $a \in A$ and $x \in f^{-1}(a)$, we have that $T_a A \oplus f_*(T_x X) = T_a Y$.

The notion of transversality is important primarily because we can use the implicit function theorem to show that if f is transverse to A then $f^{-1}(A)$ is a manifold. Furthermore the orientations of the normal bundle of A and the manifold X canonically induce an orientation on $f^{-1}(A)$. Transversality turns out to be a rather generic property, so given a map $f : X \to Y$, we can always find a homotopic map g which is transverse to a desired submanifold $A \subset Y$. Throughout the proof we shall be content with just picking maps to be transverse when it is necessary, but a reader who is interested in the details should explore the transversality theorem, and look to an introductory text in differential topology such as [7]. Note that the proof of 8.5 is identical in the oriented and unoriented cases, so we shall only carry it out in the oriented case.

Proof of 8.5. We define a map $\zeta : \pi_{m+n}(T\widetilde{\gamma}_s^m) \to \widetilde{\Omega}_n$ for $m > n + 1$ and $s > m + n + 1$. Let $[f]$ be a homotopy class in $\pi_{m+n}(T\widetilde{\gamma}_s^m)$ and pick f such that $f : S^{m+n} \to T\widetilde{\gamma}_s^m$ is transverse to $\widetilde{G}_m(\mathbb{R}^s) \subset T\widetilde{\gamma}_s^m$. Then we can let

$\zeta(f)$ be the manifold $f^{-1}(\widetilde{G}_m(\mathbb{R}^s))$. We still have to prove this is well defined. Let g be a map which is homotopic to f. Then there exists a homotopy $h : I \times S^{m+n} \to T\widetilde{\gamma}_s^m$. Picking h so that it is also transverse to $\widetilde{G}_m(\mathbb{R}^s)$, we get that $h^{-1}(\widetilde{G}_m(\mathbb{R}^s)) = f^{-1}(\widetilde{G}_m(\mathbb{R}^s)) \sqcup -g^{-1}(\widetilde{G}_m(\mathbb{R}^s))$, so that $f^{-1}(\widetilde{G}_m(\mathbb{R}^s))$ and $g^{-1}(\widetilde{G}_m(\mathbb{R}^s))$ are cobordant. Thus ζ is well-defined.

Next we show surjectivity. Let Mfd_n denote the set of all n-dimensional manifolds. We are going to define a map $\phi : Mfd_n \to \pi_{m+n}(T\widetilde{\gamma}_s^m)$ such that $\zeta \circ \phi$ sends each manifold to its cobordism class, making the following diagram commute.

$$\begin{tikzcd}
Mfd_n \arrow[d, "\pi"] \arrow[dr, "\phi"] & \\
\widetilde{\Omega}_n & \pi_{m+n}(T\widetilde{\gamma}_s^m) \arrow[l, "\zeta"]
\end{tikzcd}$$

Since π is clearly surjective, if $\pi = \zeta \circ \phi$ then we will be done with surjectivity. Given an n-manifold M, we can embed it in $\mathbb{R}^s$ by the Whitney embedding theorem. Furthermore by the tubular neighborhood theorem, we can embed its normal bundle ν_M as a open neighborhood U of M in $\mathbb{R}^s$. We know there is a canonical bundle map from ν_M to $\widetilde{\gamma}_s^m$, which induces a map $f : U \to \widetilde{\gamma}_s^m$. We take one point compactifications of both $\mathbb{R}^s$ and $\widetilde{\gamma}_s^m$ and extend f so that it maps all of $S^s - U$ to ∞. Thus we get an element $[f] \in \pi_{m+n}(T\widetilde{\gamma}_s^m)$. Note that by definition we have that $f^{-1}(\widetilde{G}_m(\mathbb{R}^s)) = M$. Thus if we pick some $g \in [f]$ which is transverse to $\widetilde{G}_m(\mathbb{R}^s)$ then $g^{-1}(\widetilde{G}_m(\mathbb{R}^s))$ must be cobordant to M. So indeed we have that $\pi = \zeta \circ \phi$, completing the proof of surjectivity.

Now we prove injectivity. Pick $[f] \in \pi_{m+n}(T\widetilde{\gamma}_s^m)$ so that $\zeta(f) = [M]$ where M is the boundary of some manifold W. Once again leaving out the bulk of the differential topology details we claim that there is a diffeomorphism $g : \nu_M \to f^{-1}(T\widetilde{\gamma}_s^m - \infty)$ which sends the zero section of ν_M to M. We can further let f be such that $f \circ g : \nu_M \to \widetilde{\gamma}_s^m$ is a bundle map. By the Whitney Embedding theorem we can embed W into D^s so that it extends the embedding of M into S^s. We can further embed the normal bundle ν_W as an open neighborhood V of D^s which extends the embedding of ν_M into S^s. Thus we have a bundle map $F : V \to \widetilde{\gamma}_s^m$ which must extend f. Sending $D^s - V$ to ∞ we get a map $F : D^s \to T\widetilde{\gamma}_s^m$ which extends f. Since D^s is contractible this means that F and therefore f are null-homotopic.

□

Note that since ζ is an isomorphism ϕ must descend to a well-defined map on cobordism classes $\bar{\phi} : \Omega_n \to \pi_{m+n}(T\widetilde{\gamma}_s^m)$ such that $\bar{\phi}$ is the inverse of ζ.

The unoriented cobordism has been computed and turns out to be freely generated over $\mathbb{Z}/2\mathbb{Z}$ with one generator of degree i for every i such that $i+1$ is not a power of 2. This derivation is beyond the scope of this paper, but it can be found in [2]. On the other hand we will partially compute the oriented cobordism ring with the help of the following theorem based on the work of Serre.

Theorem 8.7. *If X is a finite CW-complex which is n-connected for some $n \geq 1$ then for all $i < 2n+1$, the kernel and cokernel of the Hurewicz homomorphism $\pi_i(X) \to H_i(X;\mathbb{Z})$ are finite abelian groups.*

Definition 8.8. We say a homomorphism $f : G \to H$ of finitely generated abelian groups is a C-isomorphism if its kernel and cokernel are both finite. Thus it is a C-isomorphism if and only if it preserves the rank of the free part of the groups.

Theorem 8.9. *The free part of $\widetilde{\Omega}_n$ is trivial if n is not divisible by 4, and otherwise it is equal to the number of partitions of $n/4$.*

Proof. By the Thom isomorphism we know that $H^n(T\widetilde{\gamma}^m) = 0$ for $1 \leq n \leq m-1$ so by the Hurewicz Theorem $T\widetilde{\gamma}^m$ is (m - 1)-connected. Thus by Theorem 8.7 we have that the Hurewicz homomorphism $\pi_{n+m}(T\widetilde{\gamma}^m) \to H_{n+m}(T\widetilde{\gamma}^m; \mathbb{Z})$ is a C-isomorphism for $n < m - 1$. The integral Uniform Coefficient Theorem tells us that integral homology and cohomology have the same free part for all $n < m - 1$, so the free parts of $\pi_{n+m}(T\widetilde{\gamma}^m)$ and $H^{n+m}(T\widetilde{\gamma}^m; \mathbb{Z})$ have the same rank. We know there is a Thom isomorphism $\Phi : H^n(BSO(m)) \to H^{n+m}(T\widetilde{\gamma}^m)$ so $\pi_{n+m}(T\widetilde{\gamma}^m)$ has the same free part as $H^n(BSO(m))$. Since $n < m - 1$, Theorem 7.10 tells us that indeed $H^n(BSO(m))$ is generated by Pontrjagin classes. Thus the dimension of its free part is zero if n is not divisible by 4, and otherwise it is the number of partitions of $n/4$.

□

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