K-THEORY, BOTT PERIODICITY, AND ELLIPTIC OPERATORS

CAMERON KRULEWSKI

Abstract. We give the background for and a proof of the Bott periodicity theorem. Our paper develops a foundation of topological K-theory and offers a summary of Michael Atiyah's 1967 proof of the complex case and the real equivariant case of Bott periodicity, found in the paper "Bott Periodicity and the Index of Elliptic Operators" [1]. Motivation is given for steps Atiyah's proof, but many details are skipped in favor of offering an understandable overview.

Contents

Introduction 1
1. Background 2
1.1. Useful Terms 2
1.2. K-Theory 3
1.3. Important Objects and Conventions 11
2. Overview of Atiyah's 1967 Proof with Elliptic Operators 14
2.1. The Template 14
2.2. The "Formal Trick" 15
2.3. Elliptic Operators on Vector Bundles 17
2.4. The Complex Case 23
2.5. The Spinor Case 24
2.6. Conclusion 31
Acknowledgments 31
References 32

Introduction

Homotopy groups are very useful for classifying topological spaces, but are notoriously difficult to calculate. Bott's 1959 result [9] massively simplified the calculation for several classical groups by revealing a periodic structure, which in turn stimulated the development of K-theory and other fields. Written succinctly, Bott's result is that the unitary group is 2-periodic, while the orthogonal and symplectic groups are 8-periodic. That is,

$$\pi_k(U) = \pi_{k+2}(U)$$
$$\pi_k(O) = \pi_{k+8}(O)$$
$$\pi_k(Sp) = \pi_{k+8}(Sp).$$

Date: 7 January, 2018.

Since its original discovery, Bott periodicity has inspired a slew of diverse proof approaches [18], as well as several different formulations of the theorem that extend its result. While a more elementary proof of complex Bott periodicity may be found in Bott's original Morse theory proof [9] or Atiyah and Bott's K-theory approach [2], the real equivariant case of the theorem requires the use of elliptic operators. The approach we discuss below is the only one in known literature to prove the real equivariant version.

Our goal is to explicate the proof of real equivariant Bott periodicity found in Atiyah 1967 [1], where it is the first part of §6, "The Spinor Case". To build up to this case, we also treat the complex, nonequivariant version. In our explanation, we focus less on the equivariant action and more on the role of the spinor bundle and elliptic operators, because they are the more unique elements of this proof.

In this paper, we start by defining topological groups and giving an introduction to K-theory. We imagine that this will serve as a review for readers interested in periodicity. With K-theory defined, we explain the layout of [1] and explicate the general sections of the proof. Next, we develop the machinery of elliptic operators, which give rise to the index map required for each specific case of the proof. For this, we assume several results from analysis, but hope that our quick explanation is made clear by a collection of examples. Finally, we cover two cases of Bott periodicity—the complex case and the real equivariant ("spinor") case. In these, many of the more complicated calculations are summarized or cited for clarity.

The reader familiar with K-theory may skip to §2, and the reader primarily interested in the real equivariant case may focus on §2.5. Note that for the purposes of this paper, we rigorously develop Clifford algebras to show where 8-fold periodicity emerges, but give only a cursory introduction to spinors. We will refer the interested reader to [4] and [15] for more details.

1. Background

1.1. **Useful Terms.**

We begin by defining the basic terms in the statement of the theorem, and indicating some techniques that will be used later in proof.

Definition 1.1. A **topological group** is a group G that is also a topological space, and whose composition and inverse maps are continuous. That is, the maps $(g, h) \mapsto gh$ and $g \mapsto g^{-1}$ are continuous for all $g, h \in G$. Furthermore, if the topological space G is in fact a manifold, and if the composition and inverse maps are smooth instead of just continuous, the group is known as a **Lie group**.

We give a few examples to illustrate how the algebraic and topological structure correspond. Commonly encountered topological groups are those that may be written in terms of matrices, and these groups also turn out to be the ones we analyze with Bott periodicity.

Example 1.2. A basic example is the real line, $\mathbb{R}$, with the group action of addition. Consider this group, denoted $(\mathbb{R}, +)$, under the standard metric topology. It's intuitive that addition of any real number r defines a continuous map $x \mapsto x + r$, since a sum varies continuously with either addend. Negating a number to find its inverse is also a smooth operation.

Example 1.3. Perhaps the most familiar example involving matrices is the **general linear group**, $GL(n, \mathbb{R})$. This group consists of $n \times n$ invertible matrices with real coefficients, and hence corresponds to invertible, rank n linear transformations.

Matrix multiplication defines the composition law as per usual, but we can consider the group's topological structure as deriving from an embedding into $\mathbb{R}^{n\times n}$. That is, if we view the collection of $n \times n$ entries of a matrix as a point in the Euclidean space $\mathbb{R}^{n\times n}$, we again have a familiar sense of the topology by using the standard metric. Note that we can continuously find inverses through row operations.

Each matrix in $GL(n,\mathbb{R})$ can be included in $GL(n+1,\mathbb{R})$ by copying the $n \times n$ entries of the smaller matrix into a larger matrix whose last row and column are zero except for a 1 on the diagonal. In this way, the determinant, and hence invertibility, is preserved, and we get an inclusion $GL(n,\mathbb{R}) \hookrightarrow GL(n+1,\mathbb{R})$. Then, if we take the union over all n, the group becomes infinite. We write

$$GL(\mathbb{R}) = \bigcup_{n=1}^{\infty} GL(n,\mathbb{R})$$

to denote the set of all invertible linear transformations.

We could also consider $GL(n,\mathbb{C})$, the group of $n \times n$ invertible matrices with complex coefficients, and the infinite group $GL(\mathbb{C})$.

The following examples are subsets of $GL(\mathbb{R})$ or $GL(\mathbb{C})$.

Example 1.4. The **orthogonal group**, O, consists of real matrices A satisfying $A^T A = 1$. That is, the inverse of an orthogonal matrix is its transpose. Orthogonal matrices preserve both lengths and a fixed point, and the orthogonal group corresponds to transformations of rigid geometric objects. Any symmetry group of a geometric object is a subset of $O(2)$ or $O(3)$. For example, the group of rotations and flips of a 2-dimensional object such as a square forms a subspace of the orthogonal group $O(2)$. Finally, a quick argument will show that orthogonal matrices can only have determinant ± 1. Real Bott periodicity shows an 8-periodicity in $\pi_k(O)$.

Example 1.5. The **special orthogonal group**, SO, consists of orthogonal matrices that preserve orientation, corresponding to rotational symmetries. This group is described as the subset of O consisting of matrices with determinant 1.

Example 1.6. The **unitary group**, U, consists of complex matrices B satisfying $B^*B = 1$. That is, the inverse of a unitary matrix is its conjugate transpose. This implies that these matrices have determinant with norm 1, and we can see that $U \subset GL(\mathbb{C})$. The unitary group is the complex analog of O, and actually includes it: we write $O \hookrightarrow U$. Complex Bott periodicity shows a 2-periodicity in $\pi_k(U)$.

Two more important topological groups that we will need for the real equivariant case are the groups Pin and Spin. However, we delay introducing of them until §2.5, when we develop the theory of Clifford algebras.

1.2. K-Theory.

K-theory is a powerful generalized cohomology theory that was formalized in the 1960s by Hirzebruch and Atiyah soon after Bott's proof of the periodicity theorem [12]. This theory has a nice geometric formulation in terms of vector bundles, which we present next.

1.2.1. *Vector Bundles.*

To motivate the definition of a vector bundle, we present an intuitive definition of a more general space called a fiber bundle.

Definition 1.7. A **fiber bundle** with **fiber** F is a collection of two spaces and one map. It consists of a **base space** B, a **total space** E, and a continuous surjective map $p : E \to B$ that satisfies several properties. Each $x \in B$ has an open neighborhood $U \subset B$ for which the preimage in E is homeomorphic to a product of B with F on which p looks like a projection. That is, we can write $p^{-1}(U) \simeq U \times F$, and require that $p|_{p^{-1}(U)} : p^{-1}(U) \to U$ is a projection map. We call such a projection map a **local trivialization** of the bundle. Finally, fiber bundles come with a group action on the fiber; for example, homeomorphisms on the fiber.

Basically, a fiber bundle is a space that locally looks like a product of the base space B with another space F. We have not been precise about how the bundle looks globally, because fibers over the open sets U must also be compatible where they intersect. To formalize this, we would need to discuss compatibility of these coordinate patches.

Remark 1.8. An important special case of a fiber bundle that occurs when we can form the projection map over the whole space. Then the bundle looks globally like a product, and we call $E \simeq B \times F$ a **trivial bundle**.

These notions will be clearer when we focus on *vector* bundles, which are the kind of fiber bundles we are interested in for this proof. Before we begin with vector bundles, we note the convention for choice of field over which the vector spaces lie.

Remark 1.9. Traditionally, the K-theory of a space X, denoted $K(X)$, is formulated in terms of complex vector bundles. There is a real analog of K-theory known as KO-theory, which is formulated similarly, and will be necessary for proving real Bott periodicity. For ease of visualization, we present a formulation of the real theory first, so we write KO-theory below. However, the formulation is the same for K-theory proper.

Definition 1.10. A **real vector bundle** is a fiber bundle with each fiber a real vector space. Locally, we could write that a small enough open set U in the base space B satisfies $p^{-1}(U) \cong U \times V$ for V a vector space.

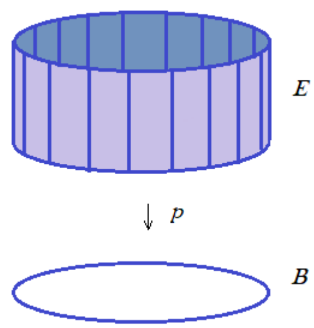
That is, for every point x in our space X, there is a vector space V_x lying above it. As x varies continuously, so does V_x, by our requirement of compatibility of coordinate patches above.

To make sure that definition of vector bundles actually makes sense, we start with some examples that are small enough to visualize.

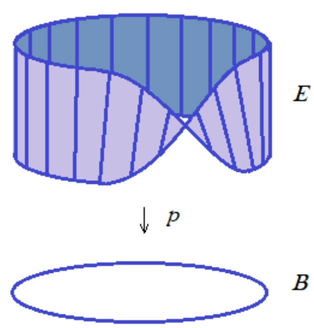
Example 1.11. Perhaps the most trivial example we can think of is when the base space is a point. Then the vector bundle is homeomorphic to the vector space that comprises the fiber. But there are still multiple options for the bundle—in fact, one for each positive integer, because we need to specify the dimension of the vector space over the point. This choice specifies the space up to isomorphism. We get a particular case of a trivial bundle of dimension n, which we can denote as ε^n.

Example 1.12. We can form slightly more exciting examples of real vector bundles over the base space S^1. If we restrict our attention to 1-dimensional bundles, there are actually only two options, which we can think of as the **cylinder** and the **Möbius strip** [12].

- The cylinder corresponds to a trivial, 1-dimensional vector bundle over S^1. That is, it is the same as the product space $S^1 \times V$ for a 1-dimensional real vector space V. We may as well pick $V = \mathbb{R}$. Then if we shrink $\mathbb{R}$ down to an open interval to make the space easier to imagine, we arrive at the product of the circle with an interval. We see that it looks like an open cylinder, or perhaps an annulus, and we denote it by ε^1.



- The Möbius strip corresponds to the nontrivial case. Locally, it is similar to the annulus, but it is nonorientable and cannot be written globally as the product $S^1 \times \mathbb{R}$ because of its famous twist in the middle.

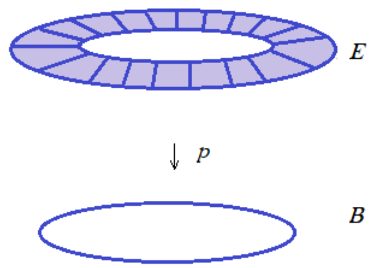


Example 1.13. Our last two introductory examples concern manifolds. For any manifold X embedded in Euclidean space, its tangent bundle TX and normal bundle NX are both vector bundles over the space.

- The **tangent bundle** of a real n-dimensional manifold X consists of a copy of $\mathbb{R}^n$ associated to each point in X such that it is tangent to the local parameterisation of the manifold. That is, for each $x \in X$, the vector space associated is the tangent space $T_x X$, which is an n-dimensional Euclidean space that locally approximates the manifold.

Consider the 2-sphere, whose tangent bundle consists of a collection of planes around the sphere. If it were the case that this bundle were trivial, we know we could continuously pick at least one of the basis vectors of the vector space as we travel around the sphere. However, this violates the Hairy Ball Theorem.

In the case of S^1, we actually can pick the vector continuously, and so get the trivial bundle $\varepsilon^1 = S^1 \times \mathbb{R}$. This time we visualize the linear space as oriented slightly differently out from the circle, forming an annulus rather than a cylinder, but it describes an isomorphic bundle.



- The **normal bundle** of a manifold is the orthogonal complement of the tangent bundle, taken pointwise. It is defined with respect to the space in which the manifold is embedded, so the higher the dimension of the ambient space, the higher the dimension of NX.

  The normal bundle of a circle is precisely the annulus we presented above, while the normal bundle of the 2-sphere could be viewed as a thickening of the sphere's surface. At each point on the n-sphere, we extend a radial vector in $\mathbb{R}^{n+1}$, completing a basis for the space.

Definition 1.14. A **line bundle** over a space B is a 1-dimensional vector bundle $p : L \to B$.

We will present a few line bundles in §1.3 that will be important for the proof. Before that, two more critical things to know about vector bundles are what sections are and how to take pullbacks.

Definition 1.15. A **section** of a vector bundle is a continuous map $s : B \to E$ such that $p \circ s = \mathrm{id}_B$, with the associated image $s(B) \subset E$.

Since a section of a vector bundle consists of a choice of vector $v \in V_x$ for each $x \in X$, it defines a vector field on the space.

Example 1.16. The **zero section** of a vector bundle $p : E \to B$ is the map $s : B \to E$ such that $s(b) = 0$ for all b. That is, we pick the vector 0 in each fiber $p^{-1}(b)$. This section $s(B)$ is canonically identified with the base space, B.

Remark 1.17. We say that a section is **global** if it is defined everywhere. If we can find, on an n-dimensional vector bundle, a collection of n linearly independent global sections, then the bundle is trivial. This is because if we have the linearly independent sections $s_1, ..., s_n : B \to E$, we can form an isomorphism from the trivial bundle $X \times \mathbb{R}^n$ by $(x, a_1, ..., a_n) \mapsto a_1 s_1(x) + ... + a_n s_n(x)$.

Finally, that fiber bundles "pull back" over continuous maps is very important, because it allows us to define bundles over new spaces, and because it guarantees an important functorial property of K-theory. When we say pullback, we mean the following:

Definition 1.18. Given a continuous map $f : Y \to X$ between two spaces, if E is a vector bundle over X, we can define a **pullback bundle**, or **induced bundle** $f^*(E)$ over Y such that $f^*(E) = \{(y, e) \in Y \times E \mid f(y) = p(e)\}$.

Then, the following diagram commutes.

$$\begin{tikzcd} f^*(E) \arrow[r] \arrow[d] & E \arrow[d, "p"] \\ Y \arrow[r, "f"] & X \end{tikzcd}$$

Note that the choice we make for $f^*(E)$ is the most direct one we could make with respect to f: we take the vector space fiber over $y \in Y$ to be the fiber over $f(y) \in X$.

Example 1.19. A nontrivial pullback is that of the Möbius bundle by the map $f : S^1 \to S^1$ such that $z \mapsto z^2$. This pullback map essentially doubles the bundle as we go around the unit circle, since $z \mapsto z^2$ wraps twice around the circle. So, the resulting vector bundle looks like a Möbius strip with two half-twists. However, a strip with two half-twists is again isomorphic to the cylinder, so the pullback under this map is ε^1.

In general, the pullback of $f_n : z \mapsto z^n$ returns the cylinder for n even and the Möbius bundle for n odd. [12]

Remark 1.20. There are several different versions of K-theory, just as there are many different proofs. As we alluded to in Remark 1.9, the one that involves *real* vector bundles is actually called KO-theory, where the "O" signifies its relevance to the orthogonal group. For completeness, we give the definition of the complex analog.

Definition 1.21. A **complex vector bundle** is fiber bundle with each fiber a complex vector space. That means for each $x \in X$, there is a complex vector space V_x lying above it, and these vector spaces vary continuously with x.

The construction of the Grothendieck ring, as explained below, works the same way for each type of bundle. But we assume that our vector bundles are real for ease of presenting examples.

1.2.2. *Formulation.* Now that we understand what vector bundles are, we examine what we can do with them. We can, in fact, define two operations on them that *almost* produce a ring structure, which we can complete with a Grothendieck construction. From there, we can use the structure to start proving Bott periodicity.

We define direct sum and tensor products on vector bundles in a straightforward way–for two bundles $p_1 : E_1 \to B$ and $p_2 : E_2 \to B$ over B, we can take their direct sum or tensor product pointwise. We say $(E_1 \oplus E_2)_x = (E_1)_x \oplus (E_2)_x$, and $(E_1 \otimes E_2)_x = (E_1)_x \otimes (E_2)_x$. As sets, then,

$$E_1 \oplus E_2 = \{(v_1, v_2) \in E_1 \times E_2 \mid p_1(v_1) = p_2(v_2) \in B\}$$
$$E_1 \otimes E_2 = \{p_1^{-1}(b) \otimes p_2^{-1}(b) \mid b \in B\}$$

with the tensor product topologized with local trivialization. See [12] for more details.

Example 1.22. For perhaps the most basic example, consider two trivial real line bundles L_1 and L_2 over a space X. Their direct sum is a trivial 2-dimensional bundle isomorphic to $\mathbb{R}^2 \times X$, since at each fiber we have a space isomorphic to $\mathrm{Span}(e_1) \oplus \mathrm{Span}(e_2) \cong \mathbb{R}^2$.

Example 1.23. For a more exciting case, consider the Möbius bundle over S^1. If we take the tensor product of it with itself, we actually get the cylinder again. In this argument, we will hint at the notion of clutching functions, which are prominent in Atiyah's proof of periodicity found in [2]. For a more precise treatment of clutching functions we refer the reader to [5].

First note that a vector bundle over a contractible space must be trivial, because we may homotopically deform the bundle to a vector space over a single, contracted point representing the base space. When we have a vector bundle over a non-contractible space, we may break that space into contractible pieces and glue the bundles over these pieces together. Then the overall bundle is determined by our choice of these "clutching functions".

Two line bundles over the circle can only be glued parallel or antiparallel to each other, meaning our gluing function is either $+1$ or -1. Intuitively, the $+1$ choice corresponds to the cylinder, while the -1 provides the half-twist to the Möbius bundle.

Finally, returning to our tensor product, we assert that the clutching function of the tensor product is the tensor product of the clutching functions, viewed as matrices. Admitting this, it is clear that because $(-1)^2 = +1$, that the tensor product of the Möbius bundle with itself is the trivial bundle, with clutching function 1.

The important takeaway from that example is not necessarily clutching functions, but that direct sum as well as tensor product allow us to make new bundles. In particular, it is possible to trivialize a bundle with the direct sum.

Example 1.24. Recall from 1.13 that the tangent and normal bundles of a manifold X are vector bundles. By its very definition, the normal bundle is the bundle that completes a basis for the ambient space when combined in a direct sum with the tangent bundle. Hence if we take the direct sum of the two bundles globally, we get a trivial bundle isomorphic to the ambient space crossed with the manifold. That is, if X is embedded in $\mathbb{R}^N$, when we take the direct sum of the bundles TX and NX, we get a trivial bundle $\varepsilon^N \cong \mathbb{R}^N \times X$.

For the 2-sphere, we can write $TS^2 \oplus NS^2 \cong S^2 \times \mathbb{R}^3$.

We would like to form a ring of isomorphism classes of vector bundles under these two operations. However, at the moment we only have a semi-ring—we have closure, associativity, and identity under $\oplus$ and $\otimes$, but we lack inverses under $\oplus$. To solve this problem, we follow a strategy developed by Grothendieck in the 1950s for group completion using formal sums. Before presenting this for vector bundles, we offer a more basic example.

Definition 1.25. The **Grothendieck group** construction is a process for forming an abelian group $Gr(M)$ out of a commutative monoid M. It has the universal property that any group receiving a map from M will also receive a map from $Gr(M)$.

Example 1.26. What is the Grothendieck completion of the natural numbers? We know that the inverses we seek probably relate to subtraction. However, differences are not unique, because for example $2 - 3 = 3 - 4$. Thus we define equivalence classes $[a - b] \sim [c - d]$ if $a, b, c, d \in \mathbb{N}$ are such that $a + d = b + c$.

Through this construction, we get the negative integers. We could write, for example, $-17 := [1 - 18]$. Hence the completion of the natural numbers is the integers, and we write $Gr(\mathbb{N}) = \mathbb{Z}$.

Analyzing the example above, we see that the addition in the completed group corresponds to the addition that we already had on the natural numbers. If we suggestively denote the addition in the group by $+$ and the addition in the monoid by $\oplus$, we see that the group completion is achieved by taking the free abelian group on the elements of M and taking the quotient of the ideal generated by elements $(m+n) - (m \oplus n)$, so as to identify $+$ and $\oplus$. Elements in this group are all formal differences.

With the process formalized, we perform it for vector bundles. We take formal differences of vector bundles, called **virtual bundles**, and identify the group addition with the $\oplus$ operation on vector bundles. With this Grothendieck completion, we finally have the ring structure of vector bundles that defines KO-theory and K-theory.

Definition 1.27. The KO-theory of X, denoted $KO(X)$, is the Grothendieck ring of the real vector bundles over X.

Definition 1.28. Correspondingly, we write $K(X)$ for the Grothendieck ring of complex vector bundles over X, and call this the K-theory of X.

Notation 1.29. We denote the isomorphism class of a vector bundle E by $[E]$. Hence a virtual bundle is something of the form $[E] - [F]$ for vector bundles E and F. For simplicity, we write the isomorphism classes of trivial bundles ε^n as $[\varepsilon^n] = [n]$, where their base space will be understood from context.

If we denote the additive and multiplicative operations in our ring by $+$ and $\times$, respectively, we can see how these operations in K-theory correspond to the operations on the representative vector bundles. Adding two virtual bundles corresponds to direct sum, so

$$([E_1] - [E_2]) + ([F_1] - [F_2]) = [E_1 \oplus F_1] - [E_2 \oplus F_2].$$

Meanwhile, multiplication corresponds to tensor product, so multiplying as we would any two binomials, we have

$$([E_1] - [E_2]) \times ([F_1] - [F_2]) = [E_1 \otimes F_1] - [E_2 \otimes F_1] - [E_1 \otimes F_2] + [E_2 \otimes F_2].$$

Next we present some calculations on example spaces.

Example 1.30. We can at least calculate the K-theory of a single point $\{x_0\}$. Recall from Ex. 1.11 that only trivial bundles exist over a point. That is, the isomorphism classes of bundles that exist over $\{x_0\}$ are just a trivial bundle ε^n of each dimension $n \in N$. Adding in formal differences of bundles, we get formal negative trivial bundles. Hence, the K-theory consists of elements $[n]$ for $n \in Gr(\mathbb{N}) = \mathbb{Z}$, and we get $K(\{x_0\}) = \mathbb{Z}$.

Example 1.31. The K-theory of a sphere depends on the parity of its dimension. We claim that

$$K(S^q) = \begin{cases} \mathbb{Z}^2, & q \text{ even} \\ 0, & q \text{ odd.} \end{cases}$$

We can argue two base cases without much difficulty. The zero-sphere is just a pair of disjoint points, each of which contributes a copy of $\mathbb{Z}$ to the K-theory. Hence $K(S^0) = \mathbb{Z}^2$. Meanwhile, the 1-sphere turns out to have trivial K-theory because all complex bundles over S^1 are trivial [8], a fact that can be found by analyzing classifying spaces. The statement for higher-dimensional spheres follows from complex Bott periodicity, to be proven below.

The reader can see [8] for computations and for further examples.

To reinforce that the structure $KO(X)$ actually gives us equivalences between non-isomorphic bundles, and that it helps us with computations, we provide the following example from [13] §1.1.

Example 1.32. Consider real vector bundles over the sphere S^2. We have seen in Example 1.13 that the tangent bundle of S^2, denoted TS^2, is *not* isomorphic to the trivial bundle $\varepsilon^2 = S^2 \times \mathbb{R}^2$. However, in terms of KO-theory, we have

$$\begin{aligned} [TS^2] &= [3] - [NS^2] \text{ by definition of normal bundle, and } [3] = S^2 \times \mathbb{R}^3 \\ &= [3] - [1] \text{ since the normal bundle is trivial} \\ &= [2] \end{aligned}$$

1.2.3. *Reduced K-theory.* When we consider pointed spaces, also called based spaces, which have distinguished base points, we arrive at a slightly different theory. Following a process common to cohomology theories, we can define the reduced K-theory of a pointed space X.

Definition 1.33. Let X be a pointed space with base point x_0. The **reduced K-theory** of X is the kernel of the map $K(X) \to K(x_0)$ induced by the inclusion map $\{x_0\} \hookrightarrow X$ of the base point into X.

The reduced K-theory can be thought of as the K-theory of X modulo the K-theory of the base point, which, as we've seen, is $\mathbb{Z}$. We can write $K(X) \cong \tilde{K}(X) \times \mathbb{Z}$. This is the same as modding out by trivial bundles, or considering only the elements of $K(X)$ with virtual dimension zero.

Example 1.34. The reduced K-theory of a point is trivial, since we have

$$\tilde{K}(X) = \mathbb{Z}/\mathbb{Z} = 0.$$

Example 1.35. The reduced theory for the sphere is then

$$\tilde{K}(S^q) = \begin{cases} \mathbb{Z}, & q \text{ even} \\ 0, & q \text{ odd.} \end{cases}$$

1.2.4. *Other K-groups.* We introduced K-theory as a cohomology theory, but so far have only discussed the zeroth level of the ring. We can write $K(X)$ as $K^0(X)$ to emphasize that. To define other levels of the theory, we can use the suspension axiom of Eilenberg-Steenrod cohomology, which holds for the reduced K-theory.

Definition 1.36. Let $I = [0, 1]$ be the unit interval and let x_0 be the base point of X. Then the **reduced suspension** of X is the space

$$\Sigma X = X \times I/(X \times \partial I \cup \{x_0\} \times I)$$

constructed by taking the quotient of $X \times I$ by the two copies of X at the boundary of I and by the copy of I associated to the base point of X.

Example 1.37. The reduced suspension of an n-sphere is homeomorphic to an $n + 1$-sphere. We write $\Sigma S^n \simeq S^{n+1}$.

Using this operation, we can define negative reduced K-groups. We set

$$\tilde{K}^{-q}(X) = \tilde{K}(\Sigma^q X).$$

Moving to the unreduced theory requires some casework. If we consider a based space X, we have that $\tilde{K}^{-q}(X)$ is the kernel of the map $K^{-q}(X) \to K^{-q}(\{p\})$, as in Def. 1.33. For an unbased space Y, we can first add a disjoint basepoint $*$ to Y by taking $Y_+ = Y \sqcup \{*\}$. We then define other K-groups by

$$K^{-q}(Y) = \tilde{K}^{-q}(Y_+).$$

These definitions are essential to our formulation of Bott periodicity, in which we show an isomorphism between different K-groups. Note that the constructions above only work to define negative K-groups—we actually require Bott periodicity to define $K^q(X)$ for positive q.

1.2.5. *Summary.* Our formulation of K-theory introduced vector bundles from fiber bundles. We explained how a vector bundle is a continuously-varying space of vector spaces V_x over the points $x \in X$, though we did not rigorously discuss the requirement of compatibility between coordinate patches. Next, we gave several examples of real vector bundles and operations on them before explaining the Grothendieck construction and corresponding equivalence relations. The KO-theory of X consists of the Grothendieck ring of real vector bundles over X, with the addition given by direct sum $\oplus$ and the multiplication given by tensor product $\otimes$, while the K-theory of X consists of that same structure but made out of complex vector bundles. Finally, we gave the reduced cohomology theory and defined other K-groups by $\tilde{K}^{-q}(X) \cong \tilde{K}(\Sigma^q X)$.

1.3. **Important Objects and Conventions.** In this section, we introduce a few more important bundles and spaces that are needed for the proof. We also clarify some notation and conventions, which includes defining K-theory with compact supports.

1.3.1. *Line Bundles.* There are a few important bundles that are commonly found in K-theoretic proofs that will be necessary below. In particular, there are two line bundles over a projective space $\mathbb{C}P^1$ that in various sources have different names and notations. To be consistent with Atiyah, we define them as follows.

Definition 1.38. The **tautological line bundle**, H^*, is the bundle whose fibers are copies of the projective line for each point of the projective space. For example, over $\mathbb{C}P^1$, the fiber of $x \in \mathbb{C}P^1$ is the one-dimensional subspace in $\mathbb{C}$ associated to that point in the projectivization. It is in this sense that the bundle is tautological—each point in a projective space is actually a line, so we choose the fiber over that point to simply be that line.

Atiyah refers to this as the standard line bundle. It is also sometimes called the universal bundle, and is denoted by $\mathcal{O}(-1)$ in algebraic geometry.

Definition 1.39. The **canonical line bundle** is the dual of the previous bundle, and hence denoted H.

It is also known as the hyperplane bundle, and it is denoted $\mathcal{O}(1)$ in algebraic geometry.

Where possible, we use H and H^* to be explicit about which bundle we refer to, because the name "canonical line bundle" has been used in literature to mean either of the above.

1.3.2. *Compact Supports.* Next, in order to follow a convention of Atiyah, we must introduce **K-theory with compact supports**. This non-standard version of K-theory allows us to define $K(X)$ for spaces X that are only locally compact, which use to discuss real vector bundles in §2.5. We do this by using the reduced K-theory of the one-point compactification of X, which we denote by X^+. We write the resulting K-theory as $K_c(X)$ to indicate that we are using a non-standard convention.

Definition 1.40. For X a locally compact space, we define

$$K_c(X) := \tilde{K}(X^+).$$

Note that when X is already compact, we have $X^+ = X_+$. That is, adding a disjoint base point is the same thing as forming the one-point compactification for compact X. For this reason, the definition above holds for compact X as well as for locally compact, non-compact X.

Example 1.41. By our definition, $K_c(\mathbb{R}^2) = \tilde{K}(S^2)$.

When X is compact, our definition corresponds with the modern version of K-theory. This is because although we add the disjoint basepoint to X, its contribution to the K-theory is removed once we take the *reduced* K-theory.

When X is not compact, however, our version does *not* agree with the modern version. To show this, we calculate the K-theory of $\mathbb{R}^q$.

Example 1.42. First, we calculate using modern conventions.

$$\begin{aligned} K(\mathbb{R}^q) &= \tilde{K}(\mathbb{R}^q_+), \text{ after adding a disjoint basepoint} \\ &= \tilde{K}(S^0), \text{ after contracting } \mathbb{R}^q \text{ to a point} \\ &= K(\{p\}), \text{ after taking the component with the basepoint} \\ &= \mathbb{Z}, \text{ as calculated in Ex. 1.30.} \end{aligned}$$

Next, we use our version, following Atiyah's convention.

$$\begin{aligned} K_c(\mathbb{R}^q) &= \tilde{K}((\mathbb{R}^q)^+), \text{ after taking the compactification} \\ &= \tilde{K}(S^q) \\ &= \begin{cases} \mathbb{Z}, & q \text{ even} \\ 0, & q \text{ odd} \end{cases} \text{ as claimed in Ex. 1.35.} \end{aligned}$$

We see that for q odd, the modern convention and Atiyah's convention disagree, so the distinction is important.

Another unintuitive fact about Atiyah's version is that his K-theory with compact supports is not a homotopy functor. That is, two homotopy equivalent spaces can have different K-theories.

Example 1.43. Euclidean space $\mathbb{R}^q$ may be contracted to a point. However, for q odd, we showed that Atiyah's convention has $K_c(\mathbb{R}^q) = 0$, and earlier in Ex. 1.30 we calculated that $K(\{p\}) = \mathbb{Z}$. So although $\mathbb{R}^q \simeq \{p\}$, their K-theories do not match.

Finally, under certain conditions for X, we can express the K-theory of the product $\mathbb{R}^q \times X$ in another way. For X compact and nondegenerately based, we have

$$K_c(\mathbb{R}^q \times X) = \tilde{K}(S^q \wedge X_+) = \tilde{K}(\Sigma^q(X_+)).$$

In general, for locally compact X, we have

$$K_c(\mathbb{R}^q \times X) = \tilde{K}((\Sigma^q X)_+).$$

Notice that the addition of a disjoint basepoint does not commute with the suspension operation for non-compact X.

From now on, we assume that we take Atiyah's convention to define K-theory for non-compact spaces, so as to be consistent with [1]. Note that a consequence of taking reduced K-theory is that our negative K-groups are now instead defined by $K_c^{-q}(X) = K_c(\mathbb{R}^q \times X)$.

1.3.3. *Bott Classes.* Using K-theory with compact supports, we define some important basis elements for K-theories used below. First, however, we need the exterior algebra.

Definition 1.44. We start with the **tensor algebra** of a vector space V,

$$T(V) = \bigoplus_{n \geq 0} V \otimes \cdots \otimes V,$$

which consists of the direct sum of n tensored copies of V, for each n. We quotient out by the ideal generated by elements of the form $v \otimes v$ to get the **exterior algebra**, denoted $\Lambda(V)$.

If V is a vector bundle over a compact space X, then the exterior algebra defines a set of other vector bundles over X. Using K-theory with compact supports to define $K_c(V) = \tilde{K}(V^+)$, we denote the element of $K_c(V)$ corresponding to the exterior algebra by λ_V.

Notation 1.45. In other sources, the one-point compactification of a vector space V might be denoted S^V instead of V^+, especially in sources that discuss the Thom Isomorphism Theorem. However, we use V^+ to be consistent with Atiyah and to avoid introducing more notation.

Definition 1.46. If we set $X = \{p\}$ and $V = \mathbb{C}$, the element $\lambda_{\mathbb{C}} \in K_c(\mathbb{C}) \cong K_c(\mathbb{R}^2)$ is a generator. Its dual, $\lambda_{\mathbb{C}}^*$, is called the **Bott class** and is denoted b.

In $K_c(\mathbb{C})$, the Bott class can be expressed as a difference of line bundles $b = 1 - H^*$. For the spinor case, we have an analog $u \in K_c(V)$ for V a spin bundle.

2. Overview of Atiyah's 1967 Proof with Elliptic Operators

We follow Atiyah's division of the paper, focusing on sections §1-3 and 6. We begin with a summary of §1, which offers a "formal trick" for simplifying the proof, and a template through which each case of periodicity is proven. We present the complex case, but the case we find most interesting is the spinor case in §6, because there is no proof of this case in the literature that does not require the use of elliptic operators. Since we do not explicitly discuss the complex equivariant case of periodicity, we introduce some of the necessary concepts from §4 to explain the spinor case from §6. We do not discuss "capital-R" Real KO-theory, but the curious reader can find it defined in [6]. Finally, we also omit a proof of the Thom isomorphism theorem, which is offered as an extension in the paper.

Note that this paper does not seek to provide a rigorous summary of Atiyah's paper; rather, we build up the relevant background, present a simplified outline and motivation for the proof, and indicate a reasonable first pass through it.

2.1. The Template.

Atiyah's paper discusses several different versions of Bott periodicity, and proves each one through a similar process. In each case, there is a version of the Bott map, which runs between two rings that we want to show are isomorphic. Isomorphism between them is what gives us the desired periodicity.

Finding the isomorphism is the tricky part. It is defined as the multiplication of some element in the appropriate K-theory, and this element must have index 1 when we apply the appropriate elliptic operator. Hence in each case, we make a choice of manifold, a choice of operator acting on the vector bundles over that manifold, and a choice of element with index 1.

For example, in the complex case, Bott periodicity involves a 2-periodic structure in the K-theory of a space X. We write the Bott map as

$$\beta : K(X) \to K^{-2}(X).$$

The map is defined by multiplication by the Bott class b, and if we can demonstrate the isomorphism we know that $K(X) \cong K^{-2}(X)$, which gives us the desired periodicity. Recall that we defined $K_c^{-n}(X) = K_c(\mathbb{R}^n \times X)$.

In each case, to show that the map β we construct is an isomorphism, we find an inverse map α, which we define as a composition with an index map. How exactly this demonstrates isomorphism involves the "formal trick" and is discussed in the next section.

In the table below, we give an outline of the choices made in each case to show how each part of the theorem follows the same template. In the following sections, we will explain the meaning of each of the entries.

Table 2.1

	Complex Case	**Spinor Case**
manifold	$\mathbb{C}P^1$	S
operator	del bar operator $\bar{\partial}$	Dirac operator D
Bott class	$b = \lambda_{\mathbb{C}}^* = 1 - H^*$	$u = S^+ - 1$

Note that S is shorthand for the spinor bundle over the sphere V^+. We will define this bundle in §2.5.4.

2.2. **The "Formal Trick".** In this section, we explain how a candidate inverse map α with certain properties can be verified as an actual inverse to β, demonstrating the desired isomorphism. Following Atiyah, we work this out for the complex case, and note that the other cases follow similarly.

2.2.1. *Desired Properties of the Map α.*

Proposition 2.1. *If $\alpha : K^{-2}(X) \to K(X)$ has the following three properties for X compact, then it is a two-sided inverse of $\beta : K(X) \to K^{-2}(X)$.*

(A1) *α is functorial in X*
(A2) *α is a $K(X)$-module homomorphism*
(A3) $\alpha(b) = 1$

The first part of the proof of this proposition is using functoriality to realize that α has most of the properties that we want. The second part is using a "trick" to show that $\alpha\beta = 1$ implies $\beta\alpha = 1$, in our setup. Note that we begin by assuming we have an α with the three listed properties; it is the job of the rest of the paper to actually construct such an α for each case.

We first want to extend to the case where X is only locally compact, so that we may use K-theory with compact supports.

Lemma 2.2. *The map α can be extended to locally compact X.*

Proof. We take the one-point compactification of X, denoted X^+, and note that we have an exact sequence

$$0 \to K_c(X) \to K(X^+) \to K(+).$$

This sequence is similar to the sequence we get when shifting from K-theory to reduced K-theory, for which we quotient by the dimension of the bundle over the base point. In this case, we quotient out over the point at infinity.

We have a similar exact sequence with $K_c^{-2}(X) = K_c(\mathbb{R}^2 \times X)$; since we used no properties other than local compactness of the base space X, our argument works just as well for $\mathbb{R}^2 \times X$ as for general X.

Drawing out the two exact sequences, and incorporating the maps α_{X^+} and α_+ over the respective spaces X^+ and $\{+\}$, we have the following diagram. It commutes by assumption of functoriality (A1).

$$\begin{tikzcd}
0 \arrow[r] & K_c^{-2}(X) \arrow[r] & K^{-2}(X^+) \arrow[r] \arrow[d, "\alpha_{X^+}"] & K^{-2}(+) \arrow[d, "\alpha_+"] \\
0 \arrow[r] & K_c(X) \arrow[r] & K(X^+) \arrow[r] & K(+)
\end{tikzcd}$$

□

Next, we show that we can consider external multiplication of elements in different K-theories, since we need some multiplicative properties for the formal trick.

Lemma 2.3. *Such an α can be extended to a functorial homomorphism*

$$\alpha_q : K^{-q-2}(X) \to K^{-q}(X)$$

that commutes with right multiplication of elements in $K^{-p}(X)$.

Proof. To extend α to K^{-q}, we simply replace X with $\mathbb{R}^q \times X$. We get a map

$$\alpha_q : K^{-q-2}(X) \to K^{-q}(X)$$

with little effort. This map retains functoriality from (A1) because all we did was specify a bit more about the form of the base space.

Next, we want to show that we can multiply by elements in $K_c^{-p}(X)$. But (A2) already ensures that our map is a $K(X)$-module homomorphism. Consider a map ϕ from $K_c(\mathbb{R}^2 \times X) \otimes K_c(Y) \to K_c(\mathbb{R}^2 \times X \times Y)$ that takes tensor products of vector bundles in $K_c(\mathbb{R}^2 \times X) = K_c^{-2}(X)$ and $K_c(Y)$ to their product in the K-theory of $\mathbb{R}^2 \times X \times Y$. This map is a $K_c(Y)$-module homomorphism, as is the map $\psi : K_c(X) \otimes K_c(Y) \to K_c(X \times Y)$. If we examine the connections between these two homomorphisms via α, we have a diagram

$$\begin{tikzcd} K_c^{-2}(X) \otimes K_c(Y) \arrow[r, "\phi"'] \arrow[d, "\alpha_X \otimes 1"] & K_c^{-2}(X \times Y) \arrow[d, "\alpha_{X \times Y}"] \\ K_c(X) \otimes K_c(Y) \arrow[r, "\psi"'] & K_c(X \times Y) \end{tikzcd}$$

This diagram commutes by the functorial condition (A1). This mapping generalizes to the locally compact case similarly.

With that formulation of multiplication, we turn to the case we want, namely $\mathbb{R}^q \times X$ and $\mathbb{R}^p \times X$ in place of X and Y. We have

$$\begin{tikzcd} K_c^{-q-2}(X) \otimes K_c^{-p}(X) \arrow[r, "\phi"'] \arrow[d, "\alpha_X \otimes 1"] & K_c^{-q-p-2}(X) \arrow[d, "\alpha_X"] \\ K_c^{-q}(X) \otimes K_c^{-p}(X) \arrow[r, "\psi"'] & K_c^{-q-p}(X) \end{tikzcd}$$

This is exactly the commutativity that we wanted to show. In terms of elements, this means that for $x \in K_c^{-q-2}(X)$, $y \in K_c^{-p}(X)$, we have $\alpha(xy) = \alpha(x)y$. ☐

Now that we know our map commutes with the proper multiplications, we can move on to the "formal trick", which requires this ability to break up spaces before and after multiplication. Recalling that β is defined by a multiplication by b, we can see that (A3) actually implies $\alpha\beta = 1$, because for any $x \in K_c(X)$,

$$\alpha\beta(x) = \alpha(bx) \overset{2.3}{=} \alpha(b)x \overset{A3}{=} 1x = x. \tag{2.4}$$

With the trick, we will show that our three conditions also give $\beta\alpha = 1$, and hence prove Prop. 2.1.

2.2.2. *The Trick.* We consider an involution on the space $K_c^{-4}(X)$, which we will use to switch the order of elements in our evaluation. Think of the ring as

$$K_c^{-4}(X) = K_c(\mathbb{R}^4 \times X) = K_c(\mathbb{R}^2 \times \mathbb{R}^2 \times X),$$

and consider $x, y \in K^{-2}(X)$, where x and y correspond to the first and second copies of $\mathbb{R}^2$, respectively. Note that their product xy is in $K_c^{-4}(X)$.

Define $\theta \in \text{Aut}\ (K_c^{-4}(X))$ to be the map that switches them. That is,

$$\theta(xy) = yx.$$

Next, we define a second involution induced by negating a coordinate in one copy of $\mathbb{R}^2$; without loss of generality, the first. Let $\varphi \in \mathrm{Aut}\ (K_c^{-2}(X))$ be induced by $u \mapsto -u$ for u in the first copy of $\mathbb{R}^2$. Note that this map negates only part of the map, so we cannot write $\varphi(x) = -x$. Following Atiyah, we thus denote $\varphi(x) = \tilde{x}$.

We define these two automorphisms θ and φ because we want to compose them. Since the map $(u, v) \mapsto (-v, u)$ is actually homotopic to the identity on $\mathbb{R}^4$, via a rotation by $\pi/2$, we know by homotopy invariance of K-theory on the compactifications of these spaces that this map induces the identity on $K_c^{-4}(X)$. Hence in K-theory, we have

$$xy = \varphi\theta(xy) = \varphi(yx) = \tilde{y}x. \tag{2.5}$$

We now have a way of switching the order of elements.

Remark 2.6. We do not simply pick an automorphism of the form induced by $(u, v) \mapsto (u, -v)$ above because this uses rotations *within* $\mathbb{R}^2$. This causes problems because the mapping would then fail to commute with the group of symmetries $O(2)$ of $\mathbb{R}^2$. Meanwhile, our compositions of θ and φ are induced by maps $(u, v) \mapsto (-v, u)$ that do not disturb our copies of $\mathbb{R}^2$; they only use that we have two copies of $\mathbb{R}^2$ in $\mathbb{R}^4$.

2.2.3. *Proof of Prop. 2.1.* Now recall Prop. 2.1. We claim that if α has the three properties listed, then it is the inverse of β.

Proof. We may assume compact X at this point, since we have shown how the maps extend to the locally compact case. Recall from eqn. 2.4 that we have $\alpha\beta(x) = x$ for all $x \in K(X)$.

In a similar way, we have for any $y \in K^{-2}(X)$ that

$$\beta\alpha(y){=}\alpha(y)b \overset{2.3}{=} \alpha(yb) \overset{2.5}{=} \alpha(b\tilde{y}) \overset{2.3}{=} \alpha(b)\tilde{y} \overset{A3}{=} 1\tilde{y} = \tilde{y},$$

where the first equality holds because multiplication of even degree elements in K-theory is commutative, by properties of the tensor product.

This does not give equality, exactly. However, since $\varphi : y \mapsto \tilde{y}$ is an automorphism of $K^{-2}(X)$, we see that both α and β must be isomorphisms—otherwise, there is no way that their composition $\beta\alpha$ could lead to the isomorphism φ. And since they are isomorphisms that satisfy $\beta\alpha = 1$ (2.4), it follows that they are inverses. □

With the formal trick complete, we see that $\alpha\beta = 1$ does imply $\beta\alpha = 1$, so to prove periodicity in the complex case below it suffices to find an α satisfying the three conditions of 2.1.

The other cases in the proof also follow the template given. However, they use slightly different analogs of the Bott map and Bott class, as shown in Table 2.1 above. The main difference in the statement of 2.1 for each case are the two spaces the Bott map runs between, and the choice of Bott class. We simply interpret A3 to mean that the image under α of the appropriate Bott class is 1.

2.3. **Elliptic Operators on Vector Bundles.** Before diving into specific cases of periodicity, we need to introduce the machinery we use for constructing appropriate maps α; namely, elliptic operators. We first define these operators over sections of our vector bundles, then extend our focus to families of operators between bundles.

When we extend the definition of the index of an operator to a family of operators parameterized by the base space of the bundle, we get a definition of index that returns a value in $K(X)$. This allows us to incorporate the index map into a composition for α. As we will see, this is why each case of the proof relies on an operator and Bott class element with index 1.

2.3.1. *Elliptic Operators and the Index Map.* Operators are functions mapping from a space to itself. In our case, we consider operators on sections of vector bundles over a manifold M. Consistent with Atiyah, we denote the set of sections of a vector bundle E as $\mathscr{D}(E)$, so if we consider an operator d between sections of the vector bundles E and F over M, we write $d : \mathscr{D}(E) \to \mathscr{D}(F)$.

Specifically, the operators we are interested in are linear elliptic partial differential operators. We follow the exposition in [13]. We can write linear partial differential operator in terms of local coordinates $x_1, ..., x_n$.

Note that below we are considering the x_i's as coordinates in $\mathbb{R}^n$, and $x \in \mathbb{R}^n$, and that this has no relation to the $x \in K(X)$ we wrote in the section above.

Definition 2.7. If $d : \mathscr{D}(E) \to \mathscr{D}(F)$ is a **linear partial differential operator**, then it is locally of the form

$$d = \sum_{r \leq n} f_{i_1 \ldots i_r} \frac{\partial^{i_1 \ldots i_r}}{\partial x_{i_1} \ldots \partial x_{i_r}},$$

where $f(x) : E_x \to F_x$ is a linear transformation between the fibers of the vector bundles that varies continuously with x.

That is, locally at x, the operator d is similar to a polynomial of the differential operators $\frac{\partial}{\partial x_i}$ whose coefficients $f_{i_1 \ldots i_r}$ are matrix-valued, with matrices corresponding to linear transformations between fibers.

To formalize this, we can introduce some dummy variables. We represent the operators $\partial/\partial x_j$ by indeterminates $i\xi_j$, and denote the corresponding polynomial by $p(x, (\xi_1, ..., \xi_n))$. In this notation, the operator d above can be written as

$$p(x, (-i\partial/\partial x_1, ..., -i\partial/\partial x_n)).$$

Remark 2.8. The factors of i are a sign convention by some authors, as they cause negatives to appear when the terms are squared.

Definition 2.9. The **symbol** of a linear partial differential operator d, denoted $\sigma(x, (\xi_1, ..., \xi_n))$, is the homogeneous polynomial consisting of only the highest order terms in $p(x, (\xi_1, ..., \xi_n))$.

We can now define what it means for an operator to be elliptic.

Definition 2.10. An operator d is elliptic if its symbol $\sigma(x, (\xi_1, ..., \xi_n))$ is invertible for all $x \in \mathscr{D}(E)$ and $(\xi_1, ..., \xi_n) \in \mathbb{R}^n \setminus \{0\}$. [13]

Example 2.11. The **Laplacian** ∇^2 (alternately denoted Δ) is a familiar second-order differential operator. To simplify our examination, imagine that we map between sections of two trivial n-dimensional real vector bundles, so that our operator essentially maps $\mathbb{R}^n \to \mathbb{R}^n$. By definition, the Laplacian on $\mathbb{R}^n$ is

$$\nabla^2 = \sum_{i=1}^{n} \left(\frac{\partial}{\partial x_i} \right)^2.$$

Note that this operator's corresponding polynomial is already homogeneous, so it is equivalent to the symbol; we write $p(x, (\xi_1, ..., \xi_n)) = \sigma(x, (\xi_1, ..., \xi_n))$.

More generally, the operator $f\nabla^2$ for any smooth, nonzero function $f : \mathbb{R}^n \to \mathbb{C}$ is also elliptic. [13] Defined on a nontrivial bundle, the Laplacian will take the form above in terms of local variabls x_i.

Example 2.12. A **Dirac operator** is like a formal square root of a second-order operator like the Laplacian ∇^2. That is, if $D^2 = \nabla^2$, then D is a Dirac operator. More specifically, an operator like $D = -i\partial/\partial x - i\partial/\partial y$, also written $D = -i\partial_x - i\partial_y$, is Dirac. These operators are generalizations of an operator used in studying the quantum mechanics of the electron [7].

Example 2.13. Another operator we are interested in is the **del bar operator**, $\bar{\partial}$. It is defined on a complex manifold, and in local coordinates $z_k = x_k + iy_k$, the operator is described as a sum

$$\bar{\partial} = \sum_{i=1}^{n} \left(\frac{\partial}{\partial x_i} + i \frac{\partial}{\partial y_i} \right) d\bar{z}_i.$$

Specifically, this operator is well-defined over sections of a **holomorphic vector bundle**, a complex vector bundle whose projection map is holomorphic. [19]

Elliptic operators are a special case of **Fredholm operators**, which are operators between Hilbert spaces whose kernels and cokernels are finite-dimensional. If we compare the dimensions of those spaces, we arrive at the notion of analytical index.

Definition 2.14. The **index** of a Fredholm operator d is

$$\text{index } d = \text{dim ker } d - \text{dim coker } d.$$

One can think of the index of an operator as a sort of measure of "how invertible" an operator is. The closer the dimension of the range is to the dimension of the codomain, the "more surjective" an operator is, and hence the smaller the dimension of the cokernel. Meanwhile, the smaller the dimension of the kernel, the "more injective" an operator is. We say that a Fredholm operator is "almost invertible," and for elliptic operators one can actually define something called a pseudoinverse.

Notation 2.15. We write $\mathfrak{F}(\mathcal{H})$ for the space of Fredholm operators acting on a Hilbert space $\mathcal{H}$.

In differential equations, one desires to study the existence and uniqueness of solutions. Existence corresponds to surjectivity, uniqueness to injectivity, and these two properties are often difficult to study separately. The index of an operator is a quantity that is easier to calculate, and still gives some relevant information. [11]

Example 2.16. Invertible Operator: If T is any invertible operator, then

$$\text{index } T = \dim \ker T - \dim \text{coker } T = 0 - 0 = 0.$$

The converse is not true.

Example 2.17. Finite-dimensional Spaces: The index of a linear operator between finite-dimensional vector spaces depends only on the dimensions of the spaces and not on the choice of map. If $T : V \to W$ for finite-dimensional vector spaces V and W, then Rank-Nullity implies that index $T = \dim V - \dim W$. [11]

Example 2.18. Shift Operator: A basic example on infinite-dimensional Hilbert spaces is the right shift operator $S : \mathcal{H} \to \mathcal{H}$. If $\mathcal{H}$ is an infinite-dimensional Hilbert space with basis $\{e_1, e_2, ...\}$, define $Se_i = e_{i+1}$. That is, the operator shifts the basis vectors down by one.

We have $\operatorname{im} S = \operatorname{Span}\{e_2, e_3, ...\}$, so $\operatorname{coker} S = \mathcal{H}/\operatorname{im} S = \operatorname{Span}\{e_1\}$, while $\ker S = \{0\}$. So, as calculated in [11],

$$\text{index } S = 0 - 1 = -1.$$

Example 2.19. Operator on the Circle:

Next, we calculate the index of an operator defined on our favorite one-dimensional compact manifold—the circle. Consider the operator $d/d\theta$ defined on the unit circle. Its kernel includes the solutions to $df/d\theta = 0$, which are constant functions f.

To calculate the cokernel, we write f in terms of its Fourier series, using the functions $e^{ik\theta}$ as a basis. We have $f(\theta) = \sum_{-\infty}^{\infty} a_k e^{ik\theta}$ and taking the derivative gives $df/d\theta = \sum_{-\infty}^{\infty} ika_k e^{ik\theta}$. This formula describes the image of f under $d/d\theta$, and includes every frequency except $k = 0$, since that term vanishes in the sum. So the codomain excludes this frequency, and the cokernel is the space of all functions quotiented by the space of functions with frequencies $k \neq 0$, which are the functions with only the frequency $k = 0$. These functions are precisely the constant functions, since $e^0 = 1$. Hence the cokernel of the map is also the space of constant functions.

We can now calculate

$$\text{index}(d/d\theta) = 1 - 1 = 0.$$

Note that this operator has index 0, but is not invertible, because of the constant of integration.

Remark 2.20. In fact, any linear elliptic operator has index zero on the circle, as explained in [13]. The Atiyah-Singer Index Theorem, which is another powerful result linking analysis and topology, demonstrates an equivalence between analytical index and topological index. In this case, the value of 0 calculated above using analytic operators corresponds to the Euler characteristic of the circle, a topological invariant.

Usefully, the index is homotopy invariant, and only depends on the highest order terms of the operator; hence it will be well-defined as we extend it to vector bundles.

2.3.2. *Extending to Families.* To prove Bott periodicity, we must extend index from a map to $\mathbb{Z}$ to a map landing in $K(X)$, so that we will be able to include index in a composition to define α.

Proposition 2.21. *If d is an elliptic differential operator on a compact manifold M and Q is any smooth vector bundle over M, then we may extend d to a map d_Q defined on Q. Then the map $Q \mapsto \text{index } d_Q$ defines a homomorphism*

$$\text{index}_d : K(M) \to \mathbb{Z}.$$

Furthermore, if X is any space, then this extends functorially to a $K(X)$-module homomorphism

$$\text{index}_d : K(M \times X) \to K(X).$$

Proof Sketch. First, we extend to d_Q. Let $d : \mathscr{D}(E) \to \mathscr{D}(F)$ be an elliptic operator between sections of vector bundles E and F over a manifold M, and let Q be another vector bundle over M. Then we can extend to an operator $d_Q : \mathscr{D}(E \otimes Q) \to \mathscr{D}(F \otimes Q)$ as follows. If Q is trivial, we take d_Q so that its symbol satisfies $\sigma(d_Q) = \sigma(d) \otimes \mathrm{id}_Q$. We can do this by taking $d_Q = d \otimes \mathrm{id}_Q$, because we do not need to worry about the lower-order terms; our construction need only be well-defined up to its symbol. If Q is nontrivial, we take that construction locally, and piece the result together using partitions of unity. We have thus argued the first extension, which gives a homomorphism from $K(M) \to \mathbb{Z}$, mapping $Q \mapsto \text{index } d_Q$.

To further generalize the notion of index, we need to consider a family of operators instead of just one. Let E now be a **family** of vector bundles over M parameterized by X, so that E_x, the restriction of E to $M \times \{x\}$, is a vector bundle over M. One can think of E itself as a vector bundle over $M \times X$.

Let F be another family of vector bundles over M parameterized by X. Then we can form a family of operators $d_x : \mathscr{D}(E_x) \to \mathscr{D}(F_x)$. These form an elliptic family if all the d_x are elliptic of the same order. One can then define an operator $d : \mathscr{D}(E) \to \mathscr{D}(F)$; with the mapping $x \to d_x$, one can one can think of this operator d as a continuous map in $[X, \mathfrak{F}(\mathcal{H})]$, where the Hilbert space $\mathcal{H}$ describes the vector bundle fibers equipped with some inner product.

We would like to define the index of d. To do so, we consider the kernel and cokernel of d. If $\dim \ker d_x$ is constant as x varies, then the family $\ker d_x$ forms a vector bundle $\ker d$ over X. Define coker d the same way. Then we write

$$\text{index } d := \ker d - \text{coker } d.$$

Since each term is a vector bundle, once we take their difference we arrive at a virtual bundle, which is an element of $K(X)$. That is, index $d \in K(X)$.

However, this definition does not always work. We cannot be guaranteed that the dimensions of $\ker d_x$ and coker d_x are constant over X, so we seek to modify d slightly. We can add a trivial, finite-dimensional bundle P over X, with a bundle map $\phi : P \to \mathscr{D}(F)$ that fills out the dimension of the range of d. That is, if $m = \dim \text{coker } d_x$, set $P = \varepsilon^m$, a trivial m-dimensional bundle, to augment the range of d to reach all of the codomain $\mathscr{D}(F)$.

Then if we define $T = d + \phi$, we have

$$T = d + \phi : \mathscr{D}(E) \oplus P \to \mathscr{D}(F)$$

surjective. This map must have kernel of constant dimension because ϕ changes dimension whenever d does. We arrive at a well-defined notion of index. [11] [1]

Definition 2.22. The **index** of an operator d between families of vector spaces $\mathscr{D}(E)$ and $\mathscr{D}(F)$ is

$$\text{index } d = (\ker T) - P \in K(X),$$

with T and P defined as above. This definition does not depend on the choice of P.

Remark 2.23. Note that our original definition of index as a morphism landing in $\mathbb{Z}$ is not incompatible with this extended definition. It corresponds to a bundle that isn't parameterized by another space X, but we could view it as parameterized trivially by a point space $X = \{\text{pt}\}$. Then it lands in $\mathbb{Z}$ because $K(\{\text{pt}\} = \mathbb{Z}$.

Example 2.24. Let $X = \mathbb{R}$ and $\mathcal{H} = \mathbb{C}$. Consider a family of multiplication operators $d_x \in \mathfrak{F}(\mathbb{C})$ defined by $d_x(z) = xz$, where $x \in \mathbb{R}$. Let E and F be two complex vector bundles over $\mathbb{R}$. We define an operator $d : \mathscr{D}(E) \to \mathscr{D}(F)$ to describe these multiplication operators parameterized by $X = \mathbb{R}$. We think of d as a map $\mathbb{R} \to \mathfrak{F}(\mathbb{C})$ by $x \mapsto d_x$, and we seek to calculate its index. Since the kernels and cokernels of d_x do not have constant dimension, we cannot automatically form vector bundles $\ker d$ and coker d [11]. See that

$$\dim \ker d_x = \begin{cases} 0 & x \neq 0 \\ 1 & x = 0 \end{cases} \qquad \dim \operatorname{coker} d_x = \begin{cases} 1 & x \neq 0 \\ 0 & x = 0 \end{cases}$$

In this case, we modify d by choosing $P = (\text{im } d_x)^\perp = \mathbb{R} \times \mathbb{C}$ for $x \neq 0$, because there d_x fails surjectivity. That is, P is a trivial complex line bundle over $\mathbb{R}$. We want this choice of P because it can complete the image of d_x when $x \neq 0$. To be compatible with that, we define $\phi : P \to \mathscr{D}(F)$ to take

$$\phi(P_x) = \begin{cases} P_x & x \neq 0 \\ \{x\} \times \{0\} & x = 0 \end{cases}$$

so that ϕ completes the range in $\mathscr{D}(F)$ where necessary.

Then $T = d + \phi$ is surjective, and has the same kernel as d. Hence

$$\text{index } d = (\ker T) - [\mathbb{R} \times \mathbb{C}] = (\ker T) - [1].$$

Note that ultimately $\ker T$ depends on what E and F are, which we leave general.

We return to our explanation of Prop. 2.21.

Proof Sketch continued: We can now discuss the index of the operator d_Q. If Q is a family of vector bundles over X, then we may define an elliptic family $d_Q : E \otimes Q \to F \otimes Q$. The map $Q \mapsto \text{index } d_Q$ extends linearly to a homomorphism $\text{index}_d : K(M \times X) \to K(X)$.

If Q over $M \times X$ is trivial, we can say that it is induced by a bundle Q_M over M. That is, we may think of Q trivial over $M \times X$ as having a copy of Q_M over each $x \in X$. When $\dim \ker d_x$ is constant, we have

$$\ker d_Q \cong \ker d \otimes Q_M$$
$$\operatorname{coker} d_Q \cong \operatorname{coker} d \otimes Q_M,$$

and we can write index $d_Q = (\text{index } d) \otimes Q_M$.

We see that the map

$$\text{index}_d : K(M \times X) \to K(X)$$

is a $K(X)$-module homomorphism, as desired for (A2). It is also functorial, which we need for (A1). For a continuous function $f : Y \to X$, we have an induced family of elliptic operators over M parameterized by Y, denoted $f^*(d)$. We have the commuting diagram as follows.

$$\begin{tikzcd} K(M \times X) \arrow[r, "\text{index}_d"'] \arrow[d, "(1\times f)^*"] & K(X) \arrow[d, "f^*"] \\ K(M \times Y) \arrow[r, "\text{index}_{f^*(d)}"'] & K(Y) \end{tikzcd}$$

Example 2.25. If $X = \{p\}$, then $M \times X$ is just a copy of M, and d is an elliptic operator on M. For any compact space Y, we get a constant family $f^*(d)$ from the constant map $f : Y \to \{p\}$.

We have now given a well-defined notion of index and verified that it is functorial and a $K(X)$-module homomorphism. It remains in the cases below to find elements of index 1 and compose to form an inverse for α. The rest of this paper will be less detailed, offering motivation but not proof for some parts of the necessary calculations.

2.4. **The Complex Case.** Complex Bott periodicity is probably the most famous version of periodicity, and has many different proofs. Admitting a few results from analysis for the calculation, the elliptic operators proof is quite quick. From Thm. (1.1) in [1], the statement is as follows.

Theorem 2.26. *Let V be a vector bundle over a compact space X, and define b to be the Bott class. Then the Bott map $\beta : K(X) \to K^{-2}(X)$ given by multiplication by b is an isomorphism.*

Proof. We will appeal to Prop. 2.1, choosing the manifold $M = P_1(\mathbb{C}) = \mathbb{C}P^1$, the complex projective line, and the del bar operator $\bar{\partial}$ from Example 2.13. For any holomorphic vector bundle Q over $\mathbb{C}P^1$, we can extend $\bar{\partial}$ to $\bar{\partial}_Q$. To help in calculating the index, we cite the following fact from algebraic geometry, which is a result from Hodge theory and Dolbeaut cohomology.

Fact 2.27. *Let $\mathscr{O}(Q)$ be the sheaf of germs of holomorphic sections of Q. Then the kernel and cokernel of the operator $\bar{\partial}_Q$ satisfy*

$$\ker \bar{\partial}_Q \cong H^0(\mathbb{C}P^1; \mathscr{O}(Q))$$
$$\operatorname{coker} \bar{\partial}_Q \cong H^1(\mathbb{C}P^1; \mathscr{O}(Q)),$$

where H^* denotes the cohomology group with coefficients in $\mathscr{O}(Q)$.

Now, we construct an element whose index is 1 under the map

$$\operatorname{index}_{\bar{\partial}} : K_c(\mathbb{C}P^1) \to \mathbb{Z}$$

defined by $Q \mapsto \bar{\partial}_Q$. Recall from Prop. 2.21 that the codomain of this map is $\mathbb{Z}$ because the the index of a map $\bar{\partial}_Q$ is defined as $\dim \ker \bar{\partial}_Q - \dim \operatorname{coker} \bar{\partial}_Q$.

The element we choose happens to be exactly the Bott class that is so useful in other proofs of this case. For $Q = 1$, the trivial complex bundle over $\mathbb{C}P^1$, we have

$$H^0(\mathbb{C}P^1; \mathscr{O}(1)) \cong \mathbb{C} \qquad \text{and} \qquad H^1(\mathbb{C}P^1; \mathscr{O}(1)) \cong 0,$$

while for $Q = H^*$ the tautological line bundle over $\mathbb{C}P^1$, we have

$$H^0(\mathbb{C}P^1; \mathscr{O}(H^*)) \cong 0 \qquad \text{and} \qquad H^1(\mathbb{C}P^1; \mathscr{O}(H^*)) \cong 0.$$

Then if we take the difference $1 - H^*$, we have

$$\begin{aligned}
\operatorname{index}_{\bar{\partial}}(1 - H^*) &= \operatorname{index}_{\bar{\partial}_1}(\mathbb{C}P^1) - \operatorname{index}_{\bar{\partial}_{H^*}}(\mathbb{C}P^1) \\
&= (1 - 0) - (0 - 0) \\
&= 1,
\end{aligned}$$

as desired.

However, to get to $K_c^{-2}(X) = K_c(\mathbb{R}^2 \times X)$, we need to relate our choice of manifold to $\mathbb{R}^2$. Recall that the complex projective line $\mathbb{C}P^1$ can be identified with the sphere S^2, if we think of $\mathbb{C}P^1$ as the Riemannian sphere or one-point compactification of $\mathbb{R}^2$. Then we may form an exact sequence

$$0 \to K_c(\mathbb{R}^2) \to K_c(\mathbb{C}P^1) \overset{\text{index}_{\bar\partial_Q}}{\to} \mathbb{Z} \to 0$$

as in Lemma 2.2. We just showed that the element $1 - H^*$ has index 1 and is thus in the kernel of $K_c(\mathbb{C}P^1) \to \mathbb{Z}$, meaning that $1 - H^* \in K_c(\mathbb{R}^2)$. Over this space, the bundle $1 - H^*$ is exactly the Bott class $b = \lambda^*_{\mathbb{C}}$, so we have shown that $\text{index}_{\bar\partial}(b) = 1$.

Finally, we can define α. By definition, $K_c^{-2}(X)$ is $K_c(\mathbb{R}^2 \times X)$, and we can map from there to the compactification $K_c(S^2 \times X)$ and then compose with the index map to land in $K_c(X)$. We set α to a composition

$$K_c^{-2}(X) \to K_c(\mathbb{R}^2 \times X) \to K_c(S^2 \times X) \overset{\text{index}_{\bar\partial}}{\to} K_c(X),$$

so that the map is functorial, a $K_c(X)$-module homomorphism, and sends b to 1. By Prop. 2.1, we are done.

□

2.5. **The Spinor Case.** The spinor case is the real equivariant version of the theorem, and the only known proof of this case of Bott periodicity requires elliptic operators. The periodicity itself relies upon the construction of the Spinor group from Clifford algebras; 8-fold and 2-fold periodicity fall out of the groups Cl_k and their complexifications, respectively. We offer an abbreviated treatment here, based off of [4] and [15].

The statement we focus on in this section is Thm. (6.1) of [1], rewritten below.

Theorem 2.28. *Let G be any compact Lie group. Then let X be a compact G-space, V a real Spin G-module of dimension divisible by 8, and let $u \in KO_G(V)$ be the Bott class of V. Then multiplication by u induces an isomorphism*

$$KO_G(X) \to KO_G(V \times X).$$

To even understand the statement, we need the notion of spinors, which arise from the Spin and Pin groups mentioned earlier. One can think of the Spin group $\text{Spin}(k)$ as the nontrivial double cover of $SO(k)$, while correspondingly the group $\text{Pin}(k)$ covers $O(n)$, for $n \geq 2$. But to see how periodicity arises, we must build up the Clifford algebra definition.

2.5.1. *Clifford Algebras and Periodicity.* Generally, a Clifford algebra $Cl(V, Q)$ is a unital, associative algebra that contains and is generated by a vector space V over a field (which we will take to be $\mathbb{R}$) where V is equipped with a quadratic form $Q : V \to \mathbb{R}$. It is the "most free" algebra generated by V subject to the condition $v^2 = Q(v) \cdot 1$ for all $v \in V$. [15]

Definition 2.29. Atiyah, Bott, and Shapiro [4] define the **Clifford algebra** as the quotient of the tensor algebra by the ideal generated by elements of the form $v \otimes v - Q(v) \cdot 1$ for $v \in V$.

The definition may seem unmotivated, but note that the main structure comes from the quadratic form Q. This gives us a notion of length from which we will find a connection to the groups $O(k)$ and $SO(k)$.

Example 2.30. When $Q = 0$, $Cl(V, Q)$ is the exterior algebra $\Lambda(V)$.

To give a more concrete picture of what a real Clifford algebra looks like, we can offer a presentation. If we take the negative definite quadratic form $Q_k = -\sum x_i^2$, then the algebra is the universal algebra with a unit and elements e_i that anticommute and square to negative one. For our purposes, we take $V = \mathbb{R}^k$. We write the kth Clifford algebra in terms of basis elements e_i as

$$Cl_k = \mathbb{R}\langle e_1, ..., e_k \mid e_i e_j = -e_j e_i,\ e_i^2 = -1\rangle.$$

Low-dimensional Clifford algebras over $\mathbb{R}$ are familiar spaces.

Example 2.31. The trivial Clifford algebra is just the ground field, $Cl_0 \cong \mathbb{R}$, generated by $\{1\}$.

Example 2.32. The first Clifford algebra is $Cl_1 \cong \mathbb{C}$, with the nonidentity element i satisfying $i^2 = -1$.

Example 2.33. The second Clifford algebra is $Cl_2 \cong \mathbb{H}$, the quaternions. We commonly write the quaternion elements as $e_1 = i$, $e_2 = j$, and $e_1 e_2 = k$.

To determine higher-dimensional Clifford algebras, we follow the process in [4]. We require a few identities to start.

Fact 2.34. *For $\mathbb{R}$, $\mathbb{C}$, and $\mathbb{H}$ the real, complex, and quaternion number fields, and $F(n)$ the algebra generated by the $n \times n$ matrices over the field $F = \mathbb{R}$, $\mathbb{C}$, or $\mathbb{H}$, we have*

$$\begin{aligned}
\mathbb{C} \otimes_{\mathbb{R}} \mathbb{C} &\cong \mathbb{C} \oplus \mathbb{C} \\
\mathbb{H} \otimes_{\mathbb{R}} \mathbb{C} &\cong \mathbb{C}(2) \\
\mathbb{H} \otimes_{\mathbb{R}} \mathbb{H} &\cong \mathbb{R}(4) \\
F(n) &\cong \mathbb{R}(n) \otimes_{\mathbb{R}} F \\
\mathbb{R}(n) \otimes_{\mathbb{R}} \mathbb{R}(m) &\cong \mathbb{R}(nm)
\end{aligned}$$

To calculate the algebras Cl_k, it helps to define algebras Cl'_k, which are quite similar but correspond to the *positive definite* quadratic form $-Q_k$. That is,

$$Cl_k = \mathbb{R}\langle e'_1, ..., e'_k \mid e'_i e'_j = -e'_j e'_i,\ e'^2_i = 1\rangle.$$

Note that elements square to the positive unit. From the two base cases $Cl_1 = \mathbb{R}$ and $Cl_2 = \mathbb{H}$, we can use the algebras Cl'_k to inductively build Cl_k.

Proposition 2.35. *There exist isomorphisms*

$$Cl_k \otimes_{\mathbb{R}} Cl'_2 \cong Cl'_{k+2} \qquad \textit{and} \qquad Cl'_k \otimes_{\mathbb{R}} Cl_2 \cong Cl_{k+2}$$

Proof. Write $R'_k = \mathrm{Span}\{e'_1, ..., e'_k\}$, the space spanned by the elements of Cl'_k but with no relations. We can define a linear map $\psi : R'_{k+2} \to Cl_k \otimes Cl'_2$ by

$$\psi(e'_i) = \begin{cases} 1 \otimes e'_i & i = 1, 2 \\ e_{i-2} \otimes e'_1 e'_2 & 2 \leq i \leq k \end{cases}$$

Since this map respects the structure of Cl'_k, it can be made into an algebra homomorphism $\psi : Cl'_{k+2} \to Cl_k \otimes Cl'_2$. Because ϕ takes basis elements to basis elements, and the domain and range have the same dimension, $k + 2$, the map is an isomorphism. We can define a similar isomorphism $Cl'_k \otimes Cl_2 \to Cl_{k+2}$. $\square$

From this result, we can derive an 8-fold periodicity in the algebras, up to a relationship called Morita equivalence. First, note that we have $Cl_4 \cong Cl_2 \otimes Cl'_2 \cong Cl'_4$. Then, alternately applying this and the two equations in 2.35, we have

$$Cl_{k+4} \;\cong\; Cl'_{k+2} \otimes_{\mathbb{R}} Cl_2 \;\cong\; Cl_k \otimes_{\mathbb{R}} Cl'_2 \otimes_{\mathbb{R}} Cl_2 \;\cong\; Cl_k \otimes_{\mathbb{R}} Cl_4.$$

From there, we finally get

$$Cl_{k+8} \;\cong\; Cl_{k+4} \otimes_{\mathbb{R}} Cl_4 \;\cong\; Cl_k \otimes_{\mathbb{R}} Cl_4 \otimes_{\mathbb{R}} Cl_4 \;\cong\; Cl_k \otimes_{\mathbb{R}} Cl_8.$$

We can calculate that $Cl_8 \cong \mathbb{R}(16)$. Substituting this, we have the main result

$$Cl_{k+8} \cong Cl_k \otimes_{\mathbb{R}} \mathbb{R}(16).$$

This implies that if $Cl_k \cong F(m)$ for some field F, then $Cl_{k+8} \cong F(16m)$. That is, in steps of eight, Clifford algebra consists of matrices over the same field but with different dimensions. It turns out that this guarantees that Cl_{k+8} and Cl_k are Morita equivalent.

Definition 2.36. Two rings R and S are **Morita equivalent** if the category of left R-modules is equivalent to the category of left S-modules. We write $R \cong_M S$. [17]

Example 2.37. For any ring R and any natural number n, $R \cong_M R(n)$. [17]

Hence our result is that $Cl_{k+8} \cong_M Cl_k$.

Now, recall that the first two algebras Cl'_1 and Cl'_2 satisfy $Cl'_1 \cong \mathbb{R} \oplus \mathbb{R}$ and $Cl'_2 \cong \mathbb{R}(2)$. From there, we can calculate all algebras Cl_k using Prop. 2.35. In the table from [4] below, we write the first eight Clifford algebras, the corresponding algebras Cl'_k, and the complexifications $Cl_k \otimes_{\mathbb{R}} \mathbb{C}$, which can be thought of as Clifford algebras of Q_k over the complex numbers. It suffices to write only the first eight, since we have periodicity. Note that the complexifications, written in the rightmost column, have period 2, which of course corresponds to the 2-fold complex Bott periodicity.

Table 2.5.1: Clifford Algebras

k	Cl_k	Cl'_k	$Cl_k \otimes_{\mathbb{R}} \mathbb{C} = Cl'_k \otimes_{\mathbb{R}} \mathbb{C}$
1	$\mathbb{C}$	$\mathbb{R} \oplus \mathbb{R}$	$\mathbb{C} \oplus \mathbb{C}$
2	$\mathbb{H}$	$\mathbb{R}(2)$	$\mathbb{C}(2)$
3	$\mathbb{H} \oplus \mathbb{H}$	$\mathbb{C}(2)$	$\mathbb{C}(2) \oplus \mathbb{C}(2)$
4	$\mathbb{H}(2)$	$\mathbb{H}(2)$	$\mathbb{C}(2)$
5	$\mathbb{C}(4)$	$\mathbb{H}(2) \oplus \mathbb{H}(2)$	$\mathbb{C}(4) \oplus \mathbb{C}(4)$
6	$\mathbb{R}(8)$	$\mathbb{H}(4)$	$\mathbb{C}(8)$
7	$\mathbb{R}(8) \oplus \mathbb{R}(8)$	$\mathbb{C}(8)$	$\mathbb{C}(8) \oplus \mathbb{C}(8)$
8	$\mathbb{R}(16)$	$\mathbb{R}(16)$	$\mathbb{C}(16)$

One more important property of Clifford algebras to explain is that they are $\mathbb{Z}/2$-graded. That is, we can write Cl_k as a direct sum $Cl_k^+ \oplus Cl_k^-$ such that multiplication respects this separation.

Notice that elements of Cl_k can all be written as strings of the form $e_{i_1}...e_{i_k}$, with no e_i repeated or with order greater than one, because we may reduce using anticommutativity and $e_i^2 = -1$. With this motivation, we define subalgebras generated by strings of k-many elements e_i,

$$Cl_k^+ = \mathbb{R}\langle e_{i_1}...e_{i_k} \mid k \text{ even}\rangle \qquad Cl_k^- = \mathbb{R}\langle e_{i_1}...e_{i_k} \mid k \text{ odd}\rangle.$$

These fit the properties we require because multiplication of elements in Cl_k preserves the "length" of an element mod 2. For example, two strings $e_{i_1}...e_{i_k}$ and $e_{j_1}...e_{j_l}$ multiply to $e_{i_1}...e_{i_k}e_{j_1}...e_{j_l}$, whose length we can only change by using anticommutativity to line up like elements e_i and eliminating two elements using $e_i^2 = -1$. This grading leads to useful decompositions in structures built out of these algebras, which we will need for the proof.

2.5.2. *Pin and Spin Groups.* Now that we have the Clifford algebras, and have determined their 8-fold periodicity, how do we arrive at the Spin group? We follow the derivation of [15], giving a simpler picture than the symbol-heavy derivation in [4]. However, our notation is consistent with the above, and our discussion is limited to the Pin and Spin groups on $\mathbb{R}^k$ with respect to our chosen quadratic form Q_k, while [15] derives them for a general vector space V and quadratic form q.

We start by considering the group of units of Cl_k, denoted Cl_k^*. Each of these units defines an automorphism of Cl_k, so we have a map called the **adjoint representation** $\mathrm{Ad} : Cl_k^* \to \mathrm{Aut}(Cl_k)$. For a given unit in Cl_k^*, the automorphism we take is that of conjugation. That is, if we let Ad_φ denote the automorphism of Cl_k corresponding to the element $\varphi \in Cl_k^*$, we define $\mathrm{Ad}_\varphi : x \mapsto \varphi x \varphi^{-1}$.

We are interested in a subgroup of Cl_k^* of elements $v \in \mathbb{R}^k \subset Cl_k^*$ for which $Q_k(v) \neq 0$. Call this group P. For the reader following [4], this subgroup serves the same purpose as Γ_k, the Clifford group.

We are interested in P because it has two nice properties. First, for $v \in P$, $\mathrm{Ad}_v(\mathbb{R}^k) = \mathbb{R}^k$, so $\mathbb{R}^k \subset Cl_k$ is preserved under the automorphism of Cl_k defined by v. Second, for $v \in P$, the adjoint transformation preserves the quadratic form Q_k. That is, for all $w \in \mathbb{R}^k$, $Q_k(\mathrm{Ad}_v(w)) = Q_k(w)$. This preservation of the quadratic form, which defines our metric, is the defining property of an orthogonal transformation. That is, all of the automorphisms corresponding to elements of P under the adjoint representation are orthogonal transformations, which correspond to orthogonal matrices in $O(k) \subset \mathrm{Aut}(\mathbb{R}^k)$. We can write

$$P \xrightarrow{\mathrm{Ad}} O(k).$$

For a proof that P has the two properties assumed, we refer the reader to [15] §1.2. But now that we see how this subgroup P corresponds to the orthogonal group, we can finally define the Pin and Spin groups.

Definition 2.38. The **Pin group** of $\mathbb{R}^k$, denoted $\mathrm{Pin}(k)$, is the subgroup of P generated by elements $v \in \mathbb{R}^k$ with $Q_k(v) = \pm 1$.

Definition 2.39. Correspondingly, the **Spin group** of $\mathbb{R}^k$, denoted $\mathrm{Spin}(k)$, is the subgroup of P generated by elements $v \in \mathbb{R}^k$ with $Q_k(v) = 1$.

The groups $\mathrm{Pin}(k)$ and $\mathrm{Spin}(k)$ are those elements of the Clifford algebras with norm one. They form nontrivial double covers of $O(k)$ and $SO(k)$, respectively, by a map called ρ, which the reader can find detailed in [4] or [15], among other sources. We have exact sequences

$$1 \to \mathbb{Z}/2 \to \mathrm{Pin}(k) \xrightarrow{\rho} O(k) \to 1$$
$$1 \to \mathbb{Z}/2 \to \mathrm{Spin}(k) \xrightarrow{\rho} SO(k) \to 1.$$

Note that just as $SO(k) \subset O(k)$, we have $\mathrm{Spin}(k) \subset \mathrm{Pin}(k)$.

Example 2.40. The smallest nontrivial examples of these groups are with $k = 1$. We have $\mathrm{Pin}(1) \cong \mathbb{Z}/4$ and $\mathrm{Spin}(1) \cong \mathbb{Z}/2$, which indeed cover $O(1) \cong \mathbb{Z}/2$ and $SO(1) \cong \{1\}$.

2.5.3. *Spinor Bundles.* It is beyond the scope of this paper to give an in-depth discussion of spin structure on manifolds, but we try to motivate the choice of manifold and operator in the spinor case by using our understanding of Spin groups and Clifford algebras. To that end, we briefly sketch a definition of a spinor bundle.

In this case of the proof, we need to incorporate an equivariant action of some group G. Hence, we need to consider representations.

Definition 2.41. A **spin representation** is just a representation of a spin group G. That is, a spin representation on a finite-dimensional vector space V is a smooth homomorphism $G \to \mathrm{Aut}(V)$ for G a spin group.

Definition 2.42. A **spinor** is an element of a spin representation.

We will give a definition of spinor bundle for completeness, although this section will be more informal. For further reading and to clarify this definition, the reader should see [15] §2.3.

First, we give some beginning intuition for spin structure. On each vector space in a vector bundle, one can make a choice of basis, called a frame. That choice of basis can be transformed by an action of the orthogonal group. If the manifold and the bundle have a preferred orientation, we can use an action of the special orthogonal group instead, in order to preserve that orientation. This defines what is called an orthonormal frame bundle on that manifold. If we then lift that bundle with respect to the map $\rho : \mathrm{Spin}(k) \to SO(k)$, we get a spin structure on the manifold. This structure defines how the bases of vector spaces in the bundle transform by the action of the spin group G.

Next, we need to see how to construct associated bundle, which is a made out of a principal bundle and homeomorphisms on a space F.

Definition 2.43. A **principal bundle** is a fiber bundle whose fiber is a group. We call a principal bundle whose fiber is G a principle G-bundle.

The **associated bundle** of a principal G bundle P over a space X involves homeomorphisms of a space F, and associates a fiber bundle to each continuous homeomorphism $\rho : G \to \mathrm{Homeo}(F)$. We start with a left free action of G on the product $P \times F$ defined by

$$(p, f) \mapsto (pg^{-1}, \rho(g)f)$$

for $(p, f) \in P \times F$, $g \in G$. Denote the space of orbits of this action by $P \times_\rho F$.

Let the projection map of the principle bundle be $\pi : P \to X$. The projection $P \times F \to P \xrightarrow{p} X$ gives a mapping

$$\pi_p : P \times_\rho F \to X,$$

defining $P \times_\rho F$ as a bundle over X. We call this bundle the **bundle associated to P by ρ.** [15]

Finally, we can give a basic definition of a spin bundle, following [15].

Definition 2.44. Let E be a vector bundle that admits a spin structure. Let M be a left module on Cl_k and $\mu : \mathrm{Spin}(k) \to SO(M)$ be a representation given by left multiplication by elements of $\mathrm{Spin}(k)$. A **real spinor bundle** of E is a bundle of the form

$$S(E) = P_{\mathrm{Spin}} \times_\mu M.$$

2.5.4. *Proof of Bott Periodicity.* Recall the statement that we seek to prove from Thm. 2.28.

Theorem 2.45. *Let X be a compact G-space, let V be a real Spin G-module of dimension $8n$, and let $u \in KO_G(V)$ be the Bott class of V. Then multiplication by u induces an isomorphism*

$$KO_G(X) \to KO_G(V \times X).$$

Note that a spin G-module is just a G-module where G is a spin group. Now that we have defined all of the terms in this statement and derived the Spin group from Clifford algebras, we can return to vector bundles. For this case of periodicity, we proceed by choosing a spinor bundle over the sphere V^+ for our manifold, and the Dirac operator for our index map.

We require V to be $8n$-dimensional in order to have the desired grading of the spinor bundle. To define this bundle, we need a manifold with a spin structure, so we compactify V to $V^+ \cong S^{8n}$. Meanwhile, we choose the Dirac operator, as seen in Example 2.12, because it is highly compatible with spin structure.

In this proof summary, we try to introduce the necessary objects of equivariant action as they appear, but for a detailed introduction to equivariant K-theory, the reader should consult [20].

Proof. Denote the spin bundle over the sphere V^+ by S. With our choice of a sphere of dimension $8n$, the spin bundle inherits a grading from the underlying 8-periodic Clifford algebra structure. That is, the bundle decomposes as $S = S^+ \oplus S^-$, and the Dirac operator maps S^+ to S^-, and vice versa. We focus on the operator whose domain is restricted to S^+, which we write as $D : S^+ \to S^-$. The other operator, mapping $S^- \to S^+$, is in fact the dual operator D^*.

To show that D is G-equivariant, we argue that V^+ can be written as the homogeneous space $\mathrm{Spin}(8n+1)/\mathrm{Spin}(8n)$, which gives us that D is a homogeneous operator. That is, the action of G factors through D as long as we can write V^+ as that quotient.

To that end, we regard $V^+ \cong S^{8n} \subset \mathbb{R}^{8n+1}$, and get an action of the group $\mathrm{Spin}(8n+1)$ deriving from its action on $\mathbb{R}^{8n+1}$. Just as $SO(8n+1)$ acts transitively on S^{8n} with stabilizer $SO(8n)$, its double cover $\mathrm{Spin}(8n+1)$ acts transitively on S^{8n} with stabilizer $\mathrm{Spin}(8n)$. Hence we can indeed write $V^+ \cong \mathrm{Spin}(8n+1)/\mathrm{Spin}(8n)$. Then since we have taken V to be a spinor bundle, we have that the action of $\mathrm{Spin}(8n)$ factors through a homomorphism $G \to \mathrm{Spin(8n)}$ by assumption. This guarantees an equivariant action by D.

With that established, we know that it makes sense to apply Prop. 2.21. We thus arrive at the homomorphism

$$\mathrm{index}_D : KO_G(V^+ \times X) \to KO_G(X).$$

This homomorphism guarantees the requirements (A1) and (A2) for Prop. 2.1 above. All that remains now is to find an element with index 1. Specifically, we seek an element $u \in KO(V^+)$ satisfying

$$\mathrm{index}_D(u) = 1 \in KO_G(\{p\}).$$

We claim that $u = S^+ - 1$ is that desired element. To prove this, we start thinking in terms of representation theory. The real representation ring of G, denoted $RO(G)$, is defined as the Grothendieck ring of real G-modules, which we can think of as real G-equivariant vector spaces. Then, $RO(G)$ corresponds exactly to the KO_G-theory of a point $\{p\}$. Hence we may write $\mathrm{index}_D(u) = 1 \in RO(G)$ [14].

It remains to show that our choice $S^+ - 1$ satisfies the desired properties. We omit the details of this calculation, referring the reader to [1] or [3] for a more thorough explanation, but give a summary.

First, Atiyah offers as an aside that one approach to calculating S^+ is to use Hodge theory to find an extended version of the Dirac operator, and relate its kernel to the cohomology of the sphere S. This method relies on something called the Riemannian connection on the sphere, which is essentially a structure that allows us to lift a differential operator from a manifold to its frame bundle and so extend the Dirac operator to act on $S \otimes S$. Once the operator has been extended, its kernel corresponds to a harmonic form, and thus, according to Hodge theory, cohomology. However, Atiyah eschews this many-step method in favor of calculating the sum and differences of the indices.

It turns out that we can appeal to a general result, which is detailed in [10]. When the base space of our vector bundles is a homogeneous space G/H, then vector bundles over G/H induced by H-modules are called **homogeneous bundles**.

Fact 2.46. *Let G be a compact Lie group and H a connected subgroup of the same rank as G. Let E and F be homogeneous vector bundles over the space G/H. Then if $d : \mathscr{D}(E) \to \mathscr{D}(F)$ is an elliptic operator acting between sections of these bundles, its index depends only on the H-modules that induce E and F, not on the choice of operator d.*

As we argued above, the Dirac operator is homogeneous, acting on the homogeneous space $\mathrm{Spin}(8n+1)/\mathrm{Spin}(8n)$. Using this fact, and character computations for the spin representations, which are detailed in [3] §§6 and 8, we arrive at two equations for the indices of S^+ and S^-. These equations specify the difference and sum of the indices, which are known as the Euler characteristic and Hirzebruch signature, respectively. They are

$$\mathrm{index}_D(S^+) - \mathrm{index}_D(S^-) = 2$$
$$\mathrm{index}_D(S^+) + \mathrm{index}_D(S^-) = 0.$$

Subtracting, we have $\mathrm{index}_D(S^+) \in RO_G(V^+)$. But since $\mathrm{Spin}(8n)$ is connected, it acts trivially on the cohomology of V^+, and we may consider the index mapping to $RO(\mathrm{Spin}(8n)) = RO(G)$ instead. Hence S^+ has index $1 \in RO(G)$, as desired.

However, we are not quite finished. The element u must be a formal difference of vector bundles in order to fall within KO_G, as required, so we seek a bundle with index 0. With a symmetry argument using the antipodal map, one can show that the trivial bundle 1 satisfies

$$\operatorname{index}_D(1) = 0 \in RO(G).$$

Because the antipodal map, like the Dirac operator restrictions D and D^* respectively, maps S^+ to S^- and vice versa, it is compatible with the Dirac operator. Then since the antipodal map acts symmetrically around the sphere, it is in particular symmetric across those two hemispheres, and hence it induces an isomorphism $\ker D \cong \ker D^*$. The antipodal map is also compatible with the action of the group $\mathrm{Spin}(8n)$, but it switches S^+ and S^-, causing the components $\ker D$ and $\ker D^*$ to cancel. In any case, we have $\operatorname{index}_D(1) = 0$.

Set $u = S^+ - 1$. We can identify this element with S^+ minus the trivial bundle with the fiber of S^+ at the point ∞, and call this element the Bott class of the module V.

With this element, we have (A3). Using exactness to pass from $K_G(V^+ \times X)$ to $K_G(V \times X)$, we have our map, which satisfies (A1) and (A2). By Prop. 2.1, we are done. □

With the hard work done, we can extract a more basic form of 8-fold periodicity.

Corollary 2.47. *The group $KO_G(X)$ is 8-periodic.*

Proof. Apply the theorem above with V a trivial bundle of dimension 8. Then we have

$$KO_G(X) \cong KO_G(\mathbb{R}^8 \times X) = KO_G^{-8}(X).$$

□

2.6. **Conclusion.** In exploring Atiyah's paper, we proved two cases of Bott periodicity. In the spinor case in particular, we tried to emphasize how the 8-fold periodicity arises in the Clifford algebras that underlie the spin bundle, to indicate why perhaps this proof method is particularly suited to the real equivariant case. Analyzing the necessary elements in this proof may give insight toward finding another method of proof.

Acknowledgments. I am very thankful to my mentor Peter May for organizing the REU, teaching part of the fascinating algebraic topology course, offering advice and resources, and answering a multitude of questions on this proof and other interesting mathematics. I would also like to thank Dylan Wilson for his thoughtful and engaging teaching, recommendations for a path through sources, and incredibly intuitive explanations for many of the inscrutable details assumed in the paper. I am also grateful to Mark Behrens for hosting a great conference at the University of Notre Dame and for generously dedicating time to explaining Clifford algebras and Spin groups, and to Shmuel Weinberger for a helpful and amusing conversation that sparked my interest in Bott periodicity. Finally, I would be remiss not to mention the incredible Aygul Galimova, with whom I collaborated in understanding Atiyah's paper and finding other sources.

References

[1] M. F. Atiyah. "Bott Periodicity and the Index of Elliptic Operators." *Quart. J. Math.* Oxford (2), 19 (1968), 113-40.

[2] M. F. Atiyah and R. Bott. "On the Periodicity Theorem for Complex Vector Bundles." Oxford University and Harvard University, 1964.

[3] M. F. Atiyah and R. Bott. "A Lefschetz Fixed Point Formula for Elliptic Complexes: I." *The Annals of Mathematics*, Second Series, Vol. 86, Issue 2 (Sept. 1967), 374-407.

[4] M. F. Atiyah, R. Bott, and A. Shapiro. "Clifford Modules." Oxford University and Harvard University, 1963.

[5] M. F. Atiyah and D.W. Anderson. *K-Theory.* W.A. Benjamin, Inc., 1967.

[6] M. F. Atiyah. "K-Theory and Reality." Quart. J. Math. Clarendon Press, Oxford (2), 17 (1996), 367-86.

[7] Berline, Nicole, Ezra Getzler, and Michèle Vergne. *Heat Kernels and Dirac Operators.* Springer-Verlag, 2004.

[8] Blair, Chris. "Some K-theory Examples." Trinity College Dublin, 2009.

[9] Bott, Raoul. "The Stable Homotopy of the Classical Groups." Annals of Mathematics, Second Series, Vol. 70, No. 2 (Sep., 1959), 313-337.

[10] Bott, Raoul. "The Index Theorem for Homogeneous Differential Operators." *Differential and Combinatorial Topology*, 167-86. Princeton University Press, 1965.

[11] Breen, Joseph. "Fredholm Operators and the Family Index." Senior Thesis, Northwestern University, 2016.

[12] Hatcher, Allen. *Vector Bundles and K-Theory.* Unpublished, 2009.

[13] Landweber, Gregory D. "K-Theory and Elliptic Operators." University of Oregon, 2008.

[14] Landweber, Gregory D. "Representation Rings of Lie Superalgebras." University of Oregon, 2005.

[15] Lawson, H. Blaine and Michelsohn, Marie-Louise. *Spin Geometry.* Princeton University Press, 1990.

[16] J. P. May. *A Concise Course in Algebraic Topology.* University of Chicago Press. 1999.

[17] "Morita Equivalence." *Encyclopedia of Mathematics.* Springer.

[18] "Proofs of Bott periodicity." *MathOverflow.*

[19] Rowland, Todd. "Del Bar Operator." From *MathWorld*–A Wolfram Web Resource, created by Eric W. Weisstein.

[20] Segal, Graeme. "Equivariant K-theory." *Publications mathématics de l'I.H.É.S.*, tome 34 (1968), 129-151.