INTRODUCTION TO KNOTS AND BRAIDS USING SEIFERT CIRCLES

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Abstract. In this paper, we will give an introduction to knots, braids, and how they relate to each other. We will first go through some introductory definitions and theorems about knots, braids, and Seifert circles. Finally, we will use these theorems to provide a proof of Alexander's theorem using Seifert diagrams and Vogel's algorithm, and work through an example of how this algorithm is used to find a closed braid representation from a knot projection.

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1. Introduction

The purpose of this paper is to provide an introduction to braid theory and knot theory, concluding in a proof of Alexander's theorem. The paper will involve examples and pictures to provide intuitive explanations of each concept as well as rigorous proofs. The first section of the paper will overview definitions of knots, explaining knot projections and knot equivalence and the three Reidemeister moves. In the next section, we explain a type of diagram constructed from a knot projection called a Seifert diagram. To understand how to form Seifert diagrams we will learn about knot orientation, and then use this concept of orientation to talk about coherency of Seifert circles in a Seifert diagram, which is related to the way each circle is oriented relative to another. The next section introduces the concept of a braid, and from there we find a type of knot called a closed braid. We will discover Alexander's theorem that any knot has a representation as a closed braid, which we will prove in the final section using Vogel's Algorithm. The algorithm involves performing a Reidemeister move on edges determined by the coherency of Seifert circles in the Seifert diagram. We will go through an example of the algorithm working to find a closed braid representation for a knot. Then we will

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prove the theorem by showing that Vogel's algorithm will conclude in a closed braid representation given any knot.

2. Knots

We usually think about knots as shoelaces, ties, and climbing ropes. The mathematical concept of a knot is really no different except that the two ends of the string are connected together, and the string has no thickness.

Definition 2.1. A mathematical knot is an embedding of the circle in $\mathbb{R}^3$.

Most of the time, we study knots by considering a projection of the knot on $\mathbb{R}^2$, a picture of the knot with crossings indicated as over or under.

Figure 1. A few examples of knot projections

Note that while we focus on knots in this paper, many theorems can be generalized to all links.

Definition 2.2. A link is a a collection of distinct knots that may also be knotted together.

One example of a link is the Borromean rings, which have the interesting property that no two rings are linked to each other, but the link of all three rings cannot be separated.

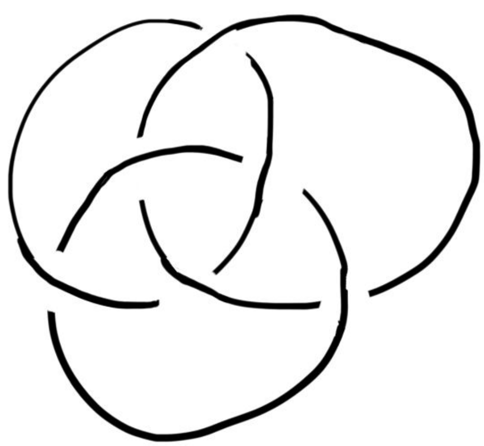

Figure 2. Borromean Rings

Two knots are equivalent if one can be deformed into the other without passing through itself. Determining when two knot projections represent the same knot can be very complicated.

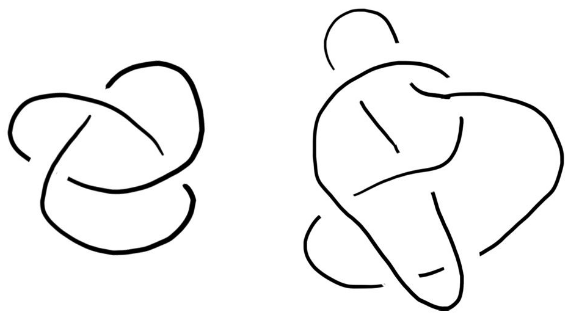

Figure 3. Two knot projections of the same knot

Since we mostly look at knot projections, it is helpful to formalize the ways we can deform a knot projection. To do this, we use Reidemeister moves.

Reidemeister move I puts a loop into an edge or takes one out, move II passes one edge over another or back, and move III passes an edge over a crossing.

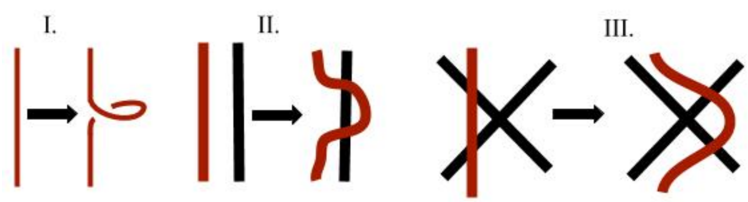

Figure 4. The three Reidemeister moves

These three moves clearly do not change the knot into a different one. And remarkably, they are sufficient to transform any projection of a knot into any other projection of that same knot.

Definition 2.3. Two knot projections are equivalent if one can be transformed into the other by a series of Reidemeister moves.

3. Seifert Circles

Given a knot, we can assign to it an orientation by following the curve around indicating the direction. With a knot, the direction can be chosen arbitrarily because we only care whether the orientations of edges of the knot are the same or different, not what that orientation is. Link orientations are a little more complicated because the different components' orientations do not depend on each other, so there will be multiple ways to orient the link which change the relative orientations of edges that are part of different components.

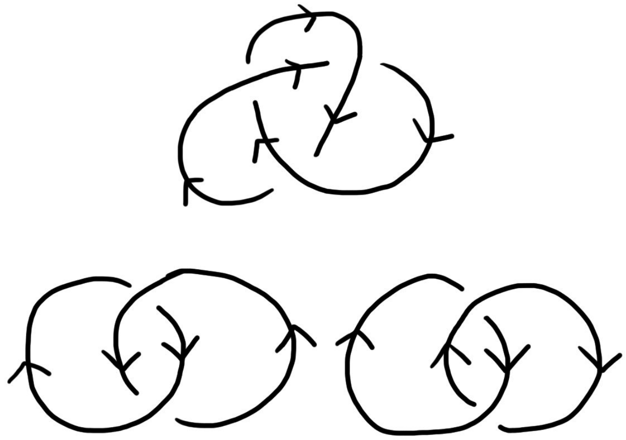

FIGURE 5. An oriented knot and two ways of orienting a link

Definition 3.1. Seifert Diagram. A Seifert diagram of a knot is constructed by eliminating all crossings such that the edges of a crossing are disconnected, then each is reconnected to the other crossing edge with compatible orientation. The resulting Seifert circles are then connected with lines where the crossings used to be.

What we mean by eliminating a crossing is shown below. We will always connect the edges by following their orientations, so they can only be reconnected in one way.

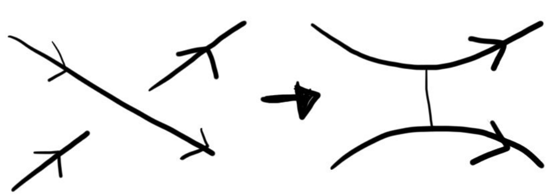

FIGURE 6. Eliminating a crossing

After eliminating all crossings in this way, we will have the Seifert diagram, which will consist of some number of circles, called Seifert circles, connected by lines. We can spread out the Seifert circles to see the regions in between them more clearly, which will be useful later on.

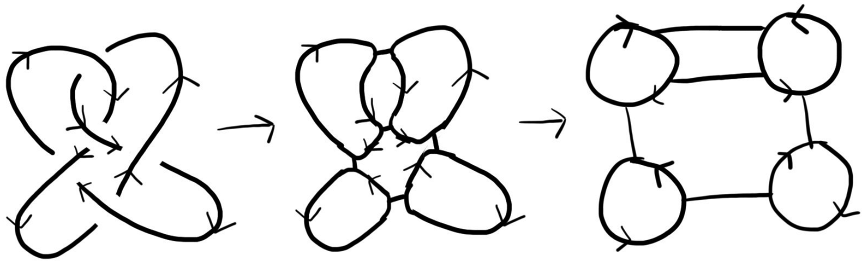

Figure 7. Creating the Seifert diagram of a knot

Notation 3.2. Let $O(C_i)$ denote the orientation of the Seifert circle C_i (clockwise or counterclockwise.)

Definition 3.3. A pair of Seifert circles are *coherent* if either

(1) they are nested and oriented in the same direction, or
(2) they are un-nested and oriented in opposite directions

If the circles satisfy neither condition, they are *incoherent*.

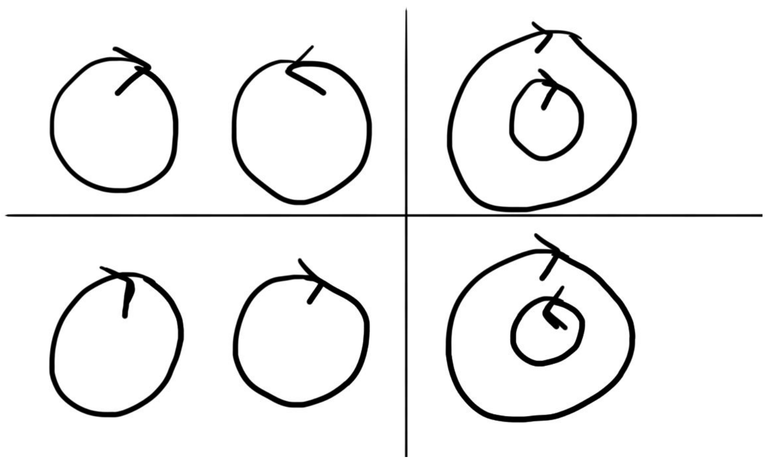

Figure 8. TOP: Coherent circles BOTTOM: Incoherent circles

Remark 3.4. Any pair of Seifert circles connected by a crossing is coherent. We see this when looking at an image of a crossing (Figure 6), and noticing that the newly disconnected parts will be oriented in coherent directions.

Theorem 3.5. *A region with more than two Seifert circles intersecting its boundary must have a pair of incoherent Seifert circles.*

Proof. Consider a region R which is bounded by only two coherent Seifert circles, C_i oriented clockwise and C_j oriented counterclockwise. Adding a third circle to the region, if it is oriented clockwise it will be incoherent with C_i, and if it is oriented counterclockwise it will be incoherent with C_j. Similarly, if C_i is nested in C_j, they must both be oriented in the same direction to be coherent, let's say clockwise. Then if C_k placed in the region between the circles is oriented clockwise, it is incoherent with C_i since they are un-nested, but if it is oriented counterclockwise it is incoherent with C_j since they are nested. □

4. Braids

Like knots, the mathematical concept of a braid is similar to our intuition. A mathematical braid is made up of a collection of n strings which travel between a horizontal starting line and end line without crossing any horizontal line in between more than once, that is, a string must always travel downhill.

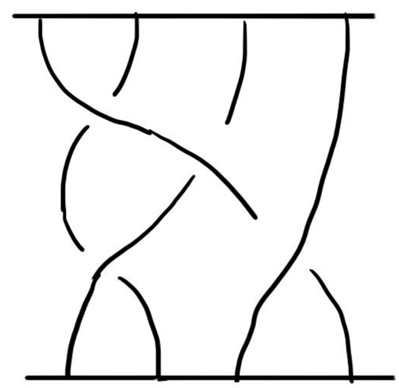

Figure 9. A braid

Definition 4.1. A closed braid is formed by connecting each starting point of a braid to the corresponding ending point.

Now, rather than disconnected strings with ends, a closed braid is a knot or link.

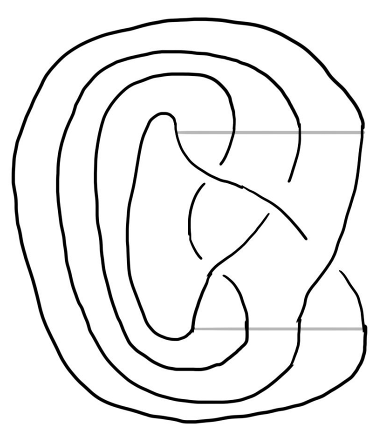

Figure 10. A closed braid

Like any other knot, this closed braid can be deformed into other knot projections, some of which are also closed braids.

Definition 4.2. The braid index of a closed braid is the least number of strands any projection of that braid has.

Every braid can be made into a closed braid, which can then be deformed into other types of knots. Surprisingly, every knot projection can also be deformed into a closed braid.

Theorem 4.3. *Every knot has a closed braid representation.*

This theorem is called Alexander's theorem, as it was first proven by J. W. Alexander in 1923. The theorem can be proven in multiple ways, but not all proofs give a computable method to find closed braid representations given an arbitrary knot, or ones that do take an unknown number of steps. Simply trying to deform a knot into a braid intuitively can be very difficult, and increasingly so with more complex knots.

5. Vogel's Algorithm

In 1990, Vogel discovered an easily computable algorithm which gives a closed braid projection from any knot projection. The proof that this algorithm is effective for any knot is also a proof of Alexander's theorem.

For the algorithm, it does not matter which part of the knot crosses over and which crosses under in a crossing, so we can consider a shadow of a projection, which is just a knot projection without over and under crossings indicated. Regions of this projection are bounded by edges, which are sections of the knot between crossings. The outside region also counts as a region of the knot, and is bounded by all the outermost edges.

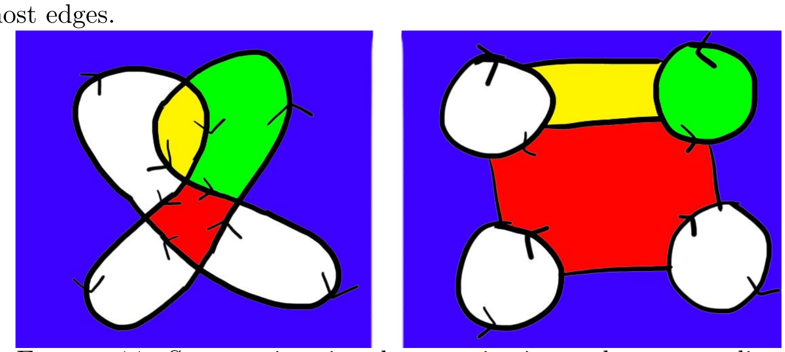

Figure 11. Some regions in a knot projection and corresponding Seifert diagram

The conclusion of the algorithm is creating a knot projection where the Seifert diagram consists of a set of coherent nested circles. A projection of this type is a closed braid representation, as seen below.

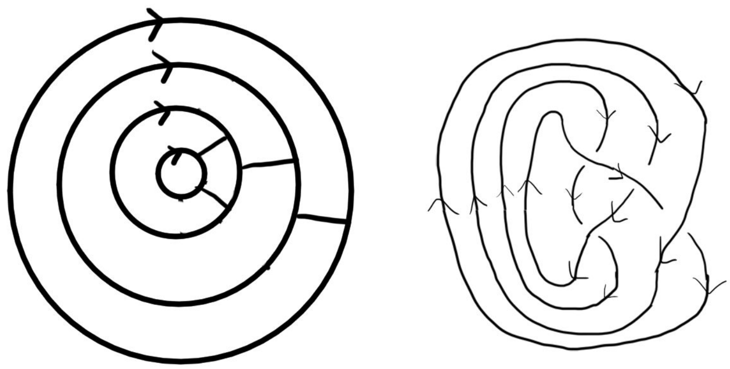

FIGURE 12. If all Seifert circles are coherent and nested, the knot projection is a closed braid.

Notation 5.1. Let $C_a(n)C_b$ mean that Seifert circle a is nested inside Seifert circle b.

Vogel's algorithm consists of only two moves on the knot projection. One is Reidemeister move II. The other is called a *change of infinity*, which can be performed on any edge which borders the outside region of the knot. Intuitively, doing a change of infinity on an arc is flipping this edge around to the other side of the knot.

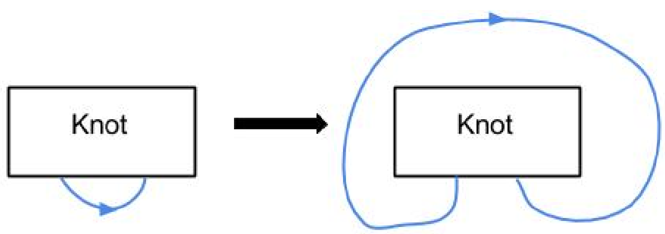

FIGURE 13. A change of infinity for an arbitrary edge bordering the outside of a knot

The algorithm involves doing Reidemeister move II on certain edges. We will use the Seifert diagram of the knot to determine which edges we should perform the move on. When we create a Seifert diagram from a knot projection, the edges of the knot become segments of Seifert circles, and crossings of the knot still contain the region by the lines connecting Seifert circles. This means that any region of a knot projection will have an equivalent region in the Seifert diagram.

Definition 5.2. A *defect region* is a region of the Seifert diagram whose border contains at least one pair of of incoherent edges, that is, edges belonging to distinct incoherent Seifert circles.

Whenever there is a defect region in the diagram, we will perform move II on the pair of incoherent edges in the knot projection. The effect this has on the Seifert diagram is to replace two disconnected incoherent Seifert circles with a nested pair of coherent Seifert circles. We will call this maneuver Q.

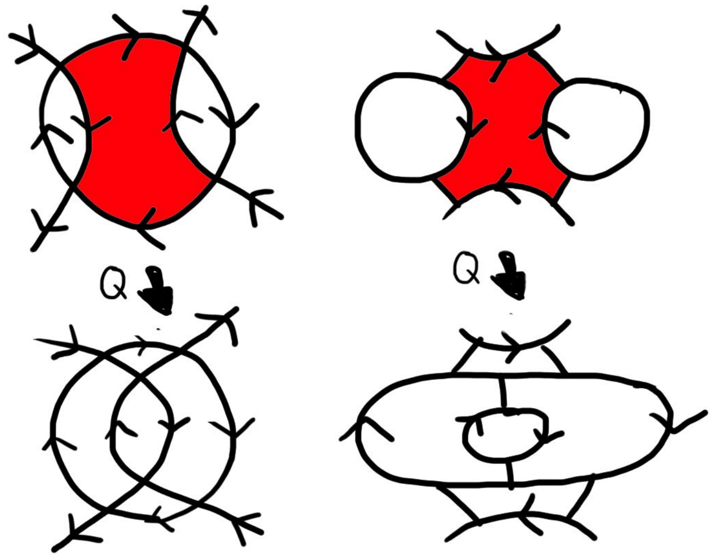

FIGURE 14. The effect of Maneuver Q on a knot projection and corresponding Seifert diagrams. The defect region is highlighted red.

When there are no more defect regions, we will perform changes of infinity until all circles are nested.

Now, we will go through an example of this algorithm working on a knot projection. After, we will prove that the algorithm will always result in entirely nested Seifert circles for any knot projection.

Example 5.3. In this example, we follow the algorithm, using maneuver Q and changes of infinity to change the knot into a form made of nested Seifert circles. Defect regions are marked as red, and the edges we will be performing Q on are markedd in green.

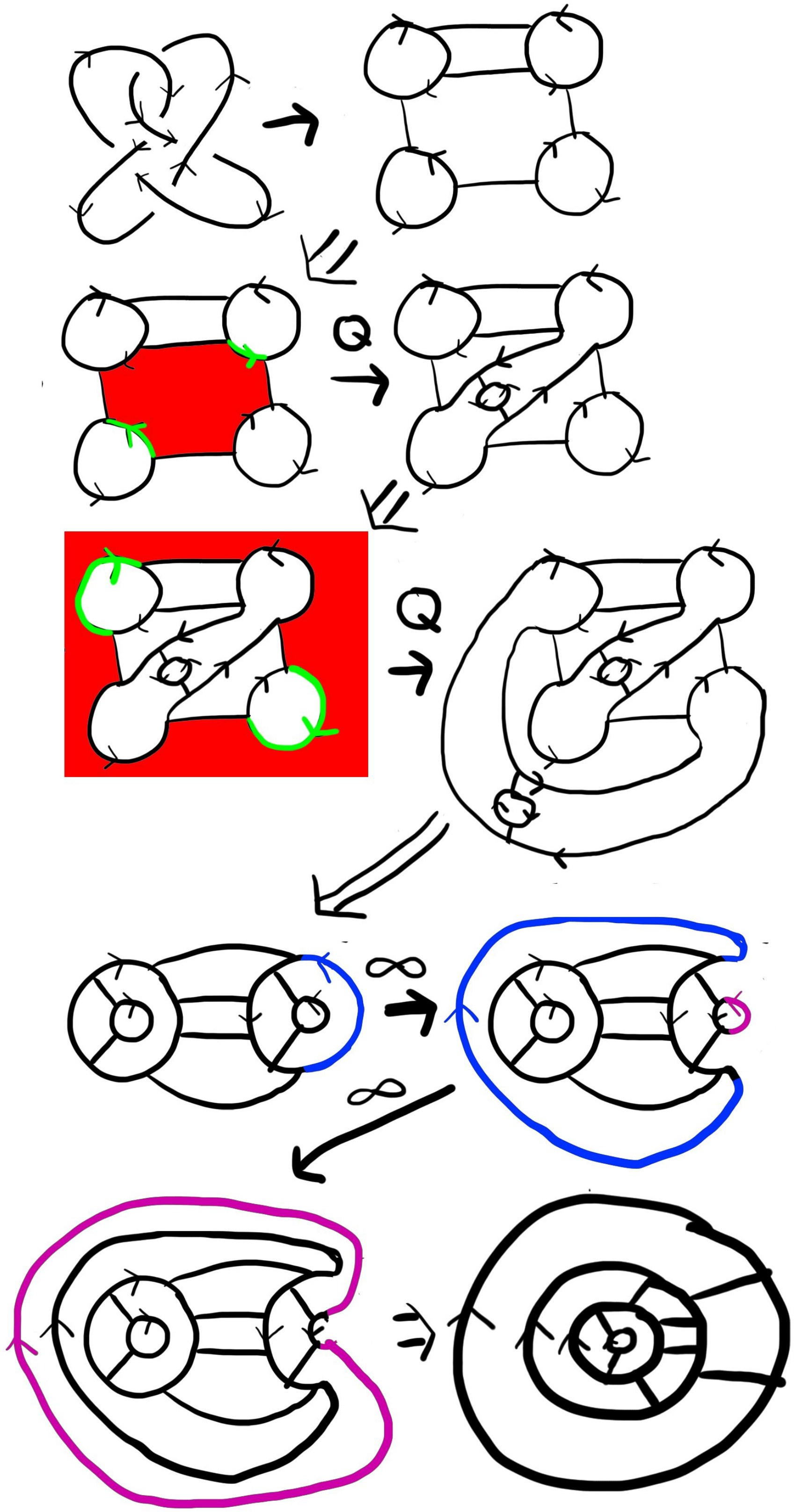

Figure 15. An example of Vogel's algorithm.

Now to prove Alexander's theorem, we first need to know that Q will always decrease the number of incoherent Seifert circles. Then we must prove that there is no knot projection whose Seifert diagram contains incoherent Seifert circles which cannot be eliminated by Q. This will mean that we can reach a diagram with all coherent Seifert circles. We will then show that in the case that all Seifert circles are coherent, we can use changes of infinity to nest all circles. All together, this will prove that Vogel's algorithm concludes in completely nested Seifert circles from any knot projection, which we know is a closed braid.

Lemma 5.4. *Performing Q on a knot diagram reduces the number of incoherent Seifert circles in the diagram by 1.*

Proof. The two Seifert circle segments we perform Q on must have the same orientation. Thus all other Seifert circles in the diagram must be either coherent to both these circles or incoherent to both. When we perform Q on these circles, we end up with two new circles oriented the same direction as the original circles. All other Seifert circles are still coherent or incoherent with these two circles. However, the two circles we perform Q on are now coherent with each other. Since this is the only change in coherency of Seifert circles, the number of incoherent Seifert circles decreases by 1. □

Lemma 5.5. *If a Seifert diagram has an incoherent pair of Seifert circles, it has a defect region.*

Proof. Suppose there are no defect regions. If $C_a(n)C_b$, then $O(C_a) = O(C_b)$, otherwise they would be incoherent and the region bounded by them would be defect.

There also cannot be a circle C_c in the region bounded by C_a and C_b or that region would be bounded by more than two circles and have an incoherent pair. So there cannot be circles C_a and C_c such that $C_a(n)C_b$ and $C_c(n)C_b$.

Consider a circle C_1 bounding the outer region. There exist Seifert circles $C_i(n)C_{i-1}(n)...(n)C_1$. These must be the only circles within C_1 since there cannot be more than one circle immediately nested within each circle. For each $k = 2, ..., i$, we have the equality $O(C_k) = O(C_{k+1})$ because they bound a region. Then we also have equality $O(C_i) = ... = O(C_1)$.

If the outside is bounded by only one circle, C_1, we are done because all the nested circles have the same orientation which makes them all coherent. Now consider if the outside is bounded by two circles, C_1 and $C_1\prime$. Similarly, for circles $C_i\prime(n)...(n)C_1\prime$, we find that $O(C_k\prime) = ... = O(C_1\prime)$. But since $C_1 \cancel{(n)} C_1\prime$, we have O$(C_1) \neq O(C_1\prime)$ otherwise they would be incoherent and the outside region would be defect. This means that for any i and j, we have $O(C_i) \neq O(C_j\prime)$. So for any two circles, if they are nested they will have the same orientation, and if they are un-nested they will have opposite orientation, thus always being coherent. If there are no defect regions, all of the Seifert circles are coherent. If there is an incoherent pair of Seifert circles, that means there is still a defect region to apply Q on, proving that all knots will arrive at a projection where all Seifert circles are coherent. □

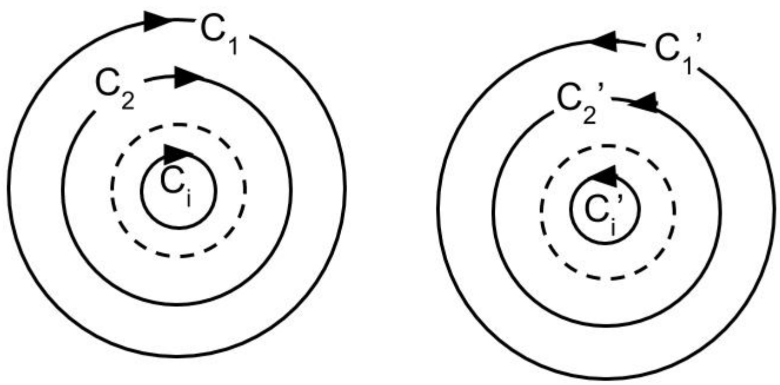

FIGURE 16. The general configuration of Seifert circles with no defect regions

Lemma 5.6. *Any knot projection where all Seifert circles are coherent can be transformed into a closed braid representation with no greater than $N/2$ changes of infinity, where N is the number of Seifert circles.*

Proof. If we have two nested sets of Seifert circles with opposite orientations, a change of infinity of an edge which bounds the outside region makes the outermost circle from one set into the new outermost circle of another set. The smaller of the two sets cannot be greater than $N/2$ Seifert circles, and so the number of circles necessary to switch will be less than or equal to $N/2$. □

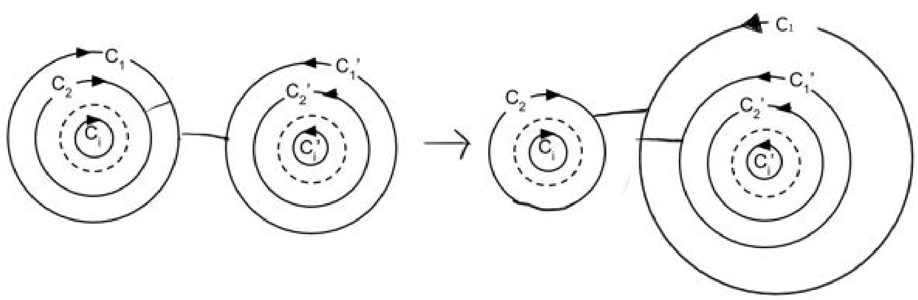

FIGURE 17. A change of infinity

Since we can always make all Seifert circles coherent, and we can nest coherent Seifert circles with changes of infinity, we can always nest all Seifert circles. Thus any knot can be represented as a closed braid.

Corollary 5.7. *The braid index is less than or equal to the minimum number of Seifert circles in any projection of the knot.*

Proof. Each Seifert circle at the end of the algorithm represents one strand on the braid. Each move in the algorithm does not add or subtract Seifert circles, only rearranges them. So the smallest possible number of strands cannot be greater than the number of Seifert circles we start with. □

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