THE FUNDAMENTAL THEOREM OF SPACE CURVES

JOSHUA CRUZ

Abstract. In this paper, we show that curves in $\mathbb{R}^3$ can be uniquely generated by their curvature and torsion. By finding conditions that guarantee the existence and uniqueness of a solution to an ODE, we will be able to solve a system of ODEs that describes curves.

Contents

1. Introduction

Maps in $\mathbb{R}^3$ with nonzero derivatives are called regular curves. By viewing the trace of a curve in $\mathbb{R}^3$, we can see how it curves and twists through space and we can numerically describe this movement with a curvature function $\kappa(s)$ and a torsion function $\tau(s)$. Interestingly enough, every curve in $\mathbb{R}^3$ is uniquely generated by its curvature and torsion alone. That is to say, given a nonzero curvature function $\kappa(s)$ and a torsion function $\tau(s)$, we can generate a regular curve that is unique up to rigid motion.

To understand how this is done, we must first understand how curves can be described at any point. This can be achieved by a set of 3 unit vectors which can be related to one another using a system of ODEs. Finding a unique curve then becomes a question of finding a unique solution to this system of equations. Using the Banach Fixed Point Theorem, we will verify conditions that guarantee a unique solution for ODEs and apply such conditions to find a unique curve.

2. Curves and the Frenet Trihedron

Definition 2.1. A *parametrized differentiable curve* is a differentiable map $\alpha : I \to \mathbb{R}^3$ where I is some open interval $(a, b) \subset \mathbb{R}$.

Example 2.2. A simple example is the helix, parametrized by

$$\alpha(t) = (\cos(t), \sin(t), t).$$

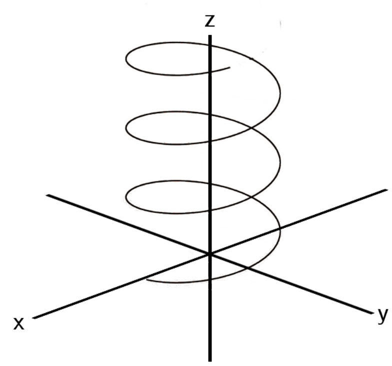

FIGURE 1. A helix in $\mathbb{R}^3$

The image of α is represented above in Figure 1.

Definition 2.3. The *tangent vector* t to a curve α at s is defined as $t(s) = \alpha'(s)$.

The tangent vector points in the direction of the curve at s as pictured below in Figure 2.

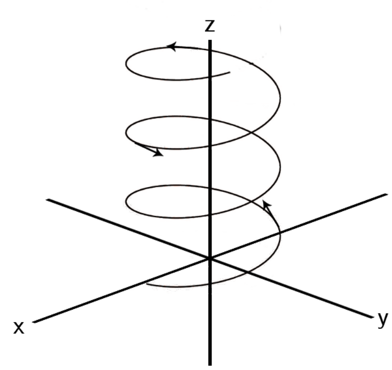

FIGURE 2. Tangent vectors of a helix at different times s

Remark 2.4. A curve α is *regular* if $\|\alpha'(s)\| \neq 0$ for all $s \in I$. All curves in this paper are assumed to be regular.

Definition 2.5. Given some curve α and $t \in (a, b)$, the *arc length* of a curve from time $t_0 \in (a, b)$ to t is defined as

$$s(t) = \int_{t_0}^{t} \|\alpha'(t)\| dt.$$

Definition 2.6. A *unit speed parametrization* of a curve α is some curve $\overline{\alpha}$ with the same image as α such that $\|\overline{\alpha}'(s)\| = 1$ for all $s \in I$.

Lemma 2.7. *For all curves α such that $\|\alpha'(t)\| > 0$ for all t, there exists a unit speed parametrization.*

Proof. Consider the arc length function,

$$s(t) = \int_{t_0}^{t} \|\alpha'(t)\| dt.$$

By differentiating both sides, we see that

$$s'(t) = \|\alpha'(t)\|.$$

Since $\|\alpha'(t)\| = s'(t) \neq 0$ for all t, there exists an inverse function $t(s)$ with

$$\frac{dt}{ds} = \frac{1}{\frac{ds}{dt}} = \frac{1}{\|\alpha'(t(s))\|}.$$

Let $\overline{\alpha}(s) = \alpha(t(s))$. We have

$$\overline{\alpha}'(s) = \alpha'(t(s))t'(s).$$

So we have

$$\|\overline{\alpha}'(s)\| = \|\alpha'(t(s))\| \frac{1}{\|\alpha'(t(s))\|} = 1.$$

This shows that α has a unit speed parametrization, namely the arc length parametrization. □

The image of the parametrization of α is called its *trace*. It is important to note that many curves have the same trace.

Example 2.8. The curves

$$\alpha(t) = (\cos(t), \sin(t), t) \text{ and } \beta(t) = (\cos(2t), \sin(2t), 2t)$$

both trace the helix shown in Figure 1.

For the rest of this paper, curves will be parametrized using arc length, s, so that we can achieve unit speed everywhere.

We now want to define the curvature of a (unit speed) curve α at the point s. Intuitively, the curvature measures the rate at which the curve turns which can be done by measuring the rate at which the tangent vector turns. For a unit speed curve, $\|\alpha'(s)\| = \|t(s)\| = 1$. This means that $t'(s)$ captures how the direction of $t(s)$ is changing near s. This motivates the following definition.

Definition 2.9. For a curve α, the *curvature* $\kappa(s)$ of α is $\kappa(s) = \|\alpha''(s)\| = \|t'(s)\|$.

Lemma 2.10. *Let $v : I \subset \mathbb{R} \to \mathbb{R}^3$ be a smooth map so that $v(s)$ is a unit vector for each s. Then for every s we have that $v(s)$ is perpendicular to its derivative $v'(s)$.*

Proof. Since $\|v(s)\| = 1$, we have $v(s) \cdot v(s) = 1$. By differentiating both sides, we can see that $v'(s) \cdot v(s) = 0$, showing that they are orthogonal. □

This means that $t(s)$ is perpendicular to $t'(s)$. We can write $t'(s) = \kappa(s)n(s)$ for some unit vector $n(s)$ that is perpendicular to $t(s)$.

Definition 2.11. The *normal vector* of a curve α is the unit vector $n(s)$ such that $t'(s) = \kappa(s)n(s)$.

Definition 2.12. The *binormal vector* b at s is defined as $b(s) = t(s) \times n(s)$.

It is important to note that since t and n are unit vectors, b is also a unit vector. Together these vectors form a geometric structure known as the *Frenet Trihedron*, pictured in Figure 3.

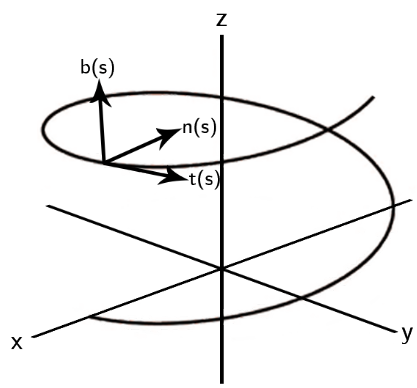

FIGURE 3. The Frenet Trihedron at some s

We should note that since each vector is perpendicular to the other two, each one is the cross product of the other two. That is to say, by the right-hand rule

$$\begin{aligned} t &= b \times n \\ n &= b \times t \\ b &= t \times n. \end{aligned}$$

Now we will examine the derivative of each of these vectors. By Lemma 2.10, we know that each vector in the Frenet Trihedron is perpendicular to its derivative. The derivative of t has already been found to equal κn, in our definition of the normal vector. Next, we will compute b'. Since $b = t \times n$, the product rule gives

$$b' = t' \times n + t \times n'.$$

By Lemma 2.10, we have $t' \times n = 0$ so we get

$$b' = t \times n'.$$

By Lemma 2.10, we know n' is not parallel to n since n' and n are perpendicular. In addition, n' cannot be parallel to t, otherwise $b'(s)$ would be 0 for all values of s. So n' must be parallel to b. So, for some scalar τ

$$b' = t \times \tau b = \tau n.$$

Definition 2.13. If $b'(s) = \tau(s)n(s)$, the value $\tau(s)$ is the *torsion* of a curve at s.

Lastly, we want to find the derivative of n'. We have that $n = b \times t$. By the product rule, we have

$$\begin{aligned} n' &= b' \times t + b \times t' \\ &= b \times \kappa n + \tau n \times t \\ &= -\kappa t - \tau b. \end{aligned}$$

Definition 2.14. The *Frenet-Serret equations* are the equations

$$\begin{aligned} t' &= \kappa n \\ n' &= -\kappa t - \tau b \\ b' &= \tau n. \end{aligned}$$

Before continuing, we must first discuss torsion a bit further. Like curvature, torsion does not measure a change in magnitude of a geometric structure, rather it measures how its orientation in space changes. This structure is known as the osculating plane.

Definition 2.15. For a curve α, the *osculating plane* at s is the 2-D plane formed by $t(s)$ and $n(s)$.

The torsion measures the change in direction of the binormal vector. Since the binormal vector is perpendicular to the osculating plane, the torsion measures how the osculating plane rotates throughout space. In this way, it measures how the curve twists through space. If a curve α has $\tau(s) = 0$ for all s, α is just a plane curve, since it does not twist through $\mathbb{R}^3$. However, if $\tau(s) \neq 0$, the curve twists through space and is no longer a plane curve.

The aim of this paper is to show that for any given values of curvature and torsion, there exists a unique curve with this curvature and torsion. To prove this, we will have to examine conditions for uniqueness and existence of solutions to ordinary differential equations.

3. Banach Fixed Point Theorem

The Frenet-Serret equations form a system of ordinary differential equations. The goal of this paper is to show that we can find a unique curve satisfying these differential equations. In order to find a unique solution, we first must find a solution. A simple example of a solution to an equation $F(x) = y$ is a fixed point, some x such that $F(x) = x$. Understanding theorems about fixed points will help us find solutions to differential equations.

Definition 3.1. Let M be a metric space with metric d. The mapping $A : M \to M$ is *Lipschitz-continous* if for all x, y in M, there exists some L such that

$$d(Ax, Ay) \leq Ld(x, y).$$

Note: The constant L is called the *Lipschitz constant.*

Remark 3.2. If a map A is Lipschitz-continuous, it immediately follows that A is continuous.

Lemma 3.3. *If A is continuously differentiable and its derivative is bounded on an interval, then A is Lipschitz-continuous on the interval.*

Proof. If A has a bounded derivative on some interval I, then for some L, $|A'(x)| \leq L$ for all $x \in I$. By the Fundamental Theorem of Calculus we have

$$\begin{aligned} |A(x) - A(y)| &\leq |\int_x^y A'(t)dt| \\ &\leq \int_x^y |A'(t)|dt \\ &\leq L|x-y| \end{aligned}$$

□

Definition 3.4. Let M be a metric space with metric d. The mapping $A : M \to M$ is a *contraction mapping* if it is Lipschitz-continous with Lipschitz constant λ such that $0 \leq \lambda < 1$.

Theorem 3.5 (Banach Fixed Point Theorem)**.** *Let M be a non-empty complete metric space and let $A : M \to M$ be a contraction mapping. Then A has a unique fixed point $x \in M$.*

Proof. Let $x, y \in M$. For some λ such that $0 \leq \lambda < 1$ we have

$$d(Ax, Ay) \leq \lambda d(x, y).$$

From this, we have the following

$$\begin{aligned} d(A^n x, A^{n-1}x) &\leq \lambda d(A^{n-1}x, A^{n-2}x) \\ &\leq \lambda^2 d(A^{n-2}x, A^{n-3}x) \\ &\vdots \\ &\leq \lambda^n d(Ax, x). \end{aligned}$$

So we have

$$\sum_{n=1}^{\infty} d(A^n x, A^{n-1}x) \leq \sum_{n=1}^{\infty} \lambda^n d(Ax, x).$$

Since $|\lambda| < 1$, we have that the geometric series

$$\sum_{n=1}^{\infty} \lambda^n$$

converges. Since $d(Ax, x)$ is a constant, we have that $\sum_{n=1}^{\infty} \lambda^n d(Ax, x)$ converges. Because of this, we can conclude that the sequence $x, Ax, A^2x, \ldots$ is a Cauchy sequence. Since M is a complete metric space the limit

$$\lim_{n\to\infty} A^n x$$

converges to some point $\widetilde{x}$ in M. Since A is continuous, we have

$$\begin{aligned} A\widetilde{x} &= A(\lim_{n\to\infty} A^n x) \\ &= \lim_{n\to\infty} A^{n+1} x \\ &= \widetilde{x}. \end{aligned}$$

Since $A\widetilde{x} = \widetilde{x}$, we have found a fixed point of the mapping and the proof is complete. □

4. Local Existence and Uniqueness of Solutions to ODEs

For the rest of this paper, I will be working in $C[0,T]$, the complete metric space of continuous functions on an interval $[0,T] \subset \mathbb{R}$ that map to $\mathbb{R}$, with metric $\|.\|_\infty$ where $\|.\|_\infty$ is the supremum norm. The supremum norm of $C[0,T]$, denoted $\|.\|_\infty$, is defined as $\|x\|_\infty = \sup\{\|x\| \text{ for } x \in C[0,T]\}$.
Consider the initial value problem

$$x(0) = x_0$$
$$x'(t) = F(x(t)).$$

If this initial value problem has a solution, by the Fundamental Theorem of Calculus, it will be of the form

$$x(t) = x_0 + \int_0^t F(x(s))ds.$$

This motivates the following definition.

Definition 4.1. Let $I \subset \mathbb{R}$ be the interval $[0,T]$ and let $x : I \to C[0,T]$. We will define the mapping $A : C[0,T] \to C[0,T]$ as

$$A(x)(t) = x_0 + \int_0^t F(x(s))ds.$$

A fixed point of A would give us a solution to the differential equation. In order to prove that a unique solution exists, we will show that A is a contraction mapping.

Theorem 4.2 (Local Existence and Uniqueness of Solutions to ODEs)**.** *Consider the initial value problem*

$$x(0) = x_0$$
$$x'(t) = F(x(t)).$$

If F is Lipschitz-continous on some interval $[0,T]$, then there exists a $T' \in [0,T]$ and a unique solution, $x(t)$ defined on $C^1[0,T']$.

Proof. Let F be Lipschitz-continuous on $C^1[0,T]$. We will show A is a contraction mapping.

For any $x, y \in C[0,T]$ we have

$$\begin{aligned}
\|A(x)(t) - A(y)(t)\|_\infty &= \left\| \int_0^t F(x(s))ds - \int_0^t F(y(s))ds \right\|_\infty \\
&= \left\| \int_0^t F(x(s)) - F(y(s))ds \right\|_\infty \\
&\leq \int_0^t \|F(x(s)) - F(y(s))\|_\infty ds \\
&\leq TL\|x(s) - y(s)\|_\infty
\end{aligned}$$

If we replace T with a possibly smaller time T' such that $T'L \leq \frac{1}{2}$ then the above inequality implies $A : C[0,T'] \to C[0,T']$ is a contraction mapping. By the Banach Fixed Point Theorem, we have found a local, unique solution to the initial value problem. □

5. Global Existence and Uniqueness of Solutions to ODEs

The existence of a local solution naturally motivates us to search for a global solution as well. Before we can prove this, we need the following lemma.

Lemma 5.1. *If M is a complete metric space and $A : M \to M$ has that for some $n \geq 1$, A^n is a contraction mapping, then A has a unique fixed point.*

Proof. Let A^n be a contraction mapping. By assumption, A^n has a unique fixed point x such that $A^n x = x$. Applying A to both sides we see that $A(A^n x) = Ax$. We can rewrite this to see $A^n(Ax) = Ax$. So Ax is a fixed point of A^n. By the uniqueness of the fixed point, we have that $Ax = x$. □

Theorem 5.2 (Global Existence and Uniqueness of Solutions to ODEs)**.** *Consider the initial value problem*

$$\begin{aligned}
x(0) &= x_0 \\
x'(t) &= F(x(t)).
\end{aligned}$$

If F is Lipschitz-continous on $[0,T]$ then there exists a unique solution, $x(t)$ on $C^1(-\infty, \infty)$ with $x(0) = x_0$.

Proof. Let F be Lipschitz-continuous on $[0,T]$ and fix $T > 0$.
Claim: For $n \geq 1$, and t in the interval $[0,T]$

$$\|A^n(x(t)) - A^n(y(t))\| \leq \frac{Lt^n}{n!}\|x - y\|_\infty$$

for some L in $\mathbb{R}$.

Proof. The $n = 1$ base case is clear from Theorem 4.2.

For the inductive step, assume for $n = k - 1$

$$\|A^{k-1}(x(t)) - A^{k-1}(y(t))\| \leq \frac{(Lt)^{k-1}}{(k-1)!}.$$

We have

$$\begin{aligned}
\|A^k(x(t)) - A^k(y(t))\| &\leq \int_0^t \|F(A^{k-1}(x(s))) - F(A^{k-1}(y(s)))\| ds \\
&\leq L \int_0^t \|A^{k-1}(x(s)) - A^{k-1}(y(s))\| ds \\
&\leq L \int_0^t \frac{(Ls)^{k-1}}{(k-1)!} \|x-y\|_\infty ds \\
&= \frac{L^k}{(k-1)!} \|x-y\|_\infty \int_0^t s^{k-1} ds \\
&= \frac{L^k}{(k-1)!} \frac{t^k}{k} \|x-y\|_\infty \\
&= \frac{(Lt)^k}{k!} \|x-y\|_\infty.
\end{aligned}$$

Since this holds for $n = k$, it holds true for all n. So the claim is proved. □

It easily follows that

$$\|A^n(x) - A^n(y)\|_\infty \leq \frac{(LT)^n}{n!} \|x-y\|_\infty.$$

The terms $\frac{(LT)^n}{n!}$ are the terms in the Taylor expansion of e^{TL}. Since the Taylor series of e^x converges uniformly,

$$\lim_{n\to\infty} \frac{(LT)^n}{n!} = 0.$$

So for sufficiently large n, it is certainly true that $\frac{(LT)^n}{n!} < 1$. This shows that for sufficiently large n, A^n is a contraction mapping. By Lemma 5.1, this means that A has a unique fixed point which is the solution to the differential equation. □

6. The Fundamental Theorem of Space Curves

Theorem 4.2 and Theorem 5.2 show the existence and uniqueness of a solution of an ordinary differential equation given an initial condition. For a curve α in $\mathbb{R}^3$, the initial condition not only considers the position of α, but also the direction in which the Frenet Trihedron is oriented. This difference is known as rigid motion.

Definition 6.1. A *translation* by a vector w in $\mathbb{R}^3$ is a map $A : \mathbb{R}^3 \to \mathbb{R}^3$ such that $A(v) = v + w$ for v in $\mathbb{R}^3$.

Definition 6.2. An *orthogonal transformation* is a map $T : \mathbb{R}^3 \to \mathbb{R}^3$ such that $T(u) \cdot T(v) = u \cdot v$ for all u, v in $\mathbb{R}^3$.

Definition 6.3. A *rigid motion* in $\mathbb{R}^3$ is a map $M : \mathbb{R}^3 \to \mathbb{R}^3$ that is the composition of an orthogonal transformation and a translation with a positive determinant. In other words, $M(v) = T(v) + c$, where $T : \mathbb{R}^3 \to \mathbb{R}^3$ is an orthogonal transformation and c is a vector in $\mathbb{R}^3$.

Proposition 6.4. *The norm of a vector and the angle θ, $0 < \theta < \pi$, between vectors are invariant under orthogonal transformations.*

Proof. Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be an orthogonal transformation and u, v be vectors in $\mathbb{R}^3$. We have $v \cdot v = T(v) \cdot T(v)$. This shows that $\|v\| = \|T(v)\|$. If θ is the angle between u and v, we have

$$cos\theta = \frac{u \cdot v}{\|u\|\|v\|} = \frac{T(u) \cdot T(v)}{\|T(u)\|\|T(v)\|}.$$

So the angle between vectors is invariant under rigid motion. $\square$

Lemma 6.5. *The arc length, curvature, and torsion of a parametrized curve are invariant under rigid motion.*

Proof. Let α be some curve from $I \to \mathbb{R}^3$ and $M : \mathbb{R}^3 \to \mathbb{R}^3$ be a rigid motion. Since M is norm-preserving,

$$s(t) = \int_{t_0}^{t} \|\alpha(t)\| dt = \int_{t_0}^{t} \|M\alpha(t)\| dt.$$

So the arc length is invariant under rigid motion.
For curvature, we have that $\kappa(s) = \|\alpha''(s)\|$. Since M is a linear transformation, $(M\alpha)''(s) = M(\alpha'')(s)$. Since M is norm-preserving, $\|\alpha''(s)\| = \|M(\alpha'')(s)\|$. So curvature is invariant under rigid motion.

For torsion, we have that

$$\begin{aligned}
\tau(s)n(s) &= b(s) \\
\tau(s)\frac{\alpha''(s)}{\kappa(s)} &= t(s) \times n(s) \\
M(\tau(s)\frac{\alpha''(s)}{\kappa(s)}) &= M(\alpha'(s) \times \frac{\alpha''(s)}{\kappa(s)}) \\
\tau(s)\frac{M\alpha''(s)}{\kappa(s)} &= (M\alpha)'(s) \times \frac{(M\alpha)''(s)}{\kappa(s)} \\
\tau(s)n_M(s) &= b_M(s)
\end{aligned}$$

where $n_M(s)$ is the normal vector of the curve $M\alpha$ at s and $b_M(s)$ is the binormal vector of $M\alpha$ at at s. So torsion is invariant under rigid motion. $\square$

Theorem 6.6 (Fundamental Theorem of Space Curves)**.** *Given differentiable functions $\kappa(s) > 0$ and $\tau(s)$, defined on some open interval $I = (a, b)$, there exists a parametrized curve $\alpha : I \to \mathbb{R}^3$ such that $\kappa(s)$ is the curvature of α and $\tau(s)$ is the torsion of α that is unique up to rigid motion.*

Proof. We have following Frenet-Serret equations:

$$\begin{aligned}
t' &= \kappa n \\
n' &= -\kappa t - \tau b \\
b' &= \tau n.
\end{aligned}$$

The vectors $t'(s), n'(s)$, and $b'(s)$ are 3-dimensional vectors. Let $\beta(s)$ be the curve in $\mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}^3$ such that

$$\beta(s) = (t(s), n(s), b(s))$$

where t, n, and b are vectors in $\mathbb{R}^3$. Let $F : \mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}^3$ be the mapping

$$F(\beta(s)) = (\kappa(s)n(s), -\kappa(s)t(s) - \tau(s)b(s), \tau(s)n(s))$$

with initial condition $F(0) = x_0 \in \mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}^3$. The mapping F is continuously differentiable, so according to Lemma 3.3, F is Lipschitz-continuous.

Since F is Lipschitz-continuous, according to Theorem 5.2, there exists a unique $\beta(s)$ in $\mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}^3$ such that $\beta'(s) = F(\beta(s))$. So, given some initial condition, there exist unique vectors, t, n, and b in $\mathbb{R}^3$ such that t, n, and b satisfy the Frenet-Serret equations. This gives us a Frenet Trihedron at each point s.

We still need to show that there exists some curve α with these tangent, normal, and binormal vectors. If we choose α such that $\alpha'(s) = t(s)$ at every s, by Theorem 5.2, we see that α is unique. Therefore, we have found a curve α with curvature κ and torsion τ that is unique up to rigid motion. This completes the proof. $\square$

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